NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA3205 SET THEORY

November/December 2009 — Time allowed: 2 hours

MATRICULATION NUMBER: ______________________________

INSTRUCTIONS TO CANDIDATE

- This examination paper consists of **TWELVE (12)** questions. It comprises **EIGHT (8)** printed pages including the cover page.
- Answer **ALL** questions. Fill in your answers in the spaces provided. For each of the multiple choice questions, put a mark ($\boxed{\text{x}}$) into **EXACTLY ONE (1)** of the given choices.
- The total number of marks for this paper is **SIXTY-SEVEN (67)**. Each question carries as many marks as it has subparts. Each correct and complete answer of a subpart gives one mark, an incorrect or incomplete answer gives zero mark.
- Candidates may use calculators.

This table is for examiner's use only.

Question	Marks	Out of	Question	Marks	Out of	Question	Marks	Out of
Q01, P2:		5	Q05, P4:		6	Q09, P6:		6
Q02, P2:		5	Q06, P4:		4	Q10, P6:		4
Q03, P3:		4	Q07, P5:		4	Q11, P7:		10
Q04, P3:		6	Q08, P5:		6	Q12, P8:		7
						Total:		67

Question 1 [5 marks]. Determine the following sets:

(a) $\bigcup\{\{0,1\},\{1,2\},\{2,3\}\} - \{1,3\} = \{\underline{\quad 0,2 \quad}\}$;

(b) $\bigcap\{\{0,1,2,3,4\},\{1,2,3\},\{2\}\} = \{\underline{\quad 2 \quad}\}$;

(c) $\mathcal{P}(\{0,1\}) - \{\emptyset,\{0,1,2\}\} = \{\underline{\quad \{0\},\{1\},\{0,1\} \quad}\}$;

(d) $\mathcal{P}(\{0\}) \cup \mathcal{P}(\{1\}) \cup \mathcal{P}(\{2\}) = \{\underline{\quad \emptyset,\{0\},\{1\},\{2\} \quad}\}$;

(e) $\{0,1,2,3,4,5,6,7,8,9\} - \mathcal{P}(\{0,1,2,3,4,5\}) = \{\underline{\quad 7,8,9 \quad}\}$.

Question 2 [5 marks]. Here are statements (a) to (e) on a given set A. For each such statement, choose among the three choice options that one which does not contradict the given statement. Here is an example.

(Example) A is a power set:
☐ $A = \{\{1,2\}\}$; $\quad$ [x] $A = \{\emptyset,\{1\},\{2\},\{1,2\}\}$; $\quad$ ☐ $A = \mathbb{N}$.

Similarly, for each of the below statements find that option for which the statement can be true. The other two (false) options contradict the given statement.

(a) There is no function $f : \mathbb{N} \to A$:
[x] A is empty; $\quad$ ☐ A has one element; $\quad$ ☐ A is countable.

(b) $\mathcal{P}(A)$ has 2048 elements:
☐ $|A| = 10$; $\quad$ [x] $|A| = 11$; $\quad$ ☐ $|A| = 12$.

(c) There is a function $f : A \to A$ which is one-to-one but not onto:
☐ A has one element; $\quad$ ☐ A has two elements; $\quad$ [x] A is countable.

(d) A is inductive and $A \neq \mathbb{N}$:
☐ $A = 0$; $\quad$ ☐ $A = \omega + 5$; $\quad$ [x] $A = \omega + \omega$.

(e) A is transitive:
☐ $A = \mathbb{N} \times \mathbb{N}$; $\quad$ ☐ $A = \{2,3,4,5\}$; $\quad$ [x] $A = V_\omega$.

Question 3 [4 marks]. Sort the strings $00, 123, 124, 34, 12, 2340, 002$ according to the following orderings:

(a) Lexicographic ordering: 00,002,12,123,124,2340,34;

(b) Length-lexicographic ordering: 00,12,34,002,123,124,2340;

(c) Kleene-Brouwer ordering: 002,00,123,124,12,2340,34;

(d) Ordering by decimal value: 00,002,12,34,123,124,2340.

Question 4 [6 marks]. Write the Cantor Normal Form for the following ordinals, choose the easiest possible form:

(a) $\omega^3 + \omega^4 + \omega^2$: $\omega^4 + \omega^2$;

(b) $\omega^\omega + \omega^{\omega+2} + \omega^{\omega\cdot 2}$: $\omega^{\omega\cdot 2}$;

(c) $\omega \cdot \omega + \omega \cdot 2 + \omega \cdot 3$: $\omega^2 + \omega \cdot 5$;

(d) $1 + \omega + \omega^2 + \omega^2 + \omega + 1$: $\omega^2 \cdot 2 + \omega + 1$;

(e) $\omega^5 \cdot 0 + \omega^4 \cdot 1 + \omega^3 \cdot 2$: $\omega^4 + \omega^3 \cdot 2$;

(f) $\omega + \omega^2 + \omega + \omega^2 + \omega$: $\omega^2 \cdot 2 + \omega$.

Question 5 [6 marks]. Let $\{f : A \to B\}$ denote the set of all functions from A to B. Which of the following sets are finite, countable and uncountable?

