Midterm Examination
MA 4207: Mathematical Logic

Thursday 17 March 2016, Duration 45 minutes

Matriculation Number: _______________

Rules
This test carries 28 marks and consists of 5 questions. Each questions carries 5 or 6 marks; full marks for a correct solution; a partial solution can give a partial credit.

Question 1 [5 marks].
Let $\oplus$ be the connective "exclusive or" and $\wedge$ be the connective "and". Consider the formula

$$\phi = A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5 \oplus (A_1 \wedge A_2 \wedge A_3 \wedge A_4) \oplus (A_1 \wedge A_2 \wedge A_3 \wedge A_5)$$

and determine how many entries in the truth-table of ϕ are evaluated to 1 and how many are evaluated to 0. The variables considered in the truth-table are A_1, A_2, A_3, A_4, A_5. Explain your answer.

Solution. The first part $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5$ is 1 iff an odd number of the variables is 1 and this is the case in 16 out of 32 entries. The second part $(A_1 \wedge A_2 \wedge A_3 \wedge A_4) \oplus (A_1 \wedge A_2 \wedge A_3 \wedge A_5)$ is 1 only on two entries, namely $(1, 1, 1, 1, 0)$ and $(1, 1, 1, 0, 1)$. Both are evaluated to 0 by the first part. Hence they give two additional 1s. Thus the overall number of 1s is **Eighteen (18)** and the number of 0s is **Fourteen (14)**.

Question 2 [5 marks].
Let $A_0, A_1, \ldots$ be the list of all atoms, $\alpha_0 = (A_0 \leftrightarrow A_1)$ and, for $n = 1, 2, \ldots$, $\alpha_n = (\alpha_{n-1} \wedge (A_n \leftrightarrow A_{n+1})) \vee (\neg\alpha_{n-1} \wedge (A_n \oplus A_{n+1}))$. Let $S = \{\alpha_n : n \in \mathbb{N}\}$. How many v satisfy $v \models S$? Explain your answer.

Here v is a function which assigns to every atom A_n a truth-value $v(A_n)$ and two truth-assignments v, w are the same iff $v(A_n) = w(A_n)$ for all $n \in \mathbb{N}$. Furthermore, $v \models S$ denotes that $\overline{v}(\alpha_n) = 1$ for all $n \in \mathbb{N}$; the function $\overline{v}$ is the extension of v from atoms to all formulas in sentential logic.

Solution. The answer is **Two (2)**. The assignments satisfying this have to satisfy $v(A_n) = v(A_0)$ for all n. So assume that $v \models S$. Then $v \models \alpha_0$ and thus $v(A_1) = v(A_0)$. Furthermore, as $v \models \alpha_{n-1}$, one has that $\overline{v}(\alpha_n) = 1$ iff $\overline{v}(\alpha_{n-1} \wedge (A_n \leftrightarrow A_{n+1})) = 1$ iff $\overline{v}(A_n \leftrightarrow A_{n+1}) = 1$ iff $v(A_n) = v(A_{n+1})$. As $\overline{v}(\alpha_n) = 1$, this gives, by induction, that $v(A_{n+1}) = v(A_n) = v(A_0)$. Although the set S enforces that $v(A_n)$ is the same for all n, it does not enforce whether this common value is 0 or 1. So there are two possibilities and thus two truth-assignments v which make all formulas in S true.

Question 3 [6 marks].
Formalise the below statements on the structure $(\mathbb{Q}, +, -, \cdot, <, f, 0, 1)$ in first order logic, where $\mathbb{Q}$ is the set of rational numbers and $+, -, \cdot$ are the usual operations and $<$ the usual order on the rational numbers. The function $f : \mathbb{Q} \to \mathbb{Q}$ maps rational numbers to rational numbers.

1. every value $f(x)$ is the sum of two squares of rational numbers;
2. f is the polynomial of degree 3 with rational coefficients;
3. f is a strictly monotonically increasing function;
4. $\lim_{x\to\infty} f(x)$ exists and is a rational number;
5. $\lim_{x\to\infty} f(x)$ exists and is a real, not necessarily rational number;
6. $\limsup_{x\to-\infty} f(x)$ is $+\infty$ and $\liminf_{x\to-\infty} f(x)$ is $-\infty$.

Solution. In the following, quantifiers range over rational numbers:

1. $\forall x\, \exists y\, \exists z\, [f(x) = y \cdot y + z \cdot z]$;
2. $\exists a\, \exists b\, \exists c\, \exists d\, \forall x\, [f(x) = a + x \cdot (b + x \cdot (c + x \cdot d))]$;
3. $\forall x\, \forall y\, [x < y \to f(x) < f(y)]$;
4. $\exists z\, \forall r\, \exists x\, \forall y\, [(r > 0 \wedge y > x) \to ((f(y) - z) \cdot (f(y) - z) < r)]$;
5. $\forall r\, \exists x\, \forall y\, \forall z\, [(r > 0 \wedge y > x \wedge z > x) \to ((f(y) - f(z)) \cdot (f(y) - f(z)) < r)]$;
6. $\forall r\, \forall x\, \exists y\, \exists z\, [y < x \wedge z < x \wedge f(y) < -r \wedge r < f(z)]$.

