Deciding Parity Games in Quasipolynomial Time

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Infinite games played on Finite Graph

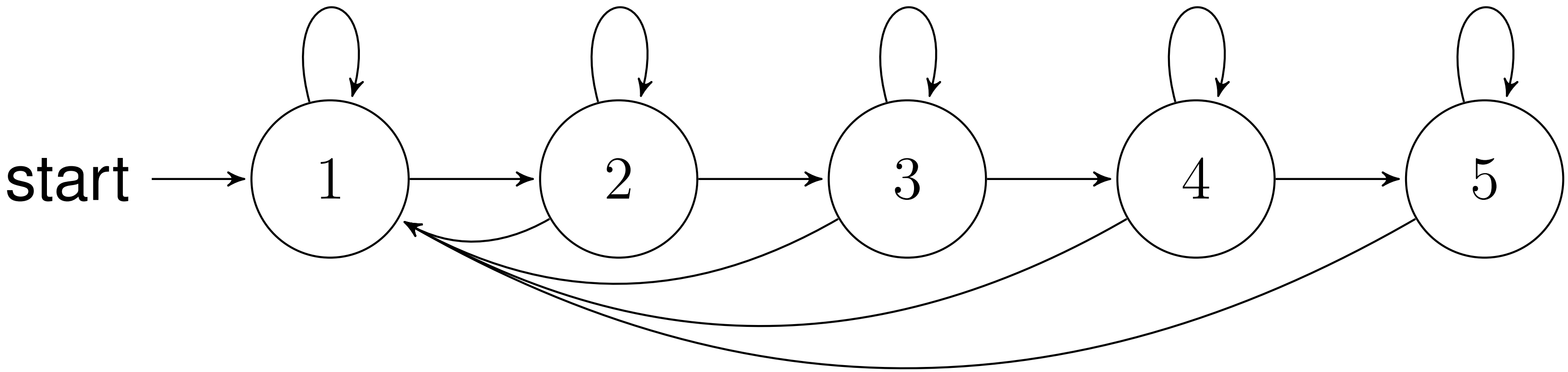
Given directed graph (V, E) with start node s and evaluation function F, where each node has at least one successor. Player $Anke$ starts to move from s and players move alternately a marker through the graph along the edges of the graph forever. Let U be the set of infinitely often visited nodes through the play. The function F maps U to one of the values $Anke$ and $Boris$ to indicate who won the play.

Parity game: Each node has a value and $F(U)$ depends on $\max\{val(u) : u \in U\}$ only. One player, say $Anke$, wins iff this maximum is odd.

Coloured Muller game: Each node has some colours and F does not depend on U directly but on $\bigcup_{u \in U} Colour(u)$. This permits more compact representation in the case that only few colours are used.