(a) $\{f : \mathbb{N} \to \{0, 1, 2\}\}$ is
☐ finite; ☐ countable; ☒ uncountable.

(b) $\{f : \{0, 1, 2\} \to \mathbb{N}\}$ is
☐ finite; ☒ countable; ☐ uncountable.

(c) $\{f : \{0, 1, 2\} \to \{1, 2, 3, 4\}\}$ is
☒ finite; ☐ countable; ☐ uncountable.

(d) $\{f : \mathbb{N} \to \mathbb{N} \wedge \forall n \in \mathbb{N}\, [f(n + 1) \leq f(n)]\}$ is
☐ finite; ☒ countable; ☐ uncountable.

(e) $\{f : \mathbb{N} \to \mathbb{N} \wedge \forall n \in \mathbb{N}\, [f(n + 1) > f(n)]\}$ is
☐ finite; ☐ countable; ☒ uncountable.

(f) $\{f : \mathbb{N} \to \mathbb{N} \wedge \forall n \in \mathbb{N}\, [f(n + 1) = f(n)] \wedge \forall n \in \mathbb{N}\, [f(n) \leq 2n + 5]\}$ is
☒ finite; ☐ countable; ☐ uncountable.

Question 6 [4 marks]. Which of the following statements about cardinals are true, false or unprovable? Here "unprovable" means that there are models of set theory where the statement is true and other models of set theory where the statement is false.

(a) $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$ for all ordinals α:
☒ true; ☐ false; ☐ unprovable.

(b) $2^{\aleph_0} = \aleph_\omega$:
☐ true; ☒ false; ☐ unprovable.

(c) $2^{\aleph_0} = \aleph_5$:
☐ true; ☐ false; ☒ unprovable.

(d) $2^{\aleph_1} = \aleph_2$:
☐ true; ☐ false; ☒ unprovable.

Question 7 [4 marks]. For functions $f, g : \mathbb{N} \to \mathbb{N}$ let $f \sqsubset g$ iff there is an n such that $f(m) < g(m)$ for all $m > n$. Now consider various subsets of $F = \{f : \mathbb{N} \to \mathbb{N}\}$ and determine what can be said about the ordering $\sqsubset$.

(a) $A = \{f \in F : f(0) = 1 \text{ and } f(n+1) \in \{2f(n), 2f(n)+1\} \text{ for all } n \in \mathbb{N}\}$:
☐ $(A, \sqsubset)$ is dense;
☒ $(A, \sqsubset)$ has end points;
☐ $(A, \sqsubset)$ is well-ordered.

(b) $B = \{f \in F : f(n) = a \cdot n^2 + b \cdot n + c \text{ for some } a, b, c \in \mathbb{N} \text{ and all } n \in \mathbb{N}\}$:
☐ $(B, \sqsubset)$ is dense;
☐ $(B, \sqsubset)$ is well-ordered and order-isomorphic to ω^2;
☒ $(B, \sqsubset)$ is well-ordered and order-isomorphic to ω^3.

(c) $C = \{f \in F : \exists n \forall m > n [f(m) = 0]\}$:
☒ all elements of $(C, \sqsubset)$ are incomparable;
☐ $(C, \sqsubset)$ is linearly ordered but not well-ordered;
☐ $(C, \sqsubset)$ is well-ordered.

(d) $D = \{f \in F : \forall n \in \mathbb{N} [f(n+2) = f(n)]\}$:
☐ $(D, \sqsubset)$ is dense and linearly ordered;
☐ $(D, \sqsubset)$ is well-ordered;
☒ $(D, \sqsubset)$ is not linearly ordered.

Question 8 [6 marks]. Answer the following questions about $\mathbb{R}$ and its ordering.

(a) Does every nonempty subset $A \subseteq \mathbb{R}$ have a least element?
☐ Yes; ☒ No.

(b) Does every bounded nonempty subset $B \subseteq \mathbb{R}$ have an infimum and a supremum?
☒ Yes; ☐ No.

(c) Is $\mathbb{R}$ a countable set?
☐ Yes; ☒ No.

(d) Is the ordering $<$ on $\mathbb{R}$ dense?
☒ Yes; ☐ No.

(e) Does the set $\{\{r \in \mathbb{R} : a < r < b\} : a, b \in \mathbb{R} \wedge a < b\}$ of all open intervals in $\mathbb{R}$ have the same cardinality as $\mathbb{R}$?
☒ Yes; ☐ No.

(f) Does the interval $\{r \in \mathbb{R} : 0 < r < 1\}$ have the same cardinality as $\mathbb{R}$?
☒ Yes; ☐ No.

Question 9 [6 marks]. Determine the rank of the following sets.

(a) $\rho(\omega \times \omega) = \underline{\quad \omega \quad}$;

(b) $\rho(V_{\omega+2} \cup \{\omega^4\}) = \underline{\quad \omega^4 + 1 \quad}$;

(c) $\rho(\{\{\emptyset, \{\{\emptyset\}\}\}\}) = \underline{\quad 4 \quad}$;

(d) $\rho(\{f : \omega^2 \to \omega^3\}) = \underline{\quad \omega^3 + 1 \quad}$;

(e) $\rho(V_\omega - \mathbb{N}) = \underline{\quad \omega \quad}$;

(f) $\rho(\{\{2, 3\}, \{4, 5\}\}) = \underline{\quad 7 \quad}$.

