6. The Growth of Functions: Big $O$, Big $\Omega$ and Big $\Theta$

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Growth of Functions

- How fast does a function grow.
- Goal is to compare functions with respect to their behaviour on inputs, in particular large inputs.
- The function \( f(n) = 20000 + n \) is on small values larger than \( g(n) = 2^n \); however, if \( n = 20 \) then \( f(n) = 20020 \) and \( g(n) = 1048576 > f(n) \). Indeed, \( g(n) > f(n) \) for all \( n \geq 20 \). So \( g \) grows faster than \( f \).
- How can this observation be formalised in general?
Function

A real function \( f \) has domain and range \( \mathbb{R} \) and is commonly denoted by the following.

\[
f : \mathbb{R} \rightarrow \mathbb{R}
\]

Some Important Functions

- **Constant**: \( f(x) = 1 \)
- **Logarithm**: \( f(x) = \log(x) \);
- **Linear**: \( f(x) = a \cdot x + b \);
- **Quadratic**: \( f(x) = a \cdot x^2 + b \cdot x + c \);
- **Polynomial**: \( f(x) = a_1 \cdot x^n + a_2 \cdot x^{n-1} + \ldots + c \);
- **Exponential**: \( f(x) = e^x \)
- **Factorial**: \( f(n) = n! \)

Online Function Plotting
Functions

- $\log(x)$
- $x$
- $x \log(x)$
- $x^2$
- $2^x$
Functions

- $\log(x)$
- $x$
- $x \log(x)$
- $x^2$
- $2^x$
Landau Notation

Big O notation and other notations derived from Landau notation are used to characterize the relative asymptotic behavior of functions. They say something about how fast a function grows or declines.
Applications in Computer Science

Big O notation, Big theta and other Landau notations are used in computer science to analyze the efficiency of algorithms. For example, when $n \to \infty$, 

$$T(n) = n^2 + 2 \cdot n + 1 \in \Theta(n^2)$$

Applications in Mathematics and Engineering

Big O notation is used in mathematics and engineering to analyze the error of an approximation. For example, when $x \to 0$, 

$$e^x = 1 + x + \frac{1}{2} \cdot x^2 + O(x^3)$$
Definition 6.3.1 (Big Omicron)

Let $g$ be a real function.

\[ O(g) = \{ f \mid \exists c \in \mathbb{R}^+ \exists x_0 \in \mathbb{R} \forall x \in \mathbb{R} \ ((x \geq x_0) \Rightarrow (0 \leq f(x) \leq c \cdot g(x))) \} \]

In English

Eventually $f$ does not grow faster than $g$: there is some $x_0$ such that, for $x$ larger than or equal to $x_0$, $f(x)$ is always lower than or equal to $c \cdot g(x)$.

We say that $g$ is an asymptotically upper bound for $f$. 
Notations

Let $g$ be a real function. Let $f$ be a real function. Let $f \in O(g)$.

- $f \in O(g)$ reads “$f$ is of the order of $g$”;
- $f \in O(g)$ reads “$f$ is big O of $g$”;
- $f \in O(g) \iff O(f) \subseteq O(g)$;
- $(f \in O(g) \land g \in O(f)) \iff O(f) = O(g)$;
- Unfortunately, $f \in O(g)$ is often written $f = O(g)$;
### Examples

- $10000 \cdot n^2 + n + 5 \in O(n^2)$;
- $n^2 + n \cdot \log(n) + 5 \in O(n^2)$;
- $n \cdot \log(n) + \log(n) \in O(n \log(n))$;
- $n \cdot \log(n) + n \notin O(n)$;
- $2^n + n^2 + 100 \cdot \log(n) \in O(2^n)$. 
Rules

Let $f, g, h$ be functions from positive real numbers to positive real numbers.

- If $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$;
- If $f, g \in O(h)$ then $c \cdot f + d \cdot g \in O(h)$ for $c, d > 0$;
- $n^i + n^j \in O(n^{\max\{i,j\}})$;
- $n^i \cdot \log^j(n) \in O(n^{i+1})$.

Warning

- $3^n \not\in O(2^n)$;
- There are $f, g$ with $f \not\in O(g)$ and $g \not\in O(f)$;
- $f \in O(g + h)$ does not imply $f \in O(g) \lor f \in O(h)$;
- Example for this: $n^2 = n^2 \cdot \sin^2(n) + n^2 \cdot \cos^2(n)$ but $n^2 \not\in O(n^2 \cdot \sin^2(n))$ and $n^2 \not\in O(n^2 \cdot \cos^2(n))$. 
Quiz

Simplify the expression by giving the best possible order.

- Example: $5 \cdot n^2 + 8 \cdot n^8 + 5 \cdot 2^n$ is $\mathcal{O}(2^n)$.
- Give the order of $3 \cdot n^2$.
- Give the order of $2877 \cdot n^2 + n^3 + n$.
- Give the order of $n^2 + n^4 + n^3 \log(n)$.
- Give the order of $18 \cdot 2^n + 8 \cdot 3^n + 0 \cdot 4^n$.
- Give the order of $(n + 1)^2 - n^2$.
- Give the order of $(n + 1)^2 - n(n + 2)$.
- Give the order of $(n + 1)^2 \cdot \log(n) - n^2$. 
Important Sets

- Constant: $O(1)$;
- Logarithmic: $O(\log(n))$;
- Polylogarithmic: $O((\log(n))^c)$;
- Linear $O(n)$;
- Pseudo-linear: $O(n \log(n))$;
- Quadratic: $O(n^2)$;
- Polynomial: $O(n^c)$, for any given $c$;
- Exponential: $O(c^n)$, for any given $c$;
- Factorial: $O(n!)$.