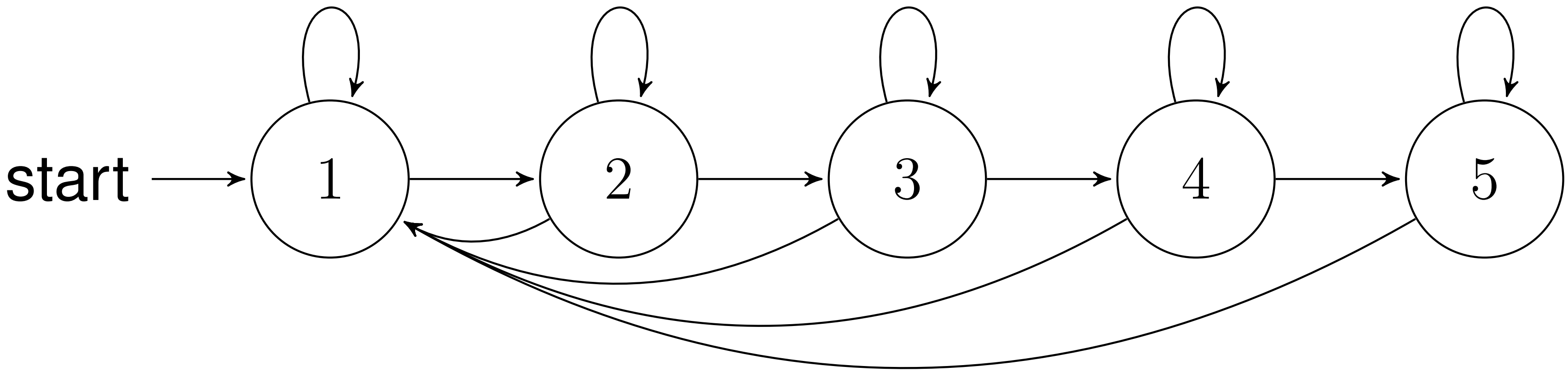
Parity Games

A parity game $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \mathbf{val})$ has a function $\mathbf{val} : \mathbf{V} \rightarrow \mathbb{N}$.
Anke wins a play $\mathbf{v_0}, \mathbf{v_1}, \mathbf{v_2}, \ldots$ iff $\mathbf{limsup}\ \mathbf{val}(\mathbf{v_k})$ is odd.



Example of parity game, node $\mathbf{q}$ is labeled with $\mathbf{val}(\mathbf{q})$. Play $\mathbf{1 - 2 - 3 - 4 - 1 - 2 - 3 - 4 - 1 - 2 - 3 - 4 - 1 - 2 - \ldots}$ is won by Boris.

Anke's Winning Strategy



Node	1	2	3	4	5
Anke's Move	1	3	3	5	5

If $\mathbf{maxval}$ is odd and one can always go from $\mathbf{n}$ to $\mathbf{n}$ and to $\mathbf{min\{maxval, n+1\}}$ and to $\mathbf{1}$ then Anke has a winning strategy.

Parity and Muller Games

Observation
A parity game with $\mathbf{n}$ nodes and values from $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}\}$ can be translated into an isomorphic Muller game with $\mathbf{n}$ nodes and $\mathbf{m}$ colours where node $\mathbf{u}$ has colours $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{val}(\mathbf{u})\}$. Anke wins play iff there is a $\mathbf{k}$ such that the set of colours of infinitely often visited nodes is of the form $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{2k} + \mathbf{1}\}$.

Theorem [Björklund, Sandberg and Vorobyov 2003]
Every coloured Muller game with $\mathbf{n}$ nodes and $\mathbf{m}$ colours can be translated into parity game with $\mathbf{m}! \cdot \mathbf{n}$ nodes and $\mathbf{2m}$ values and the same winner in time polynomial in the size of the target game.

Theorem [Hunter 2007]
A Muller game is equal to a parity game iff $\mathbf{F}$ satisfies that whenever $\mathbf{F}(\mathbf{U}) = \mathbf{F}(\mathbf{U}')$ then $\mathbf{F}(\mathbf{U} \cup \mathbf{U}') = \mathbf{F}(\mathbf{U})$.

Memoryless Strategies

A strategy is a function which tells a player how to move after a certain sequence of moves has occurred; a strategy is called a winning strategy iff a player wins whenever following the strategy's advice; a strategy is memoryless if it only depends on the current position.

A player who has a winning strategy is called the winner of a game.

Theorem [Zielonka 1998]
Player Anke has a memoryless winning strategy in a Muller game (V, E, s, F) if (a) she has a winning strategy and (b) for all $U, U' \subseteq V$ with $F(U) = \text{Boris}$ and $F(U') = \text{Boris}$ it holds that $F(U \cup U') = \text{Boris}$.

Corollary [Allen and Jutla 1991, McNaughton 1993, Mostowski 1991] The winner of a parity game can use a memoryless winning strategy.

Complexity of Parity Game

The following work provided algorithms to determine the winner of parity games; the complexity is measured by the number $\mathbf{n}$ of nodes and $\mathbf{m}$ of values.

