Regressive versions of Hindman's Theorem

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Introduction

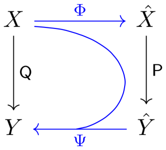
- In Reverse Mathematics, Hindman's Theorem represents an active line of research: for instance, the strength of the theorem itself is a long-standing open question.
- The same applies to many of its variants formulated over the decades.
- We isolate a new natural variant of Hindman's Theorem, called the Regressive Hindman's Theorem, modelled on Kanamori-McAloon's Regressive Ramsey's Theorem.
- We investigate its strength in terms of provability over RCA_0 and in terms of computable reductions.

Introduction

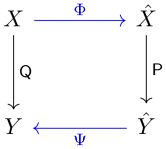
In terms of computable reductions, we focus on Weihrauch reductions.

They concern principles in the form $(\forall X)[\varphi(X) \rightarrow (\exists Y)\,\psi(X, Y)]$.
We call X s.t. $\varphi(X)$ an instance and Y s.t. $\psi(X, Y)$ a solution for X.

1. Q is Weihrauch reducible to P (denoted $\mathsf{Q} \leq_{\mathrm{W}} \mathsf{P}$) if there exist Turing functionals Φ and Ψ such that for every instance X of Q we have that $\Phi(X)$ is an instance of P, and if $\hat{Y}$ is a solution to P for $\Phi(X)$ then $\Psi(X \oplus \hat{Y})$ is a solution to Q for X.



2. Q is strongly Weihrauch reducible to P (denoted $\mathsf{Q} \leq_{\mathrm{sW}} \mathsf{P}$) if there exist Turing functionals Φ and Ψ such that for every instance X of Q we have that $\Phi(X)$ is an instance of P, and if $\hat{Y}$ is a solution to P for $\Phi(X)$ then $\Psi(\hat{Y})$ is a solution to Q for X.



Ramsey's Theorem

- Let us recall the Infinite Ramsey's Theorem (RT):

Theorem (Ramsey, 1930)

For all $n > 0$, $k > 0$ and $c : [\mathbf{N}]^n \to k$ there exists an infinite set $H \subseteq \mathbf{N}$ such that c is constant on $[H]^n$.

- The set H is called homogeneous or monochromatic for c.
- For $n > 0, k > 0$, we use RT^n_k to denote the restriction of RT to colourings of n-tuples into k colours, while we use RT^n to indicate $\forall k\, \mathrm{RT}^n_k$.

Canonical Ramsey's Theorem

- The following Erdős and Rado's Canonical Ramsey's Theorem (canRT) is a generalization of RT to infinitely many colours.

Theorem (Erdős-Rado, 1950)

For all $n > 0$ and $c : [\mathbf{N}]^n \to \mathbf{N}$ there exists an infinite set $H \subseteq \mathbf{N}$ and a subset S of $\{1, \dots, n\}$ such that for any $I \in [H]^n$, $c(I)$ is determined only by the elements of I with indexes in S.

- The set H is called canonical for c.
- We use canRT^n to denote the restriction of canRT to colourings of n-tuples.

Regressive Ramsey's Theorem

- In order to introduce a further variation of RT, we need the following definition:

Definition (Regressive functions)

Let $n > 0$. A function $c : [\mathbf{N}]^n \to \mathbf{N}$ is called regressive if and only if, for all $I \in [\mathbf{N}]^n$, $c(I) < \min(I)$ if $\min(I) > 0$, else $c(I) = 0$.

- By applying canRT to regressive functions, we obtain the Regressive Ramsey's Theorem (regRT):

Theorem (Kanamori-McAloon, 1987)

For all $n > 0$ and all regressive $c : [\mathbf{N}]^n \to \mathbf{N}$ there exists an infinite $H \subseteq \mathbf{N}$ such that, for any $I, J \in [H]^n$, $\min(I) = \min(J)$ implies $c(I) = c(J)$.

- The set H is called min-homogeneous for c.
- We denote by regRT^n the principle regRT restricted to colourings of n-tuples.

