CS4234: Optimization Algorithms Lecture 2

Introduction to Linear Programming

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Abstract

Today we introduce linear programming, a powerful tool for solving optimization problems. We discuss some terminology and notation, and we give some examples of how to use LPs to solve basic optimization problems. Then we talk about Integer Linear Programs (ILPs), and discuss how to relax ILPs to solve combinatorial optimization problems that require integral answer. We use weighted vertex cover as an example, giving a 2-approximation algorithm.

1 Linear Programming

Linear programming is a general and very powerful technique for solving optimization problems where the objective (i.e., the thing being optimized) and the constraints are *linear*. Out in the real world, this is the standard approach for solving the combinatorial optimization problems that arise all the time. This technique is so common that an LP solver is now included in most common spreadsheets, e.g., Excel and OpenOffice. (Note that the term "programming" refers not to a computer program, more to a program in the sense of an "event program," i.e., a plan for something.)

A typical linear program consists of three components:

- A list of (real-valued) variables $x_1, x_2, \ldots, x_n$. The goal of your optimization problem is to find good values for these variables.
- An objective function $f(x_1, x_2, \ldots, x_n)$ that you are trying to maximize or minimize. The goal is to find the best values for the variables so as optimize this function.
- A set of constraints that limits the feasible solution space. Each of these constraints is specified as an inequality.

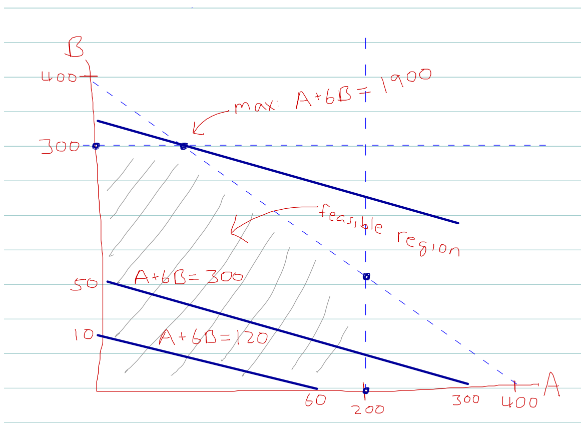
In a linear programming problem, both the objective function and the constraints are *linear* functions of the variables.

Example 1. You are employed by Acme Corporation, a company that makes two products: widgets and bobbles. Widgets sell for SGD 1/widget, and bobbles sell for SGD 6/bobble. Bobbles clearly make more money for you, and so ideally you would like to sell as many bobbles as you can. However, after doing some market research, you have discovered that there is only demand for at most 300 bobbles and at most 200 widgets. It also turns out that your factory can only produce at most 400 units, whether they are widgets or bobbles. How many widgets and bobbles should you make, in order to maximize your total revenue?

We will represent this as a linear programming problem. This problem has two variables: A and B. The variable A represents the number of widgets and the variable B represents the number of bobbles. Your revenue is equal to $A + 6B$, i.e., 1/widget and 6/bobble. Your goal is to maximize this revenue. Your constraints are that $A \leq 200$ and $B \leq 300$: that represents the demand constraints from the market. Your other constraint is a supply constraint: $A + B \leq 400$, since you can only make 400 units total. Finally, we will include two more constraints that are obvious: $A \geq 0$ and $B \geq 0$. These were not included in the problem, but are important to exclude negative solutions. Put together, this yields the following linear program:

On the left, is the LP represented mathematically, specified in terms of an objective function and a set of constraints. On the right is a picture representing the LP geometrically, where the variable A is drawn as the x-axis and the variable B is drawn as the y-axis.

$$\begin{array}{rcl} \max(A+6B) & & \text{where:} \\ A & \le & 200 \\ B & \le & 300 \\ A+B & \le & 400 \\ A & \ge & 0 \\ B & \ge & 0 \end{array}$$



The dashed lines here represent the constraints: $A \le 200$ (i.e., a vertical line), $B \le 300$ (i.e., a horizontal line), and $A + B \le 400$ (i.e., the diagonal line). Each constraint defines a ***halfspace***, i.e., it divides the universe of possible solutions in half. In two-dimensions, each constraint is a line. In higher dimensions, a constraint is defined by a hyperplane.

Everything that is beneath the three lines represents the ***feasible region***, which is defined as the values of A and B that satisfy all the constraints. In general, the feasible region is the intersection of the halfspaces defined by the hyperplanes, and from this we conclude that the feasible region is a convex polygon.

Definition 1 *The **feasible region** for a linear program with variables $x_1, x_2, \ldots, x_n$ is the set of points $(x_1, \ldots, x_n)$ that satisfy all the constraints.*

Notice that the feasible region for a linear program may be: (i) empty, (ii) a single point, or (iii) infinite.

