

## MA 4207 - Mathematical Logic

Course-Webpage <http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html>  
Homework

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### Homework 4.1

Let  $atom(\phi)$  be the set of atoms used in  $\phi$ , so  $atom(((A_1 \vee A_2) \wedge A_2)) = \{A_1, A_2\}$  and  $atom((0 \vee 1)) = \emptyset$ . Let WFF be the set of well-formed formulas. Let

$$C_1 = \{\phi \in WFF : \forall v [\text{if } v(A) = 1 \text{ for some } A \in atom(\phi) \text{ then } \bar{v}(\phi) = 1]\}.$$

For which of the connectives  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$  is  $C_1$  closed under the connective? Here one says that  $C$  is closed under the connective  $\oplus$  if all formulas  $\phi, \psi \in C$  satisfy that  $(\phi \oplus \psi) \in C$ . Similarly for other connectives.

### Homework 4.2

Let  $atom(\phi)$  and WFF be defined as in Homework 4.1. Let

$$C_2 = \{\phi \in WFF : \forall v [\text{if } v(A) = 0 \text{ for at most one } A \in atom(\phi) \text{ then } \bar{v}(\phi) = 1]\}.$$

For which of the connectives  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$  is  $C_2$  closed under the connective?

### Homework 4.3

Let  $C_3 = \{\phi \in WFF : \text{every } A \in atom(\phi) \text{ occurs in } \phi \text{ exactly once and } 0, 1 \text{ do not occur in } \phi\}$ . Prove by induction that  $C_3$  does not contain any tautology and also not contain any antitautology. Here a tautology is a formula which is always true (independent of the choice of the truth-values of the atoms) and an antitautology is a formula which is always false.

### Homework 4.4

Define on WFF by recursion the functions  $atom$  ( $\phi \mapsto atom(\phi)$ ) and  $maxatom$  ( $\phi \mapsto \max\{k : A_k \in atom(\phi)\}$ ), where the atoms  $A_1, A_2, \dots$  can be used and  $\max \emptyset = 0$ . So  $maxatom((A_1 \vee (A_4 \wedge A_5))) = 5$  and  $maxatom((0 \vee 1)) = 0$ .

### Homework 4.5

Define on WFF by recursion the function  $numcon(\phi)$  as the number of connectives  $\wedge, \vee, \rightarrow, \leftrightarrow, \oplus$  occurring in  $\phi$ . Furthermore define the function  $numneg(\phi)$  as the number of negations occurring in  $\phi$ . Determine the best-possible constants  $c, m, n$  such that

$$|\phi| \leq c \cdot numcon(\phi) + n \cdot numneg(\phi) + m$$

for all WFF  $\phi$ .

**Homework 4.6**

Let  $C_6 = \{\phi \in WFF : \text{every } A \in \text{atom}(\phi) \text{ occurs in } \phi \text{ exactly once}\}$ ; note that formulas in  $C_6$  might have occurrences of the constants 0 and 1. Define by recursion a function  $F$  from  $C_6$  into the rational numbers between 0 and 1 which returns for each formula  $\phi \in C_6$  the truth-probability  $n/2^m$  where  $n$  is the number of rows in the truth-table of  $\phi$  evaluated to 1 and  $m$  is the number of atoms used in the formula so that  $2^m$  is the overall number of rows in the truth-table of  $\phi$ . For example,  $F(1) = 1$ ,  $F((A_1 \oplus (A_2 \vee A_3))) = 1/2$  and  $F(((A_2 \vee A_5) \wedge (A_3 \vee 0))) = 3/8$ .

**Homework 4.7**

Let  $C_7 = \{\phi \in WFF : \phi \text{ can use the constants } 0, 1 \text{ and the only connectives in } \phi \text{ are } \wedge \text{ and } \vee\}$ . Prove by induction that a formula  $\phi \in C_7$  is a tautology iff  $\bar{v}(\phi) = 1$  for the truth-assignment  $v$  with  $v(A_k) = 0$  for all  $k$ .

**Homework 4.8**

Let  $C_8 = \{\phi \in WFF : \phi \text{ can use the constants } 0, 1 \text{ and the only connectives in } \phi \text{ are } \wedge \text{ and } \vee\}$ . Prove by induction that a formula  $\phi \in C_8$  is an antitautology iff  $\bar{v}(\phi) = 0$  for the truth-assignment  $v$  with  $v(A_k) = 1$  for all  $k$ .

**Homework 4.9**

Let  $U$  be a finite set of atoms and  $C_9 = \{\phi \in WFF : \text{atom}(\phi) \subseteq U\}$ . Prove that there is a finite set  $F$  of formulas such that for every formula  $\phi \in C_9$  there is a  $\psi \in F$  with  $(\psi \leftrightarrow \phi)$  being a tautology.

**Homework 4.10**

The following formulas have brackets omitted according to the rule that the binding strengths of the connectives is ordered as  $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$ . Insert back the needed brackets for getting a member of WFF.

1.  $A_1 \wedge \neg A_2 \vee A_3 \rightarrow A_4 \wedge \neg A_5$ ;
2.  $A_1 \vee \neg A_2 \wedge \neg A_3 \leftrightarrow A_4 \rightarrow A_5$ ;
3.  $\neg \neg A_1 \vee \neg A_2$ .

**Homework 4.11**

Is there a formula using the connectives  $\oplus$  and  $\neg$  but no other connectives where the value of the formula depends on the placement of brackets?

**Homework 4.12**

Let  $v(A_1) = 1$ ,  $v(A_2) = 1$ ,  $v(A_3) = 0$ . The below formulas are given in Polish notation. Write them as WFF and evaluate them according to  $v$ :

1.  $\neg \leftrightarrow \oplus A_1 A_2 A_3$ ;
2.  $\wedge \vee \neg \vee A_1 A_2 A_3 A_1$ ;
3.  $\oplus \wedge A_1 A_2 \wedge A_2 A_3$ .

**Homework 4.13**

Write the following formula in Polish notation:  $\neg((A_1 \vee \neg A_2) \wedge (A_2 \vee \neg A_3) \wedge (A_3 \vee \neg A_1))$ .

**Homework 4.14**

Consider the formula  $\phi$  in 9 atoms  $A_1, \dots, A_9$  and consider the set  $S$  of all  $(A_i \oplus A_j \oplus A_k)$  with  $A_i \in \{A_1, A_2, A_3\}$ ,  $A_j \in \{A_4, A_5, A_6\}$  and  $A_k \in \{A_7, A_8, A_9\}$ . Now let  $\phi_1$  be the conjunction of all formulas in  $S$ ; that is, if  $S = \{\psi_1, \psi_2, \dots, \psi_n\}$  then  $\phi_1 = \psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_n$  when the brackets are omitted. Determine how many assignments of these atoms make the formula true and how many make it false.

**Homework 4.15**

Let  $\phi_2$  to be taken the disjunction of the formulas in set  $S$  from Homework 4.14. Determine how many assignments of these atoms make the formula true and how many make it false.

**Homework 4.16**

The following formulas have brackets omitted according to the rule that the binding strengths of the connectives is ordered as  $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$ . Insert back the needed brackets for getting a member of WFF and evaluate the formulas for the truth-assignment  $\nu$  which sets all atoms to 1.

1.  $A_1 \vee \neg A_2 \wedge A_3 \leftrightarrow A_4 \rightarrow \neg A_5 \vee A_3 \wedge A_6 \oplus A_7$ ;
2.  $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow \neg A_5$ ;
3.  $\neg \neg \neg A_1 \oplus A_2 \oplus A_3$ .