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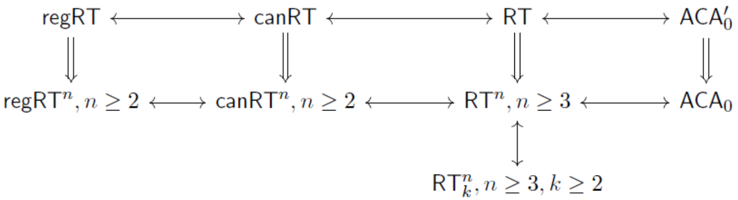
Versions of Ramsey's Theorem

We can graphically summarize the relations - over RCA_0 - between the versions of RT presented above as follows (double arrows indicate strict implications):



These results are mainly due to Clote, Hirst, Jockusch, Mileti and Simpson.

Hindman's Theorem

- We denote by $\mathrm{FS}(X)$ the set of all finite non-empty sums of distinct elements of $X \subseteq \mathbf{N}$.

Theorem (Hindman, 1972)

For all $k > 0$ and for all $c : \mathbf{N} \to k$ there exists an infinite set $H \subseteq \mathbf{N}$ such that c is constant on $\mathrm{FS}(H)$.

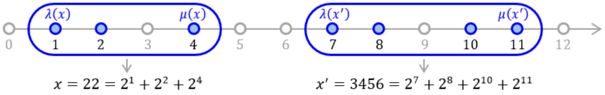
- Similarly to RT, for $n > 0, k > 0$, we use $\mathsf{HT}^{=n}_k$ and $\mathsf{HT}^{\leq n}_k$ to denote, respectively, the restrictions of HT to sums of exactly n elements ($\mathrm{FS}^{=n}_k$) and to sums of at most n elements ($\mathrm{FS}^{\leq n}_k$).
- Again, $\mathsf{HT}^{=n}$ means $\forall k\, \mathsf{HT}^{=n}_k$ and $\mathsf{HT}^{\leq n}$ means $\forall k\, \mathsf{HT}^{\leq n}_k$.

Versions of Hindman's Theorem

- For $n = 2^{t_1} + \cdots + 2^{t_p}$ with $t_1 < \cdots < t_p$, let $\lambda(n) = t_1$ and $\mu(n) = t_p$.

Definition (Apartness)

A set $X = \{x_1, x_2, \ldots\}$ satisfies the apartness condition if for all $x, x' \in X$ such that $x < x'$, we have $\mu(x) < \lambda(x')$.



- If P is a Hindman-type principle, we denote by P[ap] the principle P with the apartness condition imposed on the solution set.

Proposition

Over RCA_0, HT and HT[ap] are equivalent.

- It is unknown whether the same applies to $HT_k^{=n}$ and $HT_k^{\leq n}$.

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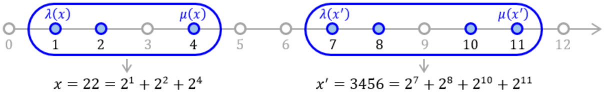

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Versions of Hindman's Theorem

- We denote by $\mathrm{FIN}(\mathbf{N})$ the set of non-empty finite subsets of $\mathbf{N}$.
- Taylor proved the analogous version of canRT for HT, i.e. the Canonical Hindman's Theorem (canHT):

Theorem (Taylor, 1976)

For all $c : \mathbf{N} \to \mathbf{N}$ there exists an infinite set $H = \{h_0 < h_1 < \dots\} \subseteq \mathbf{N}$ such that one of the following holds:

1. For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i \in I} h_i) = c(\sum_{j \in J} h_j)$.
2. For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i \in I} h_i) = c(\sum_{j \in J} h_j)$ iff $I = J$.
3. For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i \in I} h_i) = c(\sum_{j \in J} h_j)$ iff $\min(I) = \min(J)$.
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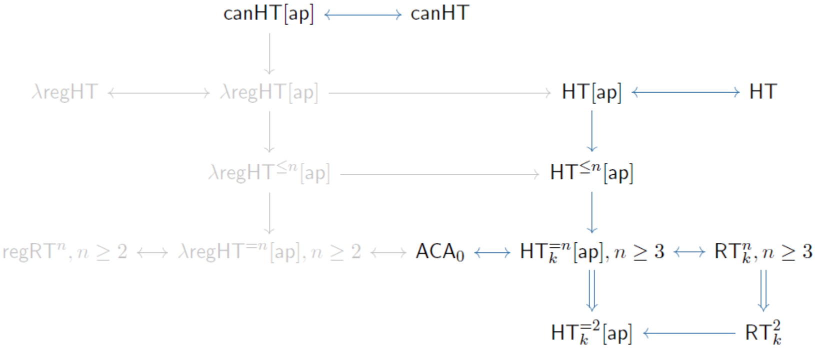
- The set H is called canonical for c.