For every point in the feasible region, we can calculate the value of the objective function: $A + 6B$. The goal is to find a point in the feasible region that maximizes this objective. For each value of c, we can draw the line for $A + 6B = c$. Our goal is to find the maximum value of c for which this line intersects the feasible region. You can see, above, we have drawn in this line for three values of c: $c = 120$, $c = 300$, and $c = 1900$. The last line, where $A + 6B = 1900$ intersects the feasible region at exactly one point: $(100, 300)$. This point, then, is the maximum value that can be achieved.

One obvious difficulty in solving LPs is that the feasible space may be infinite, and in fact, there may be an infinite number of optimal solutions. (Remember, we are considering real numbers here, so any line segment has a infinite number of points on it.) Let's think a little bit about whether we can reduce this space of possible solutions.

Imagine that I give you possible solution $(100, 300)$ and ask you to decide if it maximizes the objective function. How might you decide?

1. You draw the geometric picture and look at it. Recall that $f(100, 300) = 1900$ The line represented by the objective function $A + 6B = 1900$ cannot move up any farther. Thus, clearly 1900 is the maximum that can be achieved and hence $(100, 300)$ is a maximum.

2. Maybe we can prove algebraically that $(100, 300)$ is maximal. Recall, one of the constraints shows that $A+B \le 400$. We also know that $B \le 300$, and so $5B \le 1500$. Putting these facts together, we conclude that:

$$\begin{array}{rcl} A+B & \le & 400 \\ 5B & \le & 1500 \\ \hline A+6B & \le & 1900 \end{array}$$

Since the objective function is equal to $A + 6B$, this shows that we cannot possibly find a solution better than 1900. Since we have found such a solution, we know that $(100, 300)$ is optimal.

This may seem like a special case, but in fact it turns out that you can always generate such a set of equations to prove that you have solved your LP optimally! This amazing fact is implied by the theory of ***duality***, which we will come back to later.

3. One important fact about linear programs is that this maximum is always achieved at a vertex of the polygon defined by the constraints (if the feasible region is not empty). Notice that there may be other points (e.g., on an edge or a face) that also maximize the objective, but there is always a vertex that is at least as good. (We will not prove this fact today, but if you think about the geometric picture, it should make sense.) Therefore, one way to prove that your solution is optimal is to examine all the vertices of the polygon.

   How many vertices can there be? In two dimensions, a vertex may occur wherever two (independent) constraints intersect. In general, if here are n dimensions (i.e., there are n variables), a vertex may occur wherever n (linearly independent) hyperplanes (i.e., constraints) intersect. Recall that if you have n linearly independent equations and n variables, there is a single solution—that solution defines a vertex. Of course, if the equations are not linearly independent, you may get many solutions—in that case, there is no vertex. (For example, if the constraints define hyperplanes that are parallel and do not intersect, there is no vertex.) Or, alternatively, if the intersection point is outside the feasible region, this too is not a vertex.

   So in a system with m constraints and n variables, there are $\binom{m}{n} = O(m^n)$ vertices. (In fact, a more difficult analysis shows that the number of vertices is bounded by $O(m^{\lfloor n/2 \rfloor}$.)

   We have thus discovered an exponential time $O(m^n)$ time algorithm for solving a linear program: enumerate each of the $O(m^n)$ vertices of the polytope, calculate the value of the objective function for each point, and take the maximum.

2 Solving Linear Programs

In this class, for the most part, we will ignore the problem of solving linear programs. There exist fast and efficient LP solvers, and we will rely on these as a black-box. However, here we give a few hints as to how these LP solvers work. (It is a fascinating and well-studied problem, and if you can build a faster LP solver, you can make a lot of money!)

2.1 Simplex Method

One of the earliest techniques for solving an LP—and still one of the fastest today—is the Simplex method. It was invented by Dantzig in 1947, and remains in common use today. There are many variants, but all take exponential time in the worst-case. However, in practice, for almost every LP that anyone has ever generated, it is remarkably fast.

The basic idea behind the Simplex method is remarkably simple. Recall that if an LP is feasible, its optimum is found at a vertex. (Again, remember that the optimum may be achieved at other points as well, but this does not matter to us.) Hence, the basic algorithm can be described as follows, where the function f represents the objective function.

1. Find any (feasible) vertex v.
2. Examine all the neighboring vertices of v: $v_1, v_2, \ldots, v_k$.
3. Calculate $f(v)$ and $f(v_1), f(v_2), \ldots, f(v_k)$. If $f(v)$ is the maximum (among its neighbors), then stop and return v.
4. Otherwise, choose one of the neighboring vertices v_j where $f(v_j) > f(v)$. Let $v = v_j$.
5. Go to step (2).