Question 10 [4 marks]. Answer the following questions on the rank:

(a) Is there an ordinal α with $\rho(\omega^\alpha) = \alpha$?
$\boxed{x}$ Yes; $\square$ No.

(b) Justify your answer:

If $\alpha = \epsilon_0$ then $\omega^\alpha = \alpha$ and $\omega^\alpha = \rho(\omega^\alpha) = \rho(\alpha) = \alpha$.

(c) Is there an ordinal α with $\rho(\alpha) > \rho(S(\alpha))$?
$\square$ Yes; $\boxed{x}$ No.

(d) Justify your answer:

Note that $\alpha \subseteq S(\alpha)$ and $\rho(\beta) = \beta$ for all ordinals α and β; hence $\rho(\alpha) \leq \rho(S(\alpha))$ for all ordinals α.

Question 11 [10 marks]. (a) Are there sets A, B such that $A - B = A \cup B$?
$\boxed{\text{x}}$ Yes; $\square$ No.

(b) Justify your answer:

Take $A = \mathbb{N}$ and $B = \emptyset$ to get the equality.

(c) Are there sets C and D such that $\mathcal{P}(C) \cap \mathcal{P}(D)$ is countable?
$\square$ Yes; $\boxed{\text{x}}$ No.

(d) Justify your answer:

$\mathcal{P}(C) \cap \mathcal{P}(D) = \{U : U \subseteq C \wedge U \subseteq D\} = \mathcal{P}(C \cap D)$ and no power set of a set is countable.

(e) Is there a set E with $|E \times E| = \aleph_0$?
$\boxed{\text{x}}$ Yes; $\square$ No.

(f) Justify your answer:

Take $E = \mathbb{N}$. Then $\mathbb{N} \times \mathbb{N}$ can be mapped to $\mathbb{N}$ by Cantor's pairing function which is a bijection.

(g) Is $|\{f : G \to G\}| = |\mathcal{P}(G)|$ for every set G?
$\square$ Yes; $\boxed{\text{x}}$ No.

(h) Justify your answer:

$\mathcal{P}(\{0, 1, 2\})$ has 8 elements and $\{f : \{0, 1, 2\} \to \{0, 1, 2\}\}$ has 27 elements.

(i) Are there infinite cardinals a, b, c, d with $a < b < c < d$ and $a \cdot d < b \cdot c$?
$\square$ Yes; $\boxed{\text{x}}$ No.

(j) Justify your answer:

$a \cdot d = d$ and $b \cdot c = c$ by the Theorem of Hessenberg and $a < b < c < d$. Then $a \cdot d > b \cdot c$.

Question 12 [7 marks]. Let A be a set of three ordinals and

$$Sums(A) = \{x + y + z : x, y, z \in A \wedge x \neq y \wedge x \neq z \wedge y \neq z\}.$$

For example, $Sums(\{\omega^3, \omega^3 + \omega^2, \omega^3 + \omega\}) = \{\omega^3 \cdot 3, \omega^3 \cdot 3 + \omega, \omega^3 \cdot 3 + \omega^2\}$. The following questions study $Sums(A)$ and $|Sums(A)|$.

(a) Determine $s = \max\{|Sums(A)| : A \text{ is a set of three ordinals}\}$:
☐ $s = 3$, ☐ $s = 4$, ☒ $s = 5$.

(b) List a set A of three ordinals with $|Sums(A)| = s$ for the value s from (a).

$A = \{\omega + 1, \omega + 2, \omega^2\}$.

(c) List the elements of $Sums(A)$ for the set A from (b).

$\omega^2, \omega^2 + \omega + 1, \omega^2 + \omega + 2, \omega^2 + \omega \cdot 2 + 1, \omega^2 + \omega \cdot 2 + 2$.

(d) Explain why $|Sums(A)| < 6$ for every set A of three ordinals?

Let a, b, c be the ordinals in ascending order and let ω^d be the highest ω-power occurring in the Cantor Normal Form of c. Then $a = \omega^d \cdot x + u$, $b = \omega^d \cdot y + v$ and $c = \omega^d \cdot z + w$ where u, v, w are ordinals below ω^d and $x, y, z \in \mathbb{N}$; x and y can be 0. It follows that $a + b + c$ and $b + a + c$ are both $\omega^d \cdot (x + y + z) + w$.

(e) Is there a set A of three ordinals with $|Sums(A)| = 2$?
☒ Yes; ☐ No.

(f) Justify your answer to (e):

Take $A = \{\omega \cdot 2, \omega + 1, \omega\}$; the possible choices are $\omega \cdot 4$ and $\omega \cdot 4 + 1$.

(g) What is the value of $|Sums(\{\omega^2, \omega^3, \omega^4\})|$?
☐ 1, ☐ 2, ☐ 3, ☒ 4, ☐ 5.

END OF PAPER