Versions of Hindman's Theorem

Based on the previous propositions and on some well-known results, we can start drawing some implications.



These results are mainly due to Carlucci, Hindman, Kołodziejczyk, Lepore and Zdanowski.

Regressive Hindman's Theorem

- In order to formulate our regressive version of Hindman's Theorem, we need the following definition:

Definition (λ-regressive functions)

A function $c : \mathbf{N} \to \mathbf{N}$ is called λ-regressive if and only if, for all $n \in \mathbf{N}$, $c(n) < \lambda(n)$ if $\lambda(n) > 0$ and $c(n) = 0$ if $\lambda(n) = 0$.

- Then, by applying canHT to λ-regressive functions, we finally obtain the Regressive Hindman's Theorem (λregHT):

Theorem (Carlucci-M., 2022)

For all λ-regressive $c : \mathbf{N} \to \mathbf{N}$ there exists an infinite $H \subseteq \mathbf{N}$ such that $\mathrm{FS}(H)$ is min-term-homogeneous, i.e. for all $I, J \in \mathrm{FIN}(\mathbf{N})$, if $\min(I) = \min(J)$ then $c(\sum_{i \in I} h_i) = c(\sum_{j \in J} h_j)$.

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Regressive Hindman's Theorem

- Now, we want to investigate the strength of this novel theorem, in terms of implications over RCA_0 and of computable reductions.
- First, we can observe that canHT implies λregHT[ap], since canHT is equivalent to canHT[ap] and, by apartness and λ-regressivity, only case 1 and case 3 of canHT can occur.

Recall the five conditions of canHT are:

1. For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i\in I} h_i) = c(\sum_{j\in J} h_j)$.
2. For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i\in I} h_i) = c(\sum_{j\in J} h_j)$ iff $I = J$.
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Regressive Hindman's Theorem

- Also, similarly to HT and to canHT, we have that λregHT and λregHT[ap] are equivalent over RCA_0, since λregHT implies RT^1 and RT^1 can be used to get apartness.

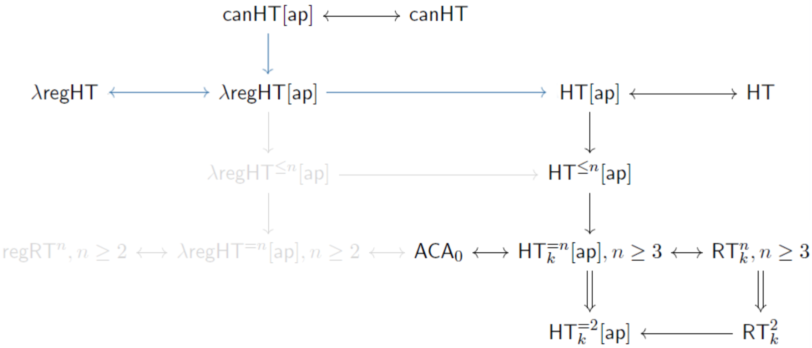
- Moreover, we can easily prove that λregHT implies HT[ap], by simply applying λregHT[ap] to the colouring:

$$g(n) = \begin{cases} f(n) & \text{if } f(n) < \lambda(n), \\ 0 & \text{otherwise.} \end{cases}$$

  where $f : \mathbf{N} \to k$ is the original colouring. Then, apartness guarantees that all but at most the first k elements of the solution of λregHT[ap] for g fall into the second case; so, we just need an application of RT^1 to obtain a solution of HT[ap] for f (that is why the argument does not witness a Weihrauch reduction).

Regressive Hindman's Theorem

Then, we can draw some additional implications (in blue) in our schema.



Then, by now we know that $ACA_0 \leq HT \leq \lambda regHT \leq canHT$ over RCA_0.

Bounded Regressive Hindman's Theorem

- Since ACA_0 is already implied by some restrictions of HT, we wonder whether this is the case for λregHT as well.

- In general, we want to investigate the strength of various natural restrictions of λregHT.