There are several things to notice about this algorithm. First, if the algorithm stops in step (3), then it really has found an optimum point. To prove this fact, we need to examine the geometry of the polytope; because the feasible region is convex and the objective function linear, we can conclude that if $f(v)$ is maximum among its neighbors, then it is maximum in the entire feasible region. This fact ensures that the Simplex Method always returns the right answer.

Second, observe that it will eventually terminate. At some point, it will have explored all the vertices of the polytope. Recall that at one of the vertices we will find an optimum, and hence we will eventually terminate. This also bounds the worst-case running time as as $O(m^n)$ for an LP with n variables and m constraints.

Third, notice that Step (4) is somewhat underspecified: which neighboring vertex should we choose? Depending on how we choose the neighboring vertex, we can get very different performance. (In the case where $n = 2$, there are not many choices. But as the dimension grows, there become a much larger number of neighboring vertices to choose among.) The rule for choosing the next vertex is known as the *pivot rule*, and a large part of designing an efficient simplex implementation is choosing the pivot rule. Even so, all known pivot rules take worst-case exponential time.

Fourth, implementing this algorithm efficiently requires some care. How should one find neighboring vertices and determine where to go next (without re-calculating all the vertices in every step)? In fact, using some basic linear algebra and matrix manipulation, this can be implemented quite efficiently (e.g., using techniques like Gaussian elimination).

2.2 An example

As an example, consider running the Simplex Method on Example 1 above (i.e., the Acme corporation optimization problem). In this case, the Simplex method might because with the feasible vertex $(0, 0)$.

First iteration. In the first iteration, it would calculate $f(0, 0) = 0$. It would also look at the two neighboring vertices, calculating that $f(0, 300) = 1800$ and $f(200, 0) = 200$. Having discovered that $(0, 0)$ is not optimal, it would choose one of the two neighbors. Assume, in this case, that the algorithm chooses next to visit neighbor $(200, 0)$.

Second iteration. In the second iteration, it would calculate $f(200, 0) = 200$. It would also look at the two neighboring vertices, calculating that $f(0, 0) = 0$ and $f(200, 200) = 1400$. In this case, there is only one neighboring vertex that is better, and it would move to $(200, 200)$.

Third iteration. In the third iteration, it would calculate $f(200, 200) = 1400$. It would also look at the two neighboring vertices, calculating that $f(200, 0) = 200$ and $f(100, 300) = 1900$. In this case, there is only one neighboring vertex that is better, and it would move to $(100, 300)$.

Fourth iteration. In the fourth iteration, it would calculate $f(100, 300) = 1900$. It would also look at the two neighboring vertices, calculating that $f(200, 200) = 400$ and $f(0, 300) = 1800$. Discover that $(100, 300)$ is better than any of its neighbors, the algorithm would stop and return $(100, 300)$ as the optimal point.

Notice that along the way, the algorithm might calculate some points that were not vertices. For example, in the second iteration, it might find the point $(400, 0)$—which is not feasible. Clearly, a critical part of any good implementation is quickly calculating the feasible neighboring vertices.

2.3 Other techniques for solving a linear program

While the Simplex Method runs in exponential time, we can solve linear programs in polynomial time! This was a huge breakthrough in the late 1970's and 1980's. The first big breakthrough came with the Ellipsoid Method, developed in 1979 by Khachiyan. The Ellipsoid method demonstrated that we could solve LPs in polynomial time, always, in the worst-case. This was a major theoretic accomplishment, but it remained quite slow in practice. A few years later in 1984, Karmarkar (who was then a PhD student as UC Berkeley) invented an Interior Point Method (often called Karmarkar's Algorithm). The interior point methods are fast, efficient, and run in polynomial time in the worst-case. Often, they are as fast as or faster (in practice) than the Simplex Method.

Let us begin with a simple idea. Imagine we had a technique for determining if there existed *any* feasible points for an LP. That is, we have a black box algorithm that returns *"yes, this LP is feasible"* or *"no, there is no feasible solution"*. Assume that we also have an upper bound MAX on the maximum value that the objective function might achieve, and a lower bound MIN on the minimum value that the objective function might achieve. Here is an algorithm for solving the LP:

1. Start with value $z = (MAX + MIN)/2$.
2. Add the constraint $f(x) \geq z$ to the LP.
3. Find a feasible point x' for the new LP.
4. If the new LP is still feasible, then set $MIN = z$ and goto Step 1.
5. If the new LP is not feasible, remove the new constraint, set $MAX = z$ and goto Step 1.

