6. The Growth of Functions: Big O, Big Ω and Big Θ

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February 13, 2014

Growth of Functions

- How fast does a function grow.
- Goal is to compare functions with respect to their behaviour on inputs, in particular large inputs.
- The function $f(n) = 20000 + n$ is on small values larger than $g(n) = 2^n$; however, if $n = 20$ then $f(n) = 20020$ and $g(n) = 1048576 > f(n)$. Indeed, $g(n) > f(n)$ for all $n \geq 20$. So g grows faster than f.
- How can this observation be formalised in general?

Function

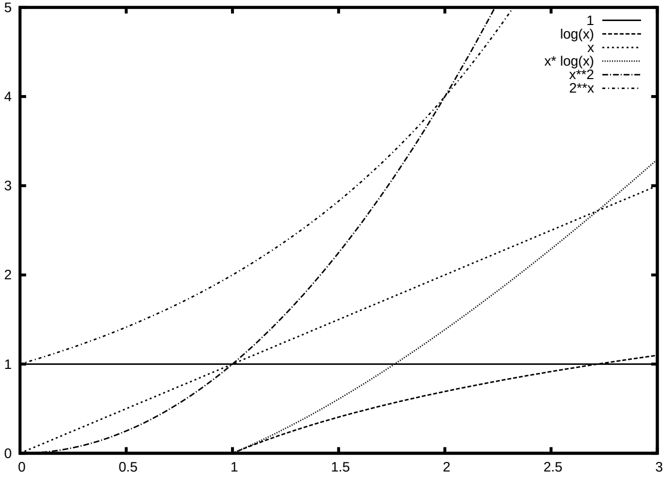
A real function f has domain and range $\mathbb{R}$ and is commonly denoted by the following.

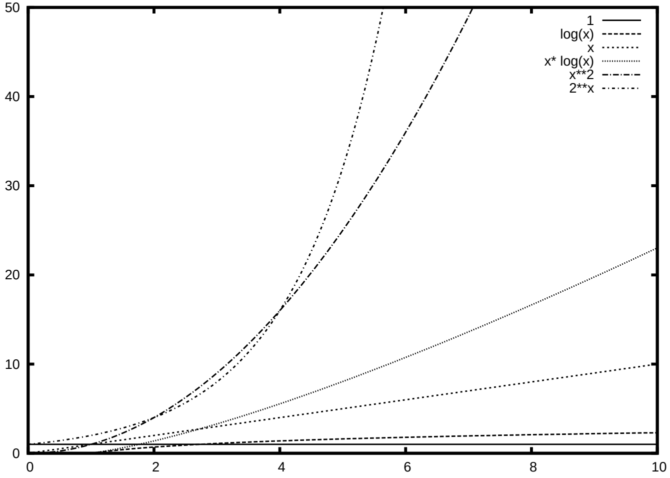
$$f : \mathbb{R} \to \mathbb{R}$$

Some Important Functions

- Constant: $f(x) = 1$
- Logarithm: $f(x) = \log(x)$;
- Linear $f(x) = a \cdot x + b$;
- Quadratic: $f(x) = a \cdot x^2 + b \cdot x + c$;
- Polynomial: $f(x) = a_1 \cdot x^n + a_2 \cdot x^{n-1} + \ldots + c$;
- Exponential: $f(x) = e^x$
- Factorial: $f(n) = n!$

Online Function Plotting


1
log(x)
x
x* log(x)
x**2
2**x
0
1
2
3
4
5
0
0.5
1
1.5
2
2.5
3


50
40
30
20
10
0
0
2
4
6
8
10
1
log(x)
x
x* log(x)
x**2
2**x

Landau Notation

Big O notation and other notations derived from Landau notation are used to characterize the relative asymptotic behavior of functions. They say something about how fast a function grows or declines.

Applications in Computer Science

Big O notation, Big theta and other Landau notations are used in computer science to analyze the efficiency of algorithms. For example, when $n \to \infty$,

$$T(n) = n^2 + 2 \cdot n + 1 \in \Theta(n^2)$$

Applications in Mathematics and Engineering

Big O notation is used in mathematics and engineering to analyze the error of an approximation. For example, when $x \to 0$,

$$e^x = 1 + x + \frac{1}{2} \cdot x^2 + O(x^3)$$

Definition 6.3.1 (Big Omicron)

Let g be a real function.

$$O(g) = \{f \mid \exists c \in \mathbb{R}^+ \; \exists x_0 \in \mathbb{R} \; \forall x \in \mathbb{R} \\ ((x \geq x_0) \Rightarrow (0 \leq f(x) \leq c \cdot g(x)))\}$$

In English

Eventually f does not grow faster than g: there is some x_0 such that, for x larger than or equal to x_0, $f(x)$ is always lower than or equal to $c \cdot g(x)$.
We say that g is an asymptotically upper bound for f.