- Then, by defining $\mathrm{FS}^{\leq n}(X)$ (resp. $\mathrm{FS}^{=n}(X)$) the set of all non-empty sums of at most (resp. exactly) $n > 0$ distinct elements of $X \subseteq \mathbf{N}$, we can formulate the Bounded Regressive Hindman's Theorems:

Definition

Let $n \geq 1$. We denote by $\lambda\mathrm{regHT}^{\leq n}$ (resp. $\lambda\mathrm{regHT}^{=n}$) the following principle: for all λ-regressive $c : \mathbf{N} \to \mathbf{N}$ there exists an infinite $H \subseteq \mathbf{N}$ such that $\mathrm{FS}^{\leq n}(H)$ (resp. $\mathrm{FS}^{=n}$) is min-term-homogeneous for c.

Bounded Regressive Hindman's Theorem

- Similarly to full λregHT, we have:

$$\mathsf{RCA}_0 \vdash \lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}] \to \mathsf{HT}^{\leq n}[\mathsf{ap}]$$

$$\mathsf{RCA}_0 \vdash \lambda\mathsf{regHT}^{= n}[\mathsf{ap}] \to \mathsf{HT}^{= n}[\mathsf{ap}]$$

- However, for these bounded versions, we also have the following reductions:

$$\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}] \geq_c \mathsf{HT}^{\leq n}[\mathsf{ap}]$$

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- By the previous implications and the fact that $\mathsf{HT}_2^{=3}[\mathsf{ap}]$ is equivalent to ACA_0, we can easily infer that $\lambda\mathsf{regHT}^{=3}[\mathsf{ap}]$ implies ACA_0.
- However, by a more careful approach, we can improve this result, thus giving a lower bound for $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ for any $n \geq 2$.

Lower bound for Bounded Regressive Hindman's Theorem

Theorem (Carlucci-M., 2022)

Let $n \geq 2$. Over RCA_0, $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ implies ACA_0.

Proof. We prove the principle RAN (equivalent to ACA_0) stating that for each injective function $f : \mathbf{N} \to \mathbf{N}$, the range of f (denoted $\rho(f)$) exists.

Since $x \in \rho(f) \iff \exists z\,(f(z) = x)$, RCA_0 can not decide $\rho(f)$.

Then, our idea is to use $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ to bound the search for z.

We define $c(m) =$ the unique $x < \lambda(m)$ such that:

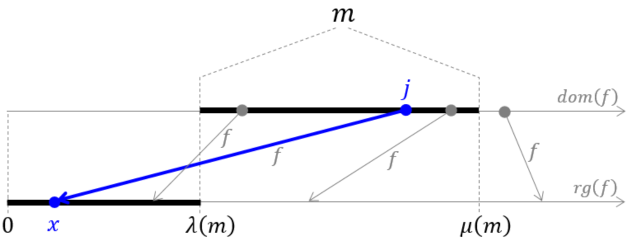
- there exists $j \in [\lambda(m), \mu(m))$ such that $f(j) = x$, and
- for all $j < j' < \mu(m)$, $f(j') \geq \lambda(m)$.

If no such x exists or m is a power of 2, we set $c(m) = 0$.

Intuitively c checks whether there are values $<\lambda(m)$ in $\rho(f\restriction[\lambda(m), \mu(m))$. If any, it returns the latest one, i.e., the one obtained as image of the maximal $j \in [\lambda(m), \mu(m))$ that is mapped by f below $\lambda(m)$.

Lower bound for Bounded Regressive Hindman's Theorem

Example:



In this case, $c(m) = x$ since:

- $x < \lambda(m)$, and
- there exists $j \in [\lambda(m), \mu(m))$ such that $f(j) = x$, and
- for all $j < j' < \mu(m)$, $f(j') \geq \lambda(m)$.

Lower bound for Bounded Regressive Hindman's Theorem

Let $H = \{h_0 < h_1 < \dots\} \subseteq \mathbf{N}^+$ be an apart solution to $\lambda\text{regHT}^{=2}$ for c.

Claim: if $x < \lambda(h_i)$ and $x \in \rho(f)$, then $x \in \rho(f \upharpoonright [0, \mu(h_{i+1}))$.