Note that addition, multiplication, order and the usage of f are allowed; except for f, the meaning of all other symbols is fixed by the model of rational numbers.

Question 4 [6 marks].
Let $(\mathbb{R}^3, +, P)$ be the set of three-dimensional real vectors where $P(x, y, z)$ is true iff x, y, z are linearly dependent, that is, if there exist $(a, b, c) \neq (0, 0, 0)$ for which $a \cdot x + b \cdot y + c \cdot z$ is the null-vector. Is there a strong homomorphism f from $(\mathbb{R}^3, +, P)$ to itself which is not one-one? If so, construct such an f; if not, explain why f does not exist.

Note that for the given structure, a strong homomorphism f must satisfy for all x, y, z that $f(x+y)$ is equal to $f(x)+f(y)$ and that $P(x, y, z)$ holds iff $P(f(x), f(y), f(z))$ holds.

Solution. The answer is **no**, that is, such an f does not exist. So consider any homomorphism f from $(\mathbb{R}^3, +, P)$ to itself and assume that this homomorphism is not one-one. The task is to show that it is not a strong homomorphism. As f is not one-one, there are two distinct vectors x, y with $f(x) = f(y)$. Though the scalar multiplication is not part of the structure, the homomorphism has still to satisfy that $f(v) + f(w) = f(v + w)$ for all vectors v, w. Thus the image of the null-vector must be the null-vector. Letting $w = -x$ and $v = x, y$, one obtains that $f(x - x) = f(x) + f(-x) = f(y) + f(-x) = f(y - x)$ and thus $f(y - x)$ is the null-vector. As $y - x$ is not the null-vector, there are two further vectors v, w such that $x - y, v, w$ are linearly independent, that is, $P(y - x, v, w)$ is not satisfied. However, $f(y - x), f(v), f(w)$ is linearly dependent as $1 \cdot f(y - x) + 0 \cdot f(v) + 0 \cdot f(w)$ is the null-vector; thus $P(f(y - x), f(v), f(w))$ is satisfied and the homomorphism f cannot be a strong homomorphism.

Question 5 [6 marks].
Let $\Gamma = \{\forall x \forall y\, [f(x) = y \to f(y) = x], \forall x [f(x) \neq x]\}$. The following proof is for $\forall x\, [f(f(x)) = x]$. Go through the proof and state which of the following rules are used: Copying axioms from Λ, copying formulas from Γ, Modus Ponens, Generalisation Theorem, Deduction Theorem, Reductio ad Absurdum, Contraposition. When axioms from Λ are copied, say which group (1–6) applies and whether universal quantifiers have been added to the axiom. If a step is faulty, indicate it as "Error" and say in a few words what is wrong.

1. $\Gamma \vdash \forall x \forall y\, [f(x) = y \to f(y) = x]$;
2. $\Gamma \vdash \forall x \forall y\, [f(x) = y \to f(y) = x] \to \forall y\, [f(x) = y \to f(y) = x]$;
3. $\Gamma \vdash \forall y\, [f(x) = y \to f(y) = x]$;
4. $\Gamma \vdash \forall y\, [f(x) = y \to f(y) = x] \to (f(x) = f(x) \to f(f(x)) = x)$;
5. $\Gamma \vdash f(x) = f(x) \to f(f(x)) = x$;
6. $\Gamma \vdash \forall y\, [y = y]$;
7. $\Gamma \vdash \forall y\, [y = y] \to f(x) = f(x)$;
8. $\Gamma \vdash f(x) = f(x)$;
9. $\Gamma \vdash f(f(x)) = x$;
10. $\Gamma \vdash \forall x\, [f(f(x)) = x]$.

Solution. The solution is as follows.

1. $\Gamma \vdash \forall x \forall y\, [f(x) = y \to f(y) = x]$;
   Copying the first formula from Γ;

2. $\Gamma \vdash \forall x \forall y\, [f(x) = y \to f(y) = x] \to \forall y\, [f(x) = y \to f(y) = x]$;
   Copying axiom from Λ (Axiom group 2);

3. $\Gamma \vdash \forall y\, [f(x) = y \to f(y) = x]$;
   Modus Ponens;

4. $\Gamma \vdash \forall y\, [f(x) = y \to f(y) = x] \to (f(x) = f(x) \to f(f(x)) = x)$;
   Copying axiom from Λ (Axiom group 2);

5. $\Gamma \vdash f(x) = f(x) \to f(f(x)) = x$;
   Modus Ponens;

6. $\Gamma \vdash \forall y\, [y = y]$;
   Copying axiom from Λ (quantified version of Axiom group 5);

7. $\Gamma \vdash \forall y\, [y = y] \to f(x) = f(x)$;
   Copying axiom from Λ (Axiom group 2);

8. $\Gamma \vdash f(x) = f(x)$;
   Modus Ponens;

9. $\Gamma \vdash f(f(x)) = x$;
   Modus Ponens;

10. $\Gamma \vdash \forall x\, [f(f(x)) = x]$;
    Generalisation Theorem.

There are no errors in the derivation.

END OF PAPER