- McNaughton 1993: $\mathbf{O((kn)^{m+1})}$ for some $\mathbf{k}$.
- Browne, Clarke, Jah, Long and Marrero 1997: $\mathbf{O(n^2 \cdot (2n/m)^{(m+3)/2})}$.
- Jurdinski, Patterson and Zwick 2006/2008: $\mathbf{n^{k \cdot \sqrt{n}}}$ for some $\mathbf{k}$.
- Schewe 2007/2016: $\mathbf{n^2 \cdot (k \cdot n \cdot m^{-2})^{m/3}}$ for some $\mathbf{k}$.
- Calude, Jain, Khoussainov, Li and Stephan 2016: $\mathbf{O(n^{\log(m)+6})}$; if $\mathbf{m \leq \log(n)}$ then $\mathbf{O(n^5)}$.
- Follow-ups: $\mathbf{O((m/\log(n))^4 \cdot n^{3.45+\log(m/\log(n)+3)})}$.

Open Problem: Can parity games be decided in polynomial time?

Overview

Introduction of Winning Statistics.

If a player plays a memoryless winning strategy then the own winning statistics will eventually indicate a win while the opponent's winning statistics will never do.

The winning statistics can be kept small.

The winning statistics permit to translate the parity game into a quasipolynomially sized reachability game where Anke has to reach a state where her winning statistics indicate a win; if she fails to do so, Boris wins.

The reachability game can be solved in time linear in the number of its edges (well-known fact).

Closer examination of special cases in order to obtain that parity games are fixed parameter tractable and to obtain furthermore some bounds for Muller games.

Winning Statistics 1

Modify parity game to parity game with winning statistics (of Anke). Winning statistics are vectors $\mathbf{b_0\, b_1\, \ldots\, b_k}$ with $\mathbf{k = \lceil \log(n) + 2 \rceil}$ with the following meaning: $\mathbf{b_i > 0}$ stands for the observation of $\mathbf{2^i}$ nodes in the play so far such that between any of these nodes the highest value was of the Anke's parity and $\mathbf{b_i}$ is the largest value observed with starting the end of the sequence. If $\mathbf{b_i, b_j > 0}$ and $\mathbf{j < i}$ then the sequence for $\mathbf{b_j}$ can only start after the $\mathbf{2^i}$ nodes of the sequence for $\mathbf{b_i}$ and also the node with value $\mathbf{b_i}$ itself have been observed.

Example (Anke has odd numbers)
Values in play: (2' 3' 2' 4 5' 2' 3' 2' 4 5') 2 6* (2' 3 5'*) (3'*)
Primed numbers have nodes of Anke's parity inbetween, three sequences, $\mathbf{b_i}$ at star.
Winning statistics: $\mathbf{b_3 = 6}$; $\mathbf{b_2 = 0}$; $\mathbf{b_1 = 5}$; $\mathbf{b_0 = 3}$.

Winning Statistics 2

Initialisation: All b_i of a winning statistics are 0.

Update rule: For each new node with value b, choose the largest i which satisfies one of the following:

- b and $b_0, b_1, \ldots, b_{i-1}$ have Anke's parity and b_i does not;
- $0 < b_i < b$.

If found, then let $b_i = b$ and $b_j = 0$ for all $j < i$, else no change.

Example (Possible Updates for Move b)
Values in play: (2' 3' 2' 4 5' 2' 3' 2' 4 5') 2 6* (2' 3 5'*) (3'*)
$b = 2$: None;
$b = 4$: $b_0 = 4$;
$b = 6$: $b_1 = 6$, $b_0 = 0$;
b odd and $b \leq 5$: $b_2 = b$, $b_1 = 0$, $b_0 = 0$;
$b \geq 7$: $b_3 = b$, $b_2 = 0$, $b_1 = 0$, $b_0 = 0$.

Winning Statistics 3

Theorem
If Anke plays a memoryless winning strategy her winning statistics will eventually have $b_k > 0$ what will indicate that she has won the play; if Boris plays a memoryless winning strategy this will never happen.