Notations

Let g be a real function. Let f be a real function. Let $f \in O(g)$.

- $f \in O(g)$ reads "f is of the order of g";
- $f \in O(g)$ reads "f is big O of g";
- $f \in O(g) \Leftrightarrow O(f) \subseteq O(g)$;
- $(f \in O(g) \ \wedge \ g \in O(f)) \Leftrightarrow O(f) = O(g)$;
- Unfortunately, $f \in O(g)$ is often written $f = O(g)$;

Examples

- $10000 \cdot n^2 + n + 5 \in O(n^2)$;
- $n^2 + n \cdot \log(n) + 5 \in O(n^2)$;
- $n \cdot \log(n) + \log(n) \in O(n \log(n))$;
- $n \cdot \log(n) + n \notin O(n)$;
- $2^n + n^2 + 100 \cdot \log(n) \in O(2^n)$.

Rules

Let f, g, h be functions from positive real numbers to positive real numbers.

- If $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$;
- If $f, g \in O(h)$ then $c \cdot f + d \cdot g \in O(h)$ for $c, d > 0$;
- $n^i + n^j \in O(n^{\max\{i,j\}})$;
- $n^i \cdot \log^j(n) \in O(n^{i+1})$.

Warning

- $3^n \notin O(2^n)$;
- There are f, g with $f \notin O(g)$ and $g \notin O(f)$;
- $f \in O(g + h)$ does not imply $f \in O(g) \vee f \in O(h)$;
- Example for this: $n^2 = n^2 \cdot sin^2(n) + n^2 \cdot cos^2(n)$ but $n^2 \notin O(n^2 \cdot sin^2(n))$ and $n^2 \notin O(n^2 \cdot cos^2(n))$.

Quiz

Simplify the expression by giving the best possible order.

- Example: $5 \cdot n^2 + 8 \cdot n^8 + 5 \cdot 2^n$ is $O(2^n)$.
- Give the order of $3 \cdot n^2$.
- Give the order of $2877 \cdot n^2 + n^3 + n$.
- Give the order of $n^2 + n^4 + n^3 \log(n)$.
- Give the order of $18 \cdot 2^n + 8 \cdot 3^n + 0 \cdot 4^n$.
- Give the order of $(n+1)^2 - n^2$.
- Give the order of $(n+1)^2 - n(n+2)$.
- Give the order of $(n+1)^2 \cdot \log(n) - n^2$.

Important Sets

- Constant: $O(1)$;
- Logarithmic: $O(\log(n))$;
- Polylogarithmic: $O((\log(n))^c)$;
- Linear $O(n)$;
- Pseudo-linear: $O(n\log(n))$;
- Quadratic: $O(n^2)$;
- Polynomial: $O(n^c)$, for any given c;
- Exponential: $O(c^n)$, for any given c,
- Factorial: $O(n!)$.

$$O(1) \subset O(\log(n)) \subset \ldots \subset O(c^n) \subset O(n!)$$

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Definition 6.4.1 (Big Omega)

Let g be a real function.

$$\Omega(g) = \{f \mid \exists c \in \mathbb{R}^+ \; \exists x_0 \in \mathbb{R} \; \forall x \in \mathbb{R} \; ((x \geq x_0) \Rightarrow (0 \leq c \cdot g(x) \leq f(x)))\}$$

In English

We say that g is an asymptotically lower bound for f.

Notations

Let g be a real function. Let f be a real function. Let $f \in \Omega(g)$. We write:

$$f(x) \in \Omega(g(x))$$

$$f(x) = \Omega(g(x))$$

Theorem 6.4.2

Lef f and g be real functions. Then

$$f \in O(g) \Leftrightarrow g \in \Omega(f).$$

Definition 6.5.1 (Big Theta)

Let g be a real function.

$$\Theta(g) = \{f \mid \exists c_1 \in \mathbb{R}^+ \; \exists c_2 \in \mathbb{R}^+ \; \exists x_0 \in \mathbb{R} \; \forall x \in \mathbb{R} \\ ((x \geq x_0) \Rightarrow (0 \leq c_1 \cdot g(x) \leq f(x) \leq c_2 \cdot g(x)))\}$$

In English

Eventually f grows as fast as g.
We say that g is an asymptotically tight bound for f.

