MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

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Homework 5.1
Consider six-valued logic with values $\{0.0, 0.2, 0.4, 0.6, 0.8, 1.0\}$ ordered in the same way as rational numbers and define that $x \wedge y$ is the minimum and $x \vee y$ is the maximum of the inputs; furthermore, $\neg x$ is $1.0 - x$ (with the usual numerical operation). Now, for any formula α, let $\min(\alpha) = \min\{\overline{\nu}(\alpha) : \nu$ is a six-valued truth-assignment of the atoms$\}$. Similarly one defines $\max(\alpha)$. Which are the possible values which $\min(\alpha)$ and $\max(\alpha)$ can take where α ranges over all formulas obtained by connecting atoms using $\wedge, \vee, \neg$?

Homework 5.2
If one uses the connectives $\wedge, \vee, \neg, \oplus$ for the six-valued logic from Homework 5.1 with $x \oplus y$ being defined as $\min\{x + y, 2 - x - y\}$, what are the possible values of $\min(\alpha)$ and $\max(\alpha)$, which are defined as in Homework 5.1.

Homework 5.3
For the six-valued logic from Homework 5.1 and 5.2, say that two formulas α, β are equivalent iff they for all six-valued truth-assignments ν to the atoms satisfy $\overline{\nu}(\alpha) = \overline{\nu}(\beta)$. Check whether the following equivalences of formulas hold in the six-valued logic:

1. $(A_1 \wedge A_2) \equiv (A_2 \wedge A_1)$;
2. $(A_1 \wedge (A_2 \vee A_3)) \equiv ((A_1 \wedge A_2) \vee (A_1 \wedge A_3))$;
3. $(A_1 \wedge (A_2 \oplus A_3)) \equiv ((A_1 \wedge A_2) \oplus (A_1 \wedge A_3))$.

Homework 5.4
For the six-valued logic from Homework 5.1 and 5.2, let $x \leftrightarrow y$ be calculated by $1 - \max\{x - y, y - x\}$. Using the equivalence definition from Homework 5.3, check whether the following formulas are equivalent:

1. $(A_1 \leftrightarrow A_2) \equiv \neg(A_1 \oplus A_2)$;
2. $(A_1 \leftrightarrow A_2) \equiv (A_1 \oplus \neg A_2)$;
3. $\neg(A_1 \wedge A_2) \leftrightarrow (\neg A_1 \vee \neg A_2) \equiv 1.0$;
4. $((A_1 \leftrightarrow A_2) \leftrightarrow A_3) \equiv (A_1 \leftrightarrow (A_2 \leftrightarrow A_3))$.

Homework 5.5
Construct a circuit for $A_1 \oplus A_2 \oplus A_3$ with the gates $\wedge, \vee, \neg$; these gates can have multiple inputs.

Homework 5.6
Construct a circuit for $A_1 \oplus A_2 \oplus A_3$ using *nand* and *nor* and *not* gates, which can have multiple inputs.

Homework 5.7
Assume that a company uses only chips which output 0 when all inputs are 0 – as all-0-inputs and outputs are considered as an "error-information" and every useful information is coded by input-vectors which are not everywhere 0. Now a vendor offers to produce the chips as specified using only "exclusive-or-gates" ($\oplus$) and "inclusive-or-gates" ($\vee$) at a very competitive price. The company boss finds it suspicious and asks the company's technician: Can this work? Provide the correct answer and prove why it can work or why it cannot work.

Homework 5.8
Recall that $maj(x, y, z)$ is 1 iff at least two of the inputs x, y, z are 1. Let C_8 consist of all formulas which are atoms or which are formed from other formulas $\alpha, \beta, \gamma \in C_8$ by taking $maj(\alpha, \beta, \gamma)$ or $\neg\alpha$. Prove by induction that each Boolean function B^n_α formed from an $\alpha \in C_8$ satisfies $B^n_\alpha(x_1, \ldots, x_n) = \neg B^n_\alpha(\neg x_1, \ldots, \neg x_n)$ for $x_1, \ldots, x_n \in \{0, 1\}$.

Homework 5.9
How many Boolean functions can be formed by using input variables $x_1, \ldots, x_n$ and the constants and connectives from $\{0, 1, \wedge\}$.

Homework 5.10
How many Boolean functions can be formed using input variables $x_1, \ldots, x_n$ and the constants and connectives from $\{0, 1, \neg, \oplus\}$.

Homework 5.11
Assume that α can use some of the atoms $A_1, \ldots, A_5$, the truth-values 0 and 1 and up to two connectives $\oplus$. How many functions of the form B^5_α can be formed using such α?

Homework 5.12
Assume that α can use some of the atoms $A_1, \ldots, A_4$, the truth-values 0 and 1 and at most one connective $\wedge$ and at most one connective $\vee$. How many functions of the form B^4_α can be formed using such α?

Homework 5.13
Use as few of "and" ($\wedge$) and "inclusive or" ($\vee$) as possible in order to make a formula α with four atoms A_1, A_2, A_3, A_4 such that the following conditions hold:

- If at least three of the atoms A_1, A_2, A_3, A_4 are true then α is true;
- If at most one of the atoms A_1, A_2, A_3, A_4 are true then α is false;

- If exactly two of the atoms A_1, A_2, A_3, A_4 are true then there is no constraint on which value α takes.

Use the last condition in order to optimise the number of connectives in the formula.

Homework 5.14
Let F be the set of all Boolean formulas with input variables x_1, x_2, x_3, x_4 which are 0 when at most one input variable is 1 and which are 1 when at least three input variables are 1. So F contains functions B^4_α for formulas α like $(A_1 \vee A_2) \wedge (A_3 \vee A_4)$. Which of the following sets of connectives satisfy to generate all formulas in F (plus some outside F): $\{\wedge, \vee\}$, $\{\wedge, \oplus\}$, $\{\oplus, \leftrightarrow\}$?

Homework 5.15
Recall that $maj(x, y, z)$ is 1 iff at least two of the inputs x, y, z are 1. Is there an $n \in \{1, 2, 3, 4\}$ for which the set $\{maj, B^n_{A_1 \oplus A_2 \oplus \ldots \oplus A_n}\}$ complete? For those n where it is incomplete, can it be made complete by adding the logical constants $0, 1$ to the set of connectives? If so, which of these are needed?