Main idea: If the lim sup of the play is of Anke's parity, say 5, and all larger numbers do no longer appear then the sum

$$\sum_{i:\, b_i \text{ has Anke's parity and } b_i \geq 5} 2^i$$

will go up between each two times a 5 is played.

If Boris is playing a memoryless winning strategy, this does not happen, as $b_k \geq 1$ indicates that the game goes through a loop with the maximum value being Anke's parity.

Reduction to Reachability Games

The game graph of the reachability game consist of all nodes $(\mathbf{v}, \mathbf{p}, \mathbf{w})$ with node $\mathbf{v}$ of parity game, player $\mathbf{p}$ to move and current winning statistics $\mathbf{w}$ of Anke. Move from $(\mathbf{v}, \mathbf{p}, \mathbf{w})$ to $(\mathbf{v}', \mathbf{p}', \mathbf{w}')$ is possible iff there is an edge from $\mathbf{v}$ to $\mathbf{v}'$ in parity game, $\mathbf{p} \neq \mathbf{p}'$ and on move with value of $\mathbf{v}'$, winning statistics $\mathbf{w}$ of Anke are updated to $\mathbf{w}'$ and $\mathbf{w}$ is not already won for Anke. Anke wins a play if it reaches a node in $\mathbf{T}$. Boris wins a play if it never goes through a node of $\mathbf{T}$ and runs forever.

The winning statistics can be written down in $\lceil \log(\mathbf{n}) + \mathbf{3} \rceil$ binary numbers of $\lceil \log(\mathbf{m}) + \mathbf{1} \rceil$ bits, the player to move needs one bit and the position needs $\lceil \log(\mathbf{n}) \rceil$ bits. The overall size can be estimated with $\mathbf{O}(\mathbf{n}^{\log(\mathbf{m})+\mathbf{5}})$ nodes and each node has up to $\mathbf{n}$ outgoing edges.

Reachability Game: Linear Time Algo

Theorem [Well-Known]
The winner of a reachability game on a graph (V, E) with set T to be reached from s can be found in time $O(|V| + |E|)$.

Assumption: Node says whether Anke or Boris moves or node is in T; algorithm marks winning nodes for Anke.

Algorithm
For all $v \in T$ and all $v \in V - T$ where Anke moves let $q_v = 1$; for all $v \in V - T$ where Boris moves let q_v be the number of successors.

Call $A(v)$ below for all $v \in T$.

$A(v)$: If $q_v = 0$ then return with no activity;
If $q_v = 1$ then let $q_v = 0$ and call $A(w)$ for all w with v being a successor of w and return;
If $q_v > 1$ then let $q_v = q_v - 1$ and return.

Anke wins iff $q_s = 0$ after running the algorithm.

More careful bounds 1

The winning statistics of $\lceil \log(\mathbf{n}) + \mathbf{3} \rceil$ numbers $\mathbf{b_0, b_1, \ldots, b_k}$ from $\mathbf{0}, \ldots, \mathbf{m}$ can be coded by $\hat{\mathbf{b}}_\mathbf{0}, \hat{\mathbf{b}}_\mathbf{1}, \ldots, \hat{\mathbf{b}}_\mathbf{k}$ with $\hat{\mathbf{b}}_\mathbf{0} = \mathbf{b_0}$ and if $\mathbf{b_{i+1}} = \mathbf{0}$ then $\hat{\mathbf{b}}_\mathbf{i+1} = \hat{\mathbf{b}}_\mathbf{i} + \mathbf{1}$ else $\hat{\mathbf{b}}_\mathbf{i+1} = \hat{\mathbf{b}}_\mathbf{i} + \mathbf{2} + \min\{\mathbf{b_{i+1}} - \mathbf{b_j} : \mathbf{j} \leq \mathbf{i}\}$. So $\hat{\mathbf{b}}_\mathbf{k} \leq \mathbf{b_k} + \mathbf{2} \cdot \mathbf{k}$.

