**MA 4207 - Mathematical Logic**
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

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**Homework 3.1**
Let $f(n)$ be the maximum number of negation symbols in a well-formed formula which does not contain any subformula of the form $(\neg(\neg\alpha))$ and which contains at most $n$ atoms. Here $(\neg(A_1 \vee (\neg(A_2 \vee (\neg A_1)))))$ has 3 atoms and $n$ is 3, as repeated atoms are counted again. Determine the value $f(n)$ in dependence of $n$.

**Homework 3.2**
Prove by induction that a well-formed formula of length $n$ contains less than $n/3$ connectives and at most $(n+3)/4$ atoms.

**Homework 3.3**
Use the truth-table method to prove that the following formulas are equivalent:

- $((\neg A_1) \vee (\neg A_2))$;
- $(\neg(A_1 \wedge A_2))$;
- $((A_1 \vee A_2) \leftrightarrow (A_1 \oplus A_2))$.

**Homework 3.4**
Use the truth-table method to check whether the following statement is correct:

$$\{(A_1 \vee A_2), (A_2 \vee A_3), (A_1 \vee A_3)\} \models ((A_1 \wedge A_2) \vee A_3).$$

**Homework 3.5**
Use the truth-table method to check whether the following statement is correct:

$$\{(A_1 \to A_2), (A_2 \to A_3), (A_3 \to A_4)\} \models ((A_1 \to A_3) \wedge (A_2 \to A_4))$$

**Homework 3.6**
Use the truth-table method to check whether the following statement is correct:

$$\{(A_1 \to A_2), (A_2 \to A_3), (A_3 \to A_4)\} \models (A_4 \to A_1)$$

**Homework 3.7**
List out the truth-table for the formula $(((A_1 \oplus A_2) \wedge (\neg A_3)) \oplus (A_1 \vee A_3))$.

**Homework 3.8**
List out the truth-table for the formula $((A_1 \oplus A_3) \vee (((A_1 \oplus A_2) \wedge (A_2 \oplus A_3)))$.

**Homework 3.9**
Consider the following formulas:

$$\begin{aligned}\phi_1 &= (((A_1 \vee A_2) \vee A_3) \wedge ((A_4 \vee A_5) \vee A_6));\\ \phi_2 &= (((A_1 \vee A_2) \wedge (A_3 \vee A_4)) \wedge (A_5 \vee A_6));\\ \phi_3 &= (((((A_1 \oplus A_2) \oplus A_3) \oplus A_4) \oplus A_5) \oplus A_6).\end{aligned}$$

There are $2^6 = 64$ ways to assign the truth-values to the sentence symbols (or atoms) $A_1, \ldots, A_6$. Determine for each of the formulas $\phi_1, \phi_2, \phi_3$, how many of these assignments make the formula true and how many of these assignments make the formula false.

**Homework 3.10**
For the formulas from Homework 3.9, is the statement

$$\{\phi_1, \phi_2, \phi_3\} \models (((((A_1 \wedge A_2) \wedge A_3) \wedge A_4) \wedge A_5) \wedge A_6)$$

true or false? Prove your answer.

**Homework 3.11**
For the formulas from Homework 3.9, is the statement

$$\{\phi_1, \phi_2, \phi_3\} \models (((((A_1 \vee A_2) \vee A_3) \vee A_4) \vee A_5) \vee A_6)$$

true or false? Prove your answer.

**Homework 3.12**
Using the connectives $\vee, \wedge, \rightarrow, \leftrightarrow, \oplus, \neg$, construct a formula using atoms $A_1, A_2, A_3, A_4$ which says that at least two and at most three of these atoms are true.

**Homework 3.13**
Using the connectives $\vee, \wedge, \rightarrow$, construct a formula using atoms $A_1, A_2, A_3, A_4, A_5, A_6$ which says that either all six atoms are false or all six atoms are true.

**Homework 3.14**
Use the truth-table method to prove the associativity of $\leftrightarrow$, that is, prove that $(A_1 \leftrightarrow (A_2 \leftrightarrow A_3))$ and $((A_1 \leftrightarrow A_2) \leftrightarrow A_3)$ are the same. Furthermore, check whether there is a truth-assignment $\nu$ with $\overline{\nu}((A_1 \leftrightarrow (A_2 \leftrightarrow A_3))) = 1$ and $\nu(A_1) \neq \nu(A_2)$.

**Homework 3.15**
Use the truth-table method to check whether $(A_1 \oplus (A_2 \oplus A_3))$ and $(A_1 \leftrightarrow (A_2 \leftrightarrow A_3))$ are equivalent.

**Homework 3.16**
Make the truth-tables of $\wedge$, $\oplus$ and $\neg$ for $\{0, u, 1\}$-valued logic where the value $u$ stands for an unknown value of 0 and 1 and where the output $u$ is taken iff one cannot derive from the inputs what the output is. Note that two inputs $u$ need not to represent the same of 0 and 1.

**Homework 3.17**
Make the truth-tables of $\rightarrow$, $\leftrightarrow$ and $\vee$ for the $\{0, u, 1\}$-valued logic from 3.16.