Essentially, we are doing a binary search on the value of the objective function. (If the initial MAX and MIN are unknown, we can use a similar exponential doubling trick to find these values as well.) Eventually, if the feasible point returned in Step 3 is a vertex (or lies on a face), we can check if it is at least as large as all neighboring vertices and terminate.

The Ellipsoid Method reduces the problem of solving a linear program to that of finding a feasible point. It then repeatedly reduce the size of the feasible region until the optimum can be found. The Ellipsoid Method solves the feasibility problem by maintaining an ellipsoid of at least some minimum volume that contains the feasible region. At each step, it chooses a point in the ellipsoid. If that point is in the feasible region, then we are done: the LP is feasible. If not, then it uses the point to generate a separating hyperplane (that separates the point from the feasible region), which then helps to construct a new ellipsoid of smaller volume. By iterating, we get smaller and smaller ellipsoids until either the volume is too small (and the LP is not feasible) or a feasible point is found. (That is a simplification, but perhaps provides some hint.)

The Interior Point Method, by contrast, is a bit harder to describe briefly. It depends on moving across the interior of the feasible region, solving a set of linear equations at every step to find the distance/direction to move. It uses a primal/dual style technique to maintain a feasible solution as it proceeds.

For the remainder of today, we will ignore the problem of *how to solve* linear programs, and instead focus on how to use linear programming to solve combinatorial graph problems. For now, here's what you need to know about solving linear programs:

- If you can represent your linear program in terms of a polynomial number of variables and a polynomial number of constraints, then there exist polynomial time algorithms for solving them. (There are also other more general solutions, involving *separation oracles* that we will ignore for now.)
- The existing LP-solvers are very efficient for almost all the LPs that you would want to solve.
- You can find an LP solver in Excel or Open Office to experiment with.

3 Weighted Vertex Cover and Integer Linear Programs

We now introduce the problem of weighted vertex cover, and show how to represent it as an integer linear program. We can then *relax* the integer linear program, leading to a linear program that we can solve. Finally, we use the LP to construct a 2-approximation algorithm.

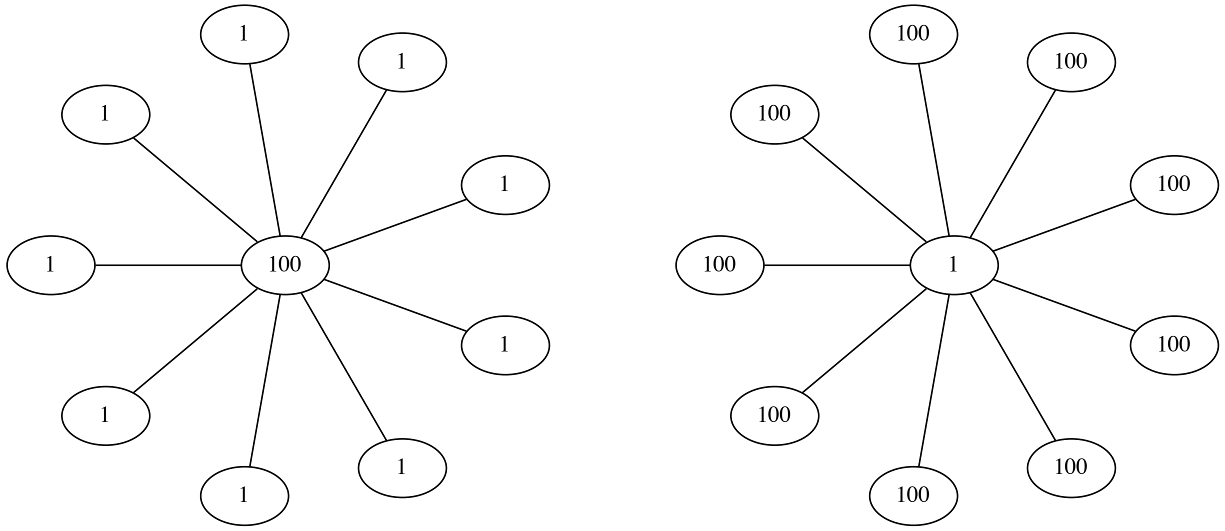


Figure 1: These are two examples of the weighted vertex cover problem, where the value in each node represents the weight. Notice in the first case, the optimal vertex cover includes all the leaves; in the second case, the optimal vertex cover includes only the center node. In both cases, the simple 2-approximate vertex cover algorithm fails badly, always including a node of cost 100.