Suppose otherwise and let $x < \lambda(h_i)$ s.t. $x \in \rho(f)$ but $x \notin f([0, \mu(h_{i+1}))$.
Let b be the true bound for the elements $< \lambda(h_i)$ in $\rho(f)$, whose existence is given by strong Σ^0_1-bounding (in RCA_0):

$$\forall n \exists b \forall i < n(\exists j(f(j) = i) \rightarrow \exists j < b(f(j) = i)).$$

Let h_j in H be such that $h_j > h_{i+1}$ and $\mu(h_j) \geq b$.

$x \notin f([0, \mu(h_{i+1}))$ but $x \in f([0, \mu(h_j))$, so $c(h_i + h_{i+1}) \neq c(h_i + h_j)$, hence contradicting min-term-homogeneity.

Then we can decide $\rho(f)$ as follows using H: given x, pick any $h_i \in H$ such that $x < \lambda(h_i)$ and check whether x appears in $f([0, \mu(h_{i+1}))$.

The previous argument also proves that $\lambda\text{regHT}^{=n}[\text{ap}] \geq_{\mathrm{W}} \mathsf{RAN}$.

Upper bound for Bounded Regressive Hindman's Theorem

- As for the upper bound, we can easily prove the reversal of the previous result:

Theorem (Carlucci-M., 2022)

Let $n \geq 2$. ACA_0 proves $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$.

- The proof is quite simple: given a λ-regressive colouring $f : \mathbf{N} \to \mathbf{N}$, we define a colouring g of n-tuples using the f-colour of the sum of the elements of the tuples, i.e. $g(x_1, \ldots, x_n) = f(x_1 + \cdots + x_n)$. Since ACA_0 implies regRT^n, we can apply it to get a min-homogeneous set for g, which is also a solution to $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ for f.

- The previous argument also proves that $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}] \leq_{\mathrm{sW}} \mathsf{regRT}^n$.

Bounds for Bounded Regressive Hindman's Theorem

- Since the lower bound and the upper bound for $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ coincide, we have that:

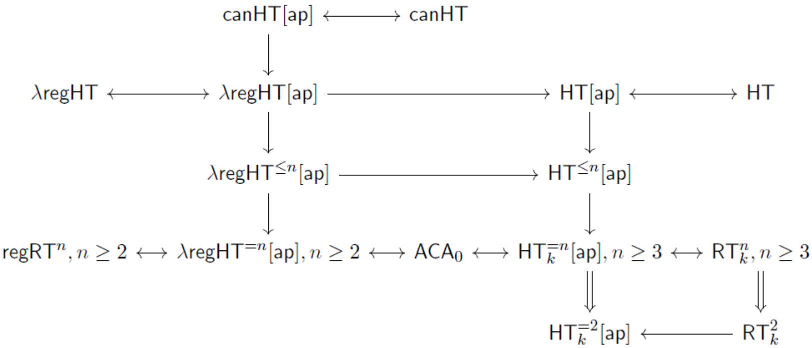
Theorem (Carlucci-M., 2022)

Over RCA_0, $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ is equivalent to ACA_0, for any $n \geq 2$.

- To sum up, we have that the following are equivalent over RCA_0:
  1. ACA_0.
  2. regRT^n, for any fixed $n \geq 2$.
  3. RT^n_k, for any fixed $n \geq 3$, $k \geq 1$.
  4. $\mathsf{HT}^{=n}_k[\mathsf{ap}]$, for any fixed $n \geq 3$, $k \geq 1$.
  5. $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$, for any fixed $n \geq 2$.

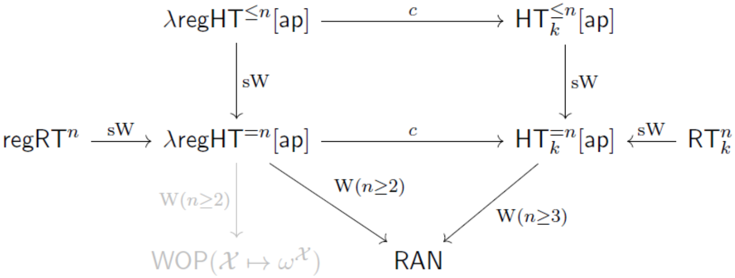
Strength of Bounded Regressive Hindman's Theorem

Then, the complete diagram of the implications over RCA_0 is the following:



Strength of Bounded Regressive Hindman's Theorem

The following diagram, instead, visualizes the known reductions:



Bounded Regressive Hindman's Theorem and WOP

Definition

The well-ordering preservation principle for base-ω exponentiation (in symbols, WOP($\mathcal{X} \mapsto \omega^{\mathcal{X}}$)) is the following Π^1_2-principle:

$$\forall \mathcal{X}(\mathrm{WO}(\mathcal{X}) \rightarrow \mathrm{WO}(\omega^{\mathcal{X}})),$$

where $\mathrm{WO}(Y)$ is the usual Π^1_1-formula stating that Y is a well-ordering.