$O(1) \subset O(\log(n)) \subset \ldots \subset O(c^n) \subset O(n!)$
**Definition 6.4.1 (Big Omega)**

Let $g$ be a real function.

$$
\Omega(g) = \{ f \mid \exists c \in \mathbb{R}^+ \exists x_0 \in \mathbb{R} \forall x \in \mathbb{R} \ ((x \geq x_0) \Rightarrow (0 \leq c \cdot g(x) \leq f(x))) \}
$$

**In English**

We say that $g$ is an *asymptotically lower bound* for $f$. 
Notations

Let $g$ be a real function. Let $f$ be a real function. Let $f \in \Omega(g)$. We write:

$$f(x) \in \Omega(g(x))$$
$$f(x) = \Omega(g(x))$$

Theorem 6.4.2

Let $f$ and $g$ be real functions. Then

$$f \in O(g) \iff g \in \Omega(f).$$
**Definition 6.5.1 (Big Theta)**

Let $g$ be a real function.

$$\Theta(g) = \{ f \mid \exists c_1 \in \mathbb{R}^+ \exists c_2 \in \mathbb{R}^+ \exists x_0 \in \mathbb{R} \forall x \in \mathbb{R}$$

$$((x \geq x_0) \Rightarrow (0 \leq c_1 \cdot g(x) \leq f(x) \leq c_2 \cdot g(x)))\}$$

**In English**

Eventually $f$ grows as fast as $g$.
We say that $g$ is an asymptotically tight bound for $f$. 
Let $g$ be a real function. Let $f$ be a real function. Let $f \in \Theta(g)$. We write:

$$f(x) \in \Theta(g(x))$$

$$f(x) = \Theta(g(x))$$
Theorem 6.5.2

Let $g$ be a real function. Let $f$ be a real function.

$$(f \in \Theta(g)) \iff (f \in O(g) \land f \in \Omega(g))$$
Quiz

Which of the following bounds are tight? That is, for which of the following examples $f, g$ is it possible to replace $f \in O(g)$ by $f \in \Theta(g)$?

- $n^2 \in O(n^5)$;
- $(22n + 5) \cdot 2^n \in O(3^n)$;
- $238 \cdot n^2 + 359 \cdot n^3 + 424 \cdot n^4 \in O(n^4)$;
- $(n + 1)^2 - n \cdot (n + 2) \in O(1)$;
- $(n + 1)^3 - n^2 \cdot (n + 3) \in O(n^2)$;
- $n^2 \cdot (\log(n))^{257} \in O(n^3)$.
Some Useful Summations

Arithmetic Series:

\[ \sum_{i=1}^{n} i = \frac{1}{2} \cdot n \cdot (n + 1) \in \Theta(n^2) \]

\[ \sum_{i=1}^{n} i^k \in \Theta(n^{k+1}) \]

Geometric Series:

\[ \sum_{i=0}^{n} k^i \in \Theta(k^n) \]

Harmonic Series:

\[ \sum_{i=1}^{n} \frac{1}{i} \in \Theta(\log(n)) \]
Theorem 6.6.1 (Master Theorem)

Let $a \geq 1$ and $b > 1$ be constants. Let $f(n)$ be a function with $f(n) \geq 1$ for all $n$. Let $T(n)$ be a function on the non-negative integers by the following recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

$T(n)$ can be bounded asymptotically as follows:

1. If $f(n) \in O(n^{\log_b(a) - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) \in \Theta(n^{\log_b(a)})$;
2. If $f(n) \in \Theta(n^{\log_b(a)})$, then $T(n) \in \Theta(n^{\log_b(a)} \cdot \log_b(n))$;
3. If $f(n) \in \Omega(n^{\log_b(a) + \varepsilon})$ for some constant $\varepsilon > 0$, and if $a \cdot f\left(\frac{n}{b}\right) \geq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large $n$, then $T(n) \in \Theta(f(n))$.

\[\frac{a \cdot n}{b}\] means either $\lceil \frac{n}{b} \rceil$ or $\lfloor \frac{n}{b} \rfloor$. 
Examples

Note that $\Theta(\log_a(n)) = \Theta(\log_b(n))$, hence the basis of the logarithm can be omitted inside $O$ and $\Theta$ expressions.

- If $T(n) = 2T(n/2) + c \cdot n$ for some $c > 0$ then $T(n) \in \Theta(n \log(n))$;
- If $T(n) = 2T(n/2)$ then $T(n) = c \cdot n$ for some constant $c$;
- If $T(n) = 8T(n/2) + c \cdot n^2$ for $c > 0$ then $T(n) \in \Theta(n^3)$;
- If $T(n) = T(n/2) + c$ for $c > 0$ then $T(n) \in \Theta(\log(n))$;
- If $T(n) = T(n/2) + n^{10}$ then $T(n) \in \Theta(n^{10})$.

Quiz

- Assume $T(n) = 3T(n/3) + 200$. What is $\Theta(T(n))$?
- Assume $T(n) = 9T(n/3) + n^3$. What is $\Theta(T(n))$?
Besides big $O$, big $\Omega$ and big $\Theta$, there are also little $o$ and little $\omega$. For positive-valued $f$, $g$, the Landau notation $f \in o(g)$ says that the limit of $f(n)/g(n)$ goes to 0. That is, for every $\varepsilon > 0$ there is an $n$ such that all $m > n$ satisfy $f(m) < \varepsilon \cdot g(m)$. Furthermore $f \in \omega(g) \iff g \in o(f)$.

**Examples**

- $287 \cdot n^2 \in o(n^3)$;
- $299^n \in o(300^n)$;
- $n \cdot \log(n) \in o(n^2)$;
- $f \notin o(f)$.

**General Rule**

If $f \in o(g(n))$ then $f \in O(g(n))$ and $f \notin \Theta(g(n))$. 
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