Notations

Let g be a real function. Let f be a real function. Let $f \in \Theta(g)$. We write:

$$f(x) \in \Theta(g(x))$$

$$f(x) = \Theta(g(x))$$

Theorem 6.5.2

Let g be a real function. Let f be a real function.

$$(f \in \Theta(g)) \Leftrightarrow (f \in O(g) \land f \in \Omega(g))$$

Quiz

Which of the following bounds are tight? That is, for which of the following examples f, g is it possible to replace $f \in O(g)$ by $f \in \Theta(g)$?

- $n^2 \in O(n^5)$;
- $(22n + 5) \cdot 2^n \in O(3^n)$;
- $238 \cdot n^2 + 359 \cdot n^3 + 424 \cdot n^4 \in O(n^4)$;
- $(n + 1)^2 - n \cdot (n + 2) \in O(1)$;
- $(n + 1)^3 - n^2 \cdot (n + 3) \in O(n^2)$;
- $n^2 \cdot (\log(n))^{257} \in O(n^3)$.

Some Useful Summations

Arithmetic Series:

$$\sum_{i=1}^{n} i = \frac{1}{2} \cdot n \cdot (n+1) \in \Theta(n^2)$$

$$\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$$

Geometric Series:

$$\sum_{i=0}^{n} k^i \in \Theta(k^n)$$

Harmonic Series:

$$\sum_{i=1}^{n} \frac{1}{i} \in \Theta(\log(n))$$

Theorem 6.6.1 (Master Theorem)

Let $a \geq 1$ and $b > 1$ be constants. Let $f(n)$ be a function with $f(n) \geq 1$ for all n. Let $T(n)$ be a function on the non-negative integers by the following recurrence[a].

$$T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

$T(n)$ can be bounded asymptotically as follows:

1. *If $f(n) \in O(n^{\log_b(a)-\varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) \in \Theta(n^{\log_b(a)})$;*
2. *If $f(n) \in \Theta(n^{\log_b(a)})$, then $T(n) \in \Theta(n^{\log_b(a)} \cdot log_b(n))$;*
3. *If $f(n) \in \Omega(n^{\log_b(a)+\varepsilon})$ for some constant $\varepsilon > 0$, and if $a \cdot f(\frac{n}{b}) \geq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) \in \Theta(f(n))$.*

[a] $\frac{n}{b}$ means either $\lceil \frac{n}{b} \rceil$ or $\lfloor \frac{n}{b} \rfloor$.

Examples

Note that $\Theta(\log_a(n)) = \Theta(\log_b(n))$, hence the basis of the logarithm can be omitted inside O and Θ expressions.

- If $T(n) = 2T(n/2) + c \cdot n$ for some $c > 0$ then $T(n) \in \Theta(n \log(n))$;
- If $T(n) = 2T(n/2)$ then $T(n) = c \cdot n$ for some constant c;
- If $T(n) = 8T(n/2) + c \cdot n^2$ for $c > 0$ then $T(n) \in \Theta(n^3)$;
- If $T(n) = T(n/2) + c$ for $c > 0$ then $T(n) \in \Theta(\log(n))$;
- If $T(n) = T(n/2) + n^{10}$ then $T(n) \in \Theta(n^1 0)$.

Quiz

- Assume $T(n) = 3T(n/3) + 200$. What is $\Theta(T(n))$?
- Assume $T(n) = 9T(n/3) + n^3$. What is $\Theta(T(n))$?

Little o

Besides big O, big Ω and big Θ, there are also little o and little ω. For positive-valued f, g, the Landau notation $f \in o(g)$ says that the limit of $f(n)/g(n)$ goes to 0. That is, for every $\varepsilon > 0$ there is an n such that all $m > n$ satisfy $f(m) < \varepsilon \cdot g(m)$.
Furthermore $f \in \omega(g) \Leftrightarrow g \in o(f)$.

Examples

- $287 \cdot n^2 \in o(n^3)$;
- $299^n \in o(300^n)$;
- $n \cdot \log(n) \in o(n^2)$;
- $f \notin o(f)$.

General Rule

If $f \in o(g(n))$ then $f \in O(g(n))$ and $f \notin \Theta(g(n))$.

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