For $\mathbf{h} = \lceil \mathbf{m} / \log(\mathbf{n}) \rceil$, one has that the $\hat{\mathbf{b}}_\mathbf{0}, \ldots, \hat{\mathbf{b}}_\mathbf{k}$ select $\lceil \log(\mathbf{n}) + \mathbf{3} \rceil$ numbers out of $\lceil \log(\mathbf{n}) + \mathbf{3} \rceil \cdot (\mathbf{h} + \mathbf{2})$. Using the entropy bound for $\binom{(\mathbf{h}+\mathbf{2}) \cdot \lceil \log(\mathbf{n})+\mathbf{3} \rceil}{\lceil \log(\mathbf{n})+\mathbf{3} \rceil}$ and the fact that $\log(\mathbf{1} + \mathbf{1}/(\mathbf{h} + \mathbf{2})) \cdot (\mathbf{h} + \mathbf{1}) \leq \mathbf{1.45}$ for all $\mathbf{h} \in \mathbb{N}$, this can be estimated by $\mathbf{O}(\mathbf{h}^4 \cdot \mathbf{n}^{\mathbf{1.45}+\log(\mathbf{h}+\mathbf{2})})$.

So the size of the reachability game is $\mathbf{O}(\mathbf{h}^4 \cdot \mathbf{n}^{\mathbf{2.45}+\log(\mathbf{h}+\mathbf{2})})$ and it has $\mathbf{O}(\mathbf{h}^4 \cdot \mathbf{n}^{\mathbf{3.45}+\log(\mathbf{h}+\mathbf{2})})$ edges.

Time $\mathbf{O}(\lceil \mathbf{m} / \log(\mathbf{n}) \rceil^4 \cdot \mathbf{n}^{\mathbf{3.45}+\log(\lceil \mathbf{m}/\log(\mathbf{n}) \rceil + \mathbf{2})})$ is sufficient to determine the winner of a parity game.

More careful bounds 2

The precise computation for the number of winning strategy values are

$$\begin{aligned}
&\mathbf{2}^{(\log(\mathbf{n})+4)\cdot(\mathbf{h}+\mathbf{2})\cdot((\mathbf{1}/(\mathbf{h}+\mathbf{2}))\cdot\log(\mathbf{h}+\mathbf{2})+((\mathbf{h}+\mathbf{1})/(\mathbf{h}+\mathbf{2}))\cdot\log((\mathbf{h}+\mathbf{2})/(\mathbf{h}+\mathbf{1})))} \\
&\quad = \mathbf{2}^{(\log(\mathbf{n})+4)\cdot(\log(\mathbf{h}+\mathbf{2})+\log(\mathbf{1}+\mathbf{1}/(\mathbf{h}+\mathbf{2}))\cdot(\mathbf{h}+\mathbf{1}))} \\
&\quad = (\mathbf{16n})^{\log(\mathbf{h}+\mathbf{2})+(\log(\mathbf{1}+\mathbf{1}/(\mathbf{h}+\mathbf{2}))\cdot(\mathbf{h}+\mathbf{1}))} \\
&\quad \leq (\mathbf{16n})^{\mathbf{1.45}+\log(\mathbf{h}+\mathbf{2})} \\
&\quad \leq \mathbf{c}\cdot(\mathbf{h}+\mathbf{2})^{\mathbf{4}}\cdot\mathbf{n}^{\mathbf{1.45}+\log(\mathbf{h}+\mathbf{2})} \\
&\quad \leq \mathbf{c}'\cdot\mathbf{h}^{\mathbf{4}}\cdot\mathbf{n}^{\mathbf{1.45}+\log(\mathbf{h}+\mathbf{2})}
\end{aligned}$$

for some constants $\mathbf{c}, \mathbf{c}'$.

For $\mathbf{h}=\mathbf{1}$ this gives size $\mathbf{O}(\mathbf{n}^{\mathbf{5.04}})$ and alternative estimations give $\mathbf{O}(\mathbf{n}^{\mathbf{5}})$ in this special case, using that $\binom{(\mathbf{h}+\mathbf{2})\cdot\lceil\log(\mathbf{n})+\mathbf{3}\rceil}{\lceil\log(\mathbf{n})+\mathbf{3}\rceil}$ is of order $\mathbf{O}(\mathbf{n}^{\mathbf{3}})$.