- It is known that WOP($\mathcal{X} \mapsto \omega^{\mathcal{X}}$) is equivalent to ACA_0 [Girard, Hirst].
- To answer questions about reducibility, we can consider the contrapositive form of WOP($\mathcal{X} \mapsto \omega^{\mathcal{X}}$): an *instance* is an infinite descending sequence in $\omega^{\mathcal{X}}$ and a *solution* is an infinite descending sequence in $\mathcal{X}$.

Bounded Regressive Hindman's Theorem and WOP

Theorem (Carlucci-M., 2022)

Let $n \geq 2$. Over RCA_0, $\lambda\mathrm{regHT}^{=n}[\mathrm{ap}]$ implies $\mathrm{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$.
Moreover, $\lambda\mathrm{regHT}^{=n}[\mathrm{ap}] \geq_{\mathrm{W}} \mathrm{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$.

Proof. The idea is to give a procedure that, at each step, extracts from the descending sequence in $\omega^{\mathcal{X}}$ the exponent of the "next" leftmost component that eventually decreases.

Example:

$\alpha_1 = \boxed{\omega^9} + \omega^8 + \omega^8 + \omega^6 + \omega^4 + \omega^3$

$\alpha_2 = \omega^8 + \omega^8 + \boxed{\omega^8} + \omega^6 + \omega^4$

$\alpha_3 = \omega^8 + \omega^8 + \omega^8 + \omega^6$

$\alpha_4 = \omega^8 + \omega^8 + \omega^8 + \omega^5 + \omega^5 + \omega^5 + \omega^5$

$\alpha_5 = \omega^8 + \omega^8 + \omega^7 + \omega^7 + \omega^7 + \omega^7$

$\dots$

$\alpha_i = \omega^8 + \omega^8 + \dots$

$\dots$

Bounded Regressive Hindman's Theorem and WOP

By strong Σ^0_1-bounding, RCA_0 knows that if a component will decrease, it will do so within ℓ steps, but RCA_0 is not able to compute such ℓ.

Thus, we adopt an approach similar to the one used to prove RAN, i.e. we use $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ to bound the research of ℓ.

First, fixed an infinite decreasing sequence α in $\omega^{\mathcal{X}}$, we define:

- $(\beta_n)_{n\in\mathbf{N}}$ the sequence of all the exponents in α (in the example above, we have $\beta = \langle 9, 8, 8, 6, 4, 3, 8, 8, 8, 6, \ldots\rangle$)
- $m(n)$ the index of the element of α from which β_n has been extracted (e.g., $m(7) = 2$ in the example above)
- $pos(n)$ the position of β_n in $\alpha_{m(n)}$ (e.g., $pos(7) = 1$ in the example above)

Then, we say that j *decreases* i (and write $dec(j, i)$) if $i < j$, $pos(i) = pos(j)$, $\beta_i > \beta_j$ and j is minimal.

Bounded Regressive Hindman's Theorem and WOP

Now we set $c(x)$ as the unique $i < \lambda(x)$ such that:

- there exists $j \in [\lambda(x), \mu(x))$ such that j decreases i, and
- for all $j < j' < \mu(x)$, if j' decreases i' then $i' \geq \lambda(x)$.

If no such i exists, we set $c(x) = 0$.

Intuitively c checks whether there are indexes below $\lambda(x)$ decreased by indexes in $[\lambda(x), \mu(x))$ and, if any, it returns the latest one.

Let $H = \{h_1 < h_2 < h_3 < \dots\}$ be an apart solution to $\lambda\text{regHT}^{=n}$ for c.

