Midterm Examination
MA 4207: Mathematical Logic

Monday 9 February 2015, Duration 45 minutes

Matriculation Number: _______________

Rules
This test carries 20 marks and consists of 5 questions. Each questions carries 4 marks; full marks for a correct solution; a partial solution can give a partial credit.

Question 1 [4 marks].
Let

$$\phi = maj(A_1 \wedge A_2 \wedge A_3\, ,\, A_4 \wedge A_5 \wedge A_6\, ,\, A_7 \wedge A_8 \wedge A_9)$$

where $\wedge$ is the operation "and" and $maj(\alpha, \beta, \gamma)$ is 1 iff at least two of α, β, γ are 1. How many of the 512 rows in the 9-column truth-table of this formula take the value 1? Explain your answer.

Solution. There are three blocks of three atoms and at least two of these blocks need to have all atoms to be 1. So there are two cases: (a) For exactly one of the blocks A_1, A_2, A_3 and A_4, A_5, A_6 and A_7, A_8, A_9 at least one of the atoms is 0 while the atoms in the other blocks are all 1; (b) all atoms are 1. In case (a), there are 7 possibilities of what values the atoms can take in the block where some of them are 0; as there are three blocks, the overall number of possibilities for case (a) is 21. In case (b), there is only 1 choice. Hence the overall number of rows which carry the value 1 in the truth-table is **22**.

Question 2 [4 marks].
In set theory, one says that two sets X and Y have the same cardinality if and only if there is a bijective (= one-one and onto) mapping f from X to Y. Select among the following five sets A, B, C, D, E two sets X, Y which have the same cardinality and make the corresponding mapping; the choice of the direction of the mapping f (whether from X to Y of from Y to X) is up to you, use the more convenient choice. The sets A, B, C, D, E are the following:

- The set $A = \{0, 1, 2, \ldots, 123456789\}$ of all natural numbers up to 123456789;
- The set $B = \{(x, y) : x, y$ are natural numbers, $x \leq 1234$ and $y \leq 56789\}$;
- The set C of all natural numbers;
- The set $D = \{(x, y) : x, y$ are natural numbers and $y \leq 2x\}$;
- The set $E = \{(x, y) : x, y$ are real numbers and $y \leq 2x\}$.

Solution. The sets **C** and **D** are both infinite and countable and thus there is a bijection between them. It is easist to make the bijection from D to C. Note that for each fixed x there are $2x + 1$ possible values for y such that (x, y) is in D, namely the $y \in \{0, 1, \ldots, 2x\}$. Now, as $(x + 1)^2 - x^2 = 2x + 1$, this permits to define the bijection $\mathbf{f}$ defined as $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{x^2} + \mathbf{y}$.

Question 3 [4 marks].
Using $\wedge$ (and), $\vee$ (or) and $\oplus$ (exclusive or), make a formula which evaluates to 1 iff exactly one or two of the three atoms A_1, A_4, A_9 are 1 and which evaluates to 0 iff these three atoms are all equal.

Solution. What the formula asks for is that at least 1 and not all of the atoms are 1. This can best be done with the formula

$$(\mathbf{A_1} \vee \mathbf{A_4} \vee \mathbf{A_9}) \oplus (\mathbf{A_1} \wedge \mathbf{A_4} \wedge \mathbf{A_9})$$

where the first term $A_1 \vee A_4 \vee A_9$ is 1 if at least one atom is 1 and the second term $A_1 \wedge A_4 \wedge A_9$ is 1 iff all atoms are 1. There are other solutions equivalent to this. For example

$$A_1 \oplus A_4 \oplus A_9 \oplus (A_1 \wedge A_4) \oplus (A_1 \wedge A_9) \oplus (A_4 \wedge A_9).$$

If one atom is 1 then exactly one of the terms combined by $\oplus$ are 1; if two atoms are 1 than three of the terms are 1 and if three atoms are 1 than all six terms are 1; thus this expression is 1 iff either 1 or 2 of the atoms A_1, A_4, A_9 are 1.

Question 4 [4 marks].
Let C be the smallest set of all formulas containing the constants 0 and 1, all atoms A_k and all formulas of the form $maj(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in C$ and where $maj(\alpha, \beta, \gamma) = 1$ iff at least two of α, β, γ are equal to 1. Furthermore, for truth-value functions v, w, say that $v \leq w$ iff for all k, $v(A_k) = 1 \Rightarrow w(A_k) = 1$; note that v, w can be incomparable with respect to $\leq$. Now call a formula ϕ "positive" if for all v, w with $v \leq w$ it holds that $\overline{v}(\phi) = 1$ implies $\overline{w}(\phi) = 1$. Prove by induction that all formulas in C are positive.

Solution. One proves the statement by structural induction.

First it is clear that all atoms and all constants are positive formulas: For constants $c \in \{0, 1\}$, it always holds that $\overline{v}(c) = \overline{w}(c)$. For atoms, $\overline{v}(A_k) = v(A_k)$ and $\overline{w}(A_k) = w(A_k)$, thus $v \leq w$ implies $\overline{v}(A_k) \leq \overline{w}(A_k)$.

Second, one has to show that whenever α, β, γ are positive so is $maj(\alpha, \beta, \gamma)$. So let v, w be truth-value functions with $v(A_k) \leq w(A_k)$ for all atoms A_k. Then, by induction hypothesis, $\overline{v}(\alpha) \leq \overline{w}(\alpha)$, $\overline{v}(\beta) \leq \overline{w}(\beta)$ and $\overline{v}(\gamma) \leq \overline{w}(\gamma)$. If now $\overline{v}(maj(\alpha, \beta, \gamma)) = 1$ then at least two of $\overline{v}(\alpha), \overline{v}(\beta), \overline{v}(\gamma)$ are 1. Thus at least two of $\overline{w}(\alpha), \overline{w}(\beta), \overline{w}(\gamma)$ are 1. It follows from the definition of maj that $\overline{w}(maj(\alpha, \beta, \gamma)) = 1$ and so it follows that $maj(\alpha, \beta, \gamma)$ is also a positive formula.

Question 5 [4 marks].
Make a set S of formulas such that for each v with $v \models S$ there are at most two atoms A_i, A_j with $v(A_i) = 1$ and $v(A_j) = 1$. Here $v \models S$ means that all formulas $\alpha \in S$ satisfy $\overline{v}(\alpha) = 1$; that is, the function v assigns the truth-values to the atoms in a way that all formulas in S are true.

Furthermore, in the case that the set S can be finite, please provide a finite S; in the case that the set S cannot be finite, please explain why S must be infinite.

Solution. If one makes a set of formulas which is not satisfied by any v then the set S has already the properties asked for in the question. For example, $S = \{0\}$ and $S = \{A_1, A_2, \neg(A_1) \wedge \neg(A_2)\}$ are such sets and both sets are finite.

If one would require that S is satisfiable (what is not asked for in the question), then one would need an infinite S. An example of such an S is $\{\neg A_i \vee \neg A_j \vee \neg A_k : i < j < k\}$. Here it is clear that no three atoms can be true; however, whenever a v makes at most two atoms true then it satisfies all formulas in S as they say that there are no three atoms which are all true. Furthermore, any satisfiable S with this property has to be infinite: a finite S would only mention finitely many atoms and so given a v satisfying S, one could use a w such that $w(A_k) = v(A_k)$ for all atoms A_k mentioned in S and $w(A_k) = 1$ for all atoms A_k not mentioned in S. This w would then make infinitely many atoms true and still satisfy S.

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