Fixed Parameter Tractability

Theorem

If $\mathbf{m} \leq \log(\mathbf{n})$ then the winner of a parity game with $\mathbf{n}$ nodes and $\mathbf{m}$ values can be found in time $\mathbf{O}(\mathbf{n}^5)$.

Corollary

Parity games are fixed parameter tractable when parameterised by the number of values.

Remark

For constant $\mathbf{m}$, the number of possible values of the winning statistics can be estimated by $\mathbf{O}(\mathbf{n}^2)$ and the number of nodes in the reachability game by $\mathbf{O}(\mathbf{n}^3)$ and the overall running time is $\mathbf{O}(\mathbf{n}^4)$. Furthermore, in the case of graphs of constant out-degree, say out-degree $\mathbf{2}$, the overall running time is $\mathbf{O}(\mathbf{n}^3)$.

Bounds for Coloured Muller Games

The reduction of coloured Muller games to parity games by Björklund, Sandberg and Vorobyov from 2003 gives the following application; note that for almost all $(\mathbf{m}, \mathbf{n})$, $\mathbf{2m} \leq \log(\mathbf{m}! \cdot \mathbf{n})$.

Theorem
A Muller game with $\mathbf{m}$ colours and $\mathbf{n}$ nodes can be solved in time $\mathbf{O}((\mathbf{m}^{\mathbf{m}} \cdot \mathbf{n})^{\mathbf{5}})$.

Theorem [Björklund, Sandberg and Vorobyov 2003]
Parity games are FPT iff Muller games are FPT.

However, there is a limitation.

Theorem
If the Exponential Time Hypothesis holds then coloured Muller games with $\mathbf{m}$ colours and $\mathbf{n}$ nodes are not solvable in $\mathbf{2}^{\mathbf{o}(\mathbf{m} \cdot \log(\mathbf{m}))} \cdot \mathbf{n}^{\mathbf{O}(\mathbf{1})}$ time.

Multi-Dimensional Parity Games

Definition. A k-dimensional parity game has in every node a k-dimensional vector of values from $\{1, 2, \ldots, m\}$. A play is win for Anke iff there is a coordinate $\tilde{k}$ such that the limit superior of the $\tilde{k}$-th value of the vectors is an odd number; a play is a win for Boris iff for every coordinate the limit superior of the values in that coordinate is even. Note that $k, m \geq 2$.

Theorem. The k-dimensional parity games with m values per dimension and n nodes can be solved in time $O((2^{k \cdot \log(k) \cdot m} \cdot n)^{5.45})$.

Theorem. The k-dimensional parity games with m values per dimension and n nodes cannot be solved in time $2^{o(k \cdot \log(k) \cdot m)} \cdot n^{O(1)}$ unless the exponential time hypothesis fails; even if one of m and k is fixed to a constant.

Summary

The winner of a parity game can be found in quasipolynomial time where the exponent depends only logarithmically on the number of values used in the parity game.

Parity games are fixed parameter tractable in the case that the number of values is fixed. If $\mathbf{m} \leq \log(\mathbf{n})$ then the parity game can be solved in $\mathbf{O}(\mathbf{n}^5)$ and one can give a general formula of the type $\mathbf{O}(2^{\mathbf{m}} \cdot \mathbf{n}^4)$ which works for all $\mathbf{n}$ and $\mathbf{m}$ [Krishnendu Chatterjee].

The bounds transfer to Muller games and show that these can be decided in $\mathbf{O}((\mathbf{m}^{\mathbf{m}} \cdot \mathbf{n})^5)$. The bound cannot be improved to $2^{\mathbf{o}(\mathbf{m} \cdot \log(\mathbf{m}))} \cdot \mathbf{n}^{\mathbf{O}(1)}$ unless the Exponential Time Hypothesis fails.