Claim: For each $h_i \in H$ and each $j < \lambda(h_i)$, if there exists k s.t. k decreases j then there exists such a k smaller than $\mu(h_{i+n-1})$.

We can prove the claim by adopting the same approach used for proving that $\lambda\text{regHT}^{=n}$ implies ACA_0 over RCA_0.

Now, we compute an infinite decreasing sequence σ in $\mathcal{X}$ as follows:

1. we look for the leftmost term of α_1 that eventually decreases, i.e. we look for the least i s.t. $m(i) = 1 \land \exists j < \mu(h_{k+n-1})$ (j decreases i), where h_k is the least element of H s.t. $i < \lambda(h_k)$
2. we fix $\sigma_1 = \beta_i$, $p = pos(i)$ and $d = m(j)$
3. we repeat the procedure, this time starting from α_d.

Note that we are assuming that we can always find a term that eventually decreases, i.e. $\forall i\ \exists i'\ (\exists j < lh(\alpha_i))\ \big(i' > i\ \land\ e((\alpha_i)_j) >_{\mathcal{X}} e((\alpha_{i'})_j)\big)$, where $e((\alpha_m)_n)$ is the exponent of the n-th term of α_m.

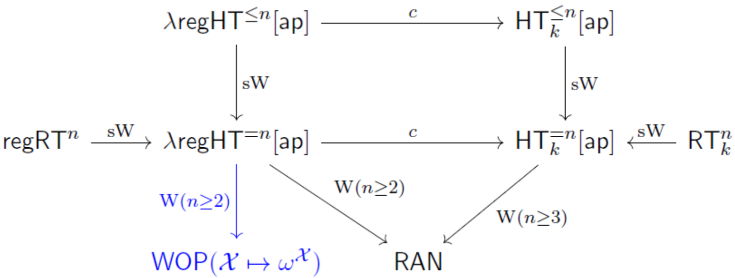
This is true, otherwise for some i we could prove by Δ^0_1-induction that:

$$\forall m\ (m \geq i \rightarrow \alpha_{m+1} \text{ is an initial segment of both } \alpha_m \text{ and } \alpha_i)$$

which implies $\forall m\ (m \geq i \rightarrow lh(\alpha_m) > lh(\alpha_{m+1}))$, contradicting $\mathrm{WO}(\omega)$.

Strength of Bounded Regressive Hindman's Theorem

Then, we can add this last result to our diagram of computable reductions:



Conclusions

- We formulated a novel Regressive Hindman's Theorem as a corollary of Taylor's Canonical Hindman's Theorem restricted to a suitable class of regressive functions.
- We studied the strength of this principle and of its restrictions in terms of provability over RCA_0 and computable reductions.
- In particular, we showed that the weakest non-trivial restriction of our Regressive Hindman's Theorem, $\lambda\mathsf{regHT}^{=2}[\mathsf{ap}]$, is equivalent to ACA_0.
- This contrasts with the standard restrictions of Hindman's Theorem, which require at least sums of exactly 3 elements to reach ACA_0.
- This situation is analogous to that of regRT^2 when compared to RT^3_2.
- Also, we proved that, for $n \geq 2$, $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ computably reduces the corresponding restrictions of Hindman's Theorem $\mathsf{HT}^{=n}[\mathsf{ap}]$.
- Finally, we proved that $\lambda\mathsf{regHT}^{=2}[\mathsf{ap}] \geq_{\mathrm{W}} \mathsf{WOP}(\mathcal{X} \to \omega^{\mathcal{X}})$, the well-ordering preservation principle that characterizes ACA_0.

Conclusions

Open questions remain about the strength of the Regressive Hindman's Theorem, of its restrictions, and of related principles, for instance:

1. What are the optimal upper bounds for canHT, for λregHT and for $\lambda\mathrm{regHT}^{\leq n}$?
2. Does HT imply/reduces λregHT (and similarly for bounded versions)?
3. What is the strength of $\lambda\mathrm{regHT}^{=2}$ without apartness? More generally, how do the bounded Regressive Hindman's Theorems behave with respect to apartness?
4. Can the reductions presented above be improved to stronger reductions?

Finally, it would be interesting to investigate relations between λregHT and other principles dealing with colourings with unboundedly many colours, like Hindman-type variants of the Thin Set Theorem recently investigated by Hirschfeldt and Reitzes.