MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework

Frank Stephan. Departments of Mathematics and Computer Science,
10 Lower Kent Ridge Road, S17#07-04 and 13 Computing Drive, COM2#03-11,
National University of Singapore, Singapore 119076.
Email fstephan@comp.nus.edu.sg
Telephone office 65162759 and 65164246
Office hours Tuesday 13.00-14.00h at Mathematics S17#07-04

Homework 13.1
Are the following sets of sentences effectively enumerable:

1. $\{\alpha \in T :$ every group satisfies $\alpha\}$; 2. $\{\alpha \in T :$ every Abelian group satisfies $\alpha\}$?

Here T is the set of all sentences in the logical language with one operation $\circ$ and one canstant e and one function f to denote the group operation, neutral element and inversion, respectively.

Homework 13.2
Let the logical language contain exactly one predicate P and no function symbols; the predicate P is unary (one input only). Recall that a sentence is a formula with no free variables. Make a sentence α such that, for each n, there are, up to isomorphism, exactly $n-1$ models of α with n elements.

Homework 13.3
Let the logical language contain the predicates $P_0, P_1, \ldots$ and let Γ for all n, m with $m < n$ contain the following formulas:

$$\exists x \, \forall y \, [P_n(x) \wedge (P_n(y) \rightarrow y = x)], \forall x \, [\neg P_n(x) \vee \neg P_m(x)].$$

How many models of finite cardinality, of cardinality $\aleph_0$ and or cardinality $\aleph_1$ does Γ have? Here isomorphic models should not be double counted.

Homework 13.4
Two structures are elementarily equivalent iff they satisfy the same sentences. Is there a structure which is elementarily equivalent to the real numbers with addition and multiplication, but not isomorphic to it? Explain your answer.

Homework 13.5
Assume that two sets of sentences Γ and Δ do not have any structure in common, that is, any structure of Γ fails to satisfy all formulas in Δ and every structure of Δ fails to satisfy all formulas of Γ, but both sets Γ and Δ are consistent. Is there a single sentence α such that all structures of Γ satisfy α and none of Δ does?

Homework 13.6
Let a structure $\mathcal{Z} = (\mathbb{Z}, \ldots, -2, P_{-2}, -1, P_{-1}, 0, P_0, 1, P_1, 2, P_2, \ldots)$ contain all integers and constants for all integers so that if c_n is the constant for n and P_n the predicate

for n then $P_n(x)$ is true in the model iff $x \leq c_n$. Note that $\leq$ itself is not part of the logical language. Up to isomorphism, how many countable models are there which are elementarily equivalent to $\mathcal{Z}$? 0 or 1 or ... or countably infinite or uncountably infinite models?

Homework 13.7
Let a structure $\mathcal{Q}$ contain the domain $\mathbb{Q}$ and for each rational number q a constant c_q and a predicate P_q such that $P_q(x)$ is true iff $x \leq q$. Note that $\leq$ itself is not part of the logical language. Up to isomorphism, how many countable models are there which are elementarily equivalent to $\mathcal{Q}$? 0 or 1 or ... or countably infinite or uncountably infinite models?

Homework 13.8
Recall that a theory is $\aleph_0$-categorical iff it has an infinite model and every two countable infinite models are isomorphic. Let the logical language have only one unary predicate P and equality $=$. Show that every complete theory of this logical language either has only a finite model or has an infinite model and is $\aleph_0$-categorical.

Homework 13.9
Let $Mod(S)$ denote the set of models of S. Show the following for sets S, T of sentences:

1. If $S \subseteq T$ then $Mod(T) \subseteq Mod(S)$;
2. $Mod(S \cup T) \subseteq Mod(S) \cap Mod(T)$;
3. If $Mod(S) = Mod(T)$ then $Mod(S) = Mod(S \cup T)$.

Homework 13.10
Is there a sentence α such that α has a model with κ members in the domain iff $\kappa = n^2$ for some $n \in \{1, 2, 3, \ldots\}$ or $\kappa \geq \aleph_0$, where the underlying logical language has one unary predicate P and one binary operation $\circ$ (α can use these).

Homework 13.11
Let $(G, \circ, e)$ be a group with 8 elements. Show that every group $(H, \bullet, d)$ which is elementarily equivalent to $(G, \circ, e)$ is also isomorphic to $(G, \circ, e)$.

Homework 13.12
Provide an example of an infinite group $(G, \circ, e)$ such that every group which is elementarily equivalent to $(G, \circ, e)$ and has the same number of elements as $(G, \circ, e)$ is also isomorphic to $(G, \circ, e)$. Hint: Use an Abelian group also satisfying some torsion axiom, say $\forall x\, [x \circ x \circ x = e]$. It does not really matter which of these axioms is chosen.

Homework 13.13
Let the logical language have one unary predicate P and equality. Furthermore, assume that a theory T has for each n an axiom which says that at least n elements x satisfy $P(x)$ and another n elements satisfy $\neg P(x)$. Show that this theory is not $\aleph_1$-categorical and determine the number of models of cardinality $\aleph_1$ it has – note that one can split a set of cardinality $\aleph_1$ into two sets of cardinality κ, λ iff $\max\{\kappa, \lambda\} = \aleph_1$. The cardinals up to $\aleph_1$ are $0, 1, 2, \ldots, \aleph_0, \aleph_1$.