MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

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Homework 7.1
Recall that a set is recursively enumerable iff it is empty or is the range of a function computed by an effective procedure (also called recursive function). Consider now three X, Y, Z be effectively enumerable sets which all contain 0 and which are the ranges of functions F_X, F_Y, F_Z which are given by effective procedures. Make recursive functions G, H such that the range of G is $X \cup Y \cup Z$ and the range of H is $(X \cap Y) \cup (X \cap Z) \cup (Y \cap Z) = \{u : u \text{ is in at least two of the sets } X, Y, Z\}$.

Homework 7.2
Prove that the following set S is decidable: S is the set of all n such that there are infinitely many natural numbers m for which both m and $m+n$ are powers of 2.

Homework 7.3
Prove that the following set S is decidable: S contains all numbers x for which there are infinitely many pairs y, z of prime numbers satisfying that $y < z \leq y + x$.

Homework 7.4
A binary tree T is a set of binary strings such that whenever $\sigma\tau \in T$ then $\sigma \in T$ (where $\sigma\tau$ is the concatenation of σ and τ). König's Lemma says that every infinite binary tree contains an infinite branch. Now let $A_1, A_2, \ldots$ be the atoms and let $S = \{\alpha_1, \alpha_2, \ldots\}$ be a set of formulas. Now let T be a binary tree which on level n contains all those $\sigma \in \{0,1\}^n$ which satisfy for all formulas $\beta \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, if no atom A_k with $k > n$ occurs in β then every v with $v(A_k) = \sigma(k)$ makes β true. Prove the following: If T is infinite then T has an infinite branch and each infinite branch defines a v with $v \models S$; if T is finite then there is a first level n on which T has no nodes and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is not satisfiable.

Homework 7.5
Let $S = \{\alpha : \alpha \text{ is a well-formed formula and } \overline{v}(\alpha) = 1 \text{ iff the majority of the atoms } A_k \in atom(\alpha) \text{ are } 1\}$. Prove that S is decidable.

Homework 7.6
Let ν be computed by an effective procedure mapping each $k \in \mathbb{N}$ to the truth-value assigned to atom A_k. Let $S = \{\alpha \in WFF : \overline{\nu}(\alpha) = 1\}$. Which of the following options is correct?
(a) S is decidable; (b) S is recursively enumerable but not decidable; (c) S is not

recursively enumerable.

Homework 7.7
Assume that you know that addition, subtraction and multiplication are effectively computable. Use now recursion in one variable to show that (a) the integer division $n, m \mapsto \max\{k : k \cdot m \leq n\}$ and (b) $n \mapsto \binom{2n}{n}$ are effectively computable functions. Note that the recursion can use case-distinctions; for example, the inductive definition of the remainder $f(a, b)$ of a by b is $f(0, b) = 0$ and if $f(a, b)+1 < b$ then $f(a+1, b) = f(a, b) + 1$ else $f(a + 1, b) = 0$.

Homework 7.8
Prove that if S is a satisfiable set of formulas then $WFF - S$ is not a satisfiable set of formulas.

Homework 7.9
Assume that S_1, S_2, S_3 are satisfiable sets of formulas. What about the set $T = (S_1 \cup S_2) \cap (S_1 \cup S_3) \cap (S_2 \cup S_3)$? Prove that T is satisfiable or give an example of S_1, S_2, S_3 where the resulting T is not satisfiable.

Homework 7.10
Let $S = \{\alpha \in WFF : \overline{\nu}(\alpha) = 1\}$ for some ν and $T = \{\alpha : (\alpha \vee A_1), (\alpha \vee (\neg A_1)) \in S\}$. Is T satisfiable? Is $S = T$?

Homework 7.11
Call a set S of formulas almost-zero-satisfiable (azs) iff there is a ν with $\nu(A_k) = 0$ for almost all atoms and $\overline{\nu}(\alpha) = 1$ for all $\alpha \in S$. Does the notion "azs" satisfy the compactness theorem? That is, for any infinite set $S \subseteq WFF$, if every finite subset is almost-zero-satisfiable, is then S itself also almost-zero-satisfiable?

Homework 7.12
Are there infinite sets S, T of wff such that every finite subset T' of T there is a finite subset S' of S such that $S' \models \alpha$ for all $\alpha \in T'$ but it does not hold that $S \models T$, that is, there is some ν which is true on all members of S but not all members of T.

Homework 7.13
Assume that there are infinitely many logical atoms. Is there a set S of formulas such that for all ν mapping atoms to $\{0, 1\}$, $\nu \models S$ iff there are exactly three atoms A, B, C with $\nu(A) = 1, \nu(B) = 1, \nu(C) = 1$?