

MA 4207 - Mathematical Logic

Course-Webpage <http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html>

Homework due in Week 13 (for both tutorial groups).

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Homework 13.1

Are the following sets of sentences effectively enumerable:

1. $\{\alpha \in T : \text{every group satisfies } \alpha\}$;
2. $\{\alpha \in T : \text{every Abelian group satisfies } \alpha\}$?

Here T is the set of all sentences in the logical language with one operation \circ and one constant e and one function f to denote the group operation, neutral element and inversion, respectively.

Homework 13.2

Let the logical language contain exactly one predicate P and no function symbols; the predicate P is unary (one input only). Recall that a sentence is a formula with no free variables. Make a sentence α such that, for each n , there are, up to isomorphism, exactly $n - 1$ models of α with n elements.

Homework 13.3

Let the logical language contain the predicates P_0, P_1, \dots and let Γ for all n, m with $m < n$ contain the following formulas:

$$\exists x \forall y [P_n(x) \wedge (P_n(y) \rightarrow y = x)], \forall x [\neg P_n(x) \vee \neg P_m(x)].$$

How many models of finite cardinality, of cardinality \aleph_0 and or cardinality \aleph_1 does Γ have? Here isomorphic models should not be double counted.

Homework 13.4

Two structures are elementarily equivalent iff they satisfy the same sentences. Is there a structure which is elementarily equivalent to the real numbers with addition and multiplication, but not isomorphic to it? Explain your answer.

Homework 13.5

Assume that two sets of sentences Γ and Δ do not have any structure in common, that is, any structure of Γ fails to satisfy all formulas in Δ and every structure of Δ fails to satisfy all formulas of Γ , but both sets Γ and Δ are consistent. Is there a single sentence α such that all structures of Γ satisfy α and none of Δ does?

Homework 13.6

Let a structure $\mathcal{Z} = (\mathbb{Z}, \dots, -2, P_{-2}, -1, P_{-1}, 0, P_0, 1, P_1, 2, P_2, \dots)$ contain all integers and constants for all integers so that if c_n is the constant for n and P_n the predicate

for n then $P_n(x)$ is true in the model iff $x \leq c_n$. Note that \leq itself is not part of the logical language. Up to isomorphism, how many countable models are there which are elementarily equivalent to \mathcal{Z} ? 0 or 1 or ... or countably infinite or uncountably infinite models?

Homework 13.7

Let a structure \mathcal{Q} contain the domain \mathbb{Q} and for each rational number q a constant c_q and a predicate P_q such that $P_q(x)$ is true iff $x \leq q$. Note that \leq itself is not part of the logical language. Up to isomorphism, how many countable models are there which are elementarily equivalent to \mathcal{Q} ? 0 or 1 or ... or countably infinite or uncountably infinite models?

Homework 13.8

Recall that a theory is \aleph_0 -categorical iff it has an infinite model and every two countable infinite models are isomorphic. Let the logical language have only one unary predicate P and equality $=$. Show that every complete theory of this logical language either has only a finite model or has an infinite model and is \aleph_0 -categorical.

Homework 13.9

Let $Mod(S)$ denote the set of models of S . Show the following for sets S, T of sentences:

1. If $S \subseteq T$ then $Mod(T) \subseteq Mod(S)$;
2. $Mod(S \cup T) \subseteq Mod(S) \cap Mod(T)$;
3. If $Mod(S) = Mod(T)$ then $Mod(S) = Mod(S \cup T)$.

Homework 13.10

Are there sets S, T of sentences such that a model \mathcal{A} is finite iff $\mathcal{A} \models S$ or $\mathcal{A} \models T$?

Homework 13.11

Let (G, \circ, e) be a group with 8 elements. Show that every group (H, \bullet, d) which is elementarily equivalent to (G, \circ, e) is also isomorphic to (G, \circ, e) .

Homework 13.12

Provide an example of an infinite group (G, \circ, e) such that every group which is elementarily equivalent to (G, \circ, e) and has the same number of elements as (G, \circ, e) is also isomorphic to (G, \circ, e) . Hint: Use an Abelian group also satisfying some torsion axiom, say $\forall x [x \circ x \circ x = e]$. It does not really matter which of these axioms is chosen.

Homework 13.13

Let the logical language have one unary predicate P and equality. Furthermore, assume that a theory T has for each n an axiom which says that at least n elements x satisfy $P(x)$ and another n elements satisfy $\neg P(x)$. Show that this theory is not \aleph_1 -categorical and determine the number of models of cardinality \aleph_1 it has – note that one can split a set of cardinality \aleph_1 into two sets of cardinality κ, λ iff $\max\{\kappa, \lambda\} = \aleph_1$. The cardinals up to \aleph_1 are $0, 1, 2, \dots, \aleph_0, \aleph_1$.