3.1 Weighted Vertex Cover

To this point, we have looked at the *unweighted* version of vertex cover: every node has the same cost. It is natural to consider a weighted version of the problem where different vertices have different costs.

Definition 2 *A **weighted vertex cover** for a graph $G = (V, E)$ where each vertex $v \in V$ has **weight** $w(v)$, is a set $S \subseteq V$ such that for every edge $e = (u, v) \in E$, either $u \in S$ or $v \in S$. The cost of vertex cover S is the sum of the weights, i.e., $\sum_{v \in S} w(v)$.*

For vertex cover, however, the 2-approximation algorithm presented earlier is no longer sufficient. See the examples in Figure 1. In both examples, the simple 2-approximation algorithm examines exactly one edge and adds both endpoints, incurring a cost of 101. And yet in both cases, there is a much better solution, of cost 9 in the first case and of cost 1 in the second case.

There are several methods for solving the Weighted Vertex Cover problem, and our goal over the rest of this lecture is to develop one such solution. In order to do so, we will use linear programming.

3.2 Vertex Cover as an Integer Linear Program

Assume we are given a graph $G = (V, E)$ with weight function $w : V \to \mathbb{R}$, and we want to find a minimum cost vertex cover.

The first step is to define a set of variables. In this case, it is natural to define one variable for each node in the graph. Assuming there are n nodes in the graph, we define variables $x_1, x_2, \ldots, x_n$. These variables should be interpreted as follows: $x_j = 1$ implies that node v_j is included in the vertex cover; $x_j = 0$ implies that node v_j is *not* included in the vertex cover.

Given these variables, we can now define the objective function. We want to minimize the sum of the weights of the nodes included in the vertex cover. That is, we want to minimize: $\sum_{j=1}^{n} w(v_j) \cdot x_j$.

Finally, we define the constraints. First, we need to ensure that every edge is covered: for every edge $e = (v_i, v_j)$, we need to ensure that either v_i or v_j is in the vertex cover. We need to represent this constraint as a linear function. Here is one way to do that: $x_i + x_j \geq 1$. This ensures that either x_i or x_j is included in the vertex cover.

We need one more constraint: for each variable x_j, we need to ensure that either $x_j = 1$ or $x_j = 0$. What would it mean if the linear programming solver returned that $x_j = 0.4$? This would not useful. Unfortunately, there is no good way to represent this constraint as a linear function.

And it should not be surprising to you that we cannot represent the problem of vertex cover as a linear program! We know, already, that vertex cover is NP-hard. We also know that linear programs can be solved in polynomial time. Thus, if we found a linear programming formulation for vertex cover, that would imply that $P = NP$.

Instead, we formulate an *integer* linear program:

Definition 3 *An **integer linear program** is a linear program in which all the variables are constrained to be integer values.*

Thus, we can now represent weighted vertex cover as an integer linear program as follows:

$$\min\left(\sum_{j=1}^{n} w(v_j) \cdot x_j\right) \qquad \text{where:}$$

$$\begin{aligned} x_i + x_j &\geq 1 \quad \text{for all } (i,j) \in E \\ x_j &\geq 0 \quad \text{for all } j \in V \\ x_j &\leq 1 \quad \text{for all } j \in V \\ x_j &\in \mathbb{Z} \quad \text{for all } j \in V \end{aligned}$$

Notice that the objective function is a linear function of the variables x_j, where the weights are simply constants. (There is no term in the expression that looks like $x_i x_j$, i.e., multiplying variables together.) Similarly, each of the constraints is a linear function of the variables. The only constraint that cannot be expressed as a linear function is the last one, where we assert that each of the variables must be integral. (We will often abbreviate the last three lines by simply stating that $x_j \in \{0, 1\}$.)

3.3 Relaxation

Unfortunately, there is no polynomial time algorithm for solving integer linear programs (abbreviated ILP). We have already effectively shown that solving ILPs is NP-hard. If there were a polynomial time algorithm, we would have proved that $P = NP$. Instead, we will *relax* the integer linear program to a (regular) linear program. That is, we will consider the same optimization problem, dropping the constraint that the variables be integers. Here, for example, is the vertex cover relaxation:

$$\min\left(\sum_{j=1}^{n} w(v_j) \cdot x_j\right) \qquad \text{where:}$$

$$\begin{aligned} x_i + x_j &\geq 1 \quad \text{for all } (i,j) \in E \\ x_j &\geq 0 \quad \text{for all } j \in V \\ x_j &\leq 1 \quad \text{for all } j \in V \\ x_j &\in \mathbb{Z} \quad \text{for all } j \in V \end{aligned}$$

Notice that the solution to this LP is no longer guaranteed to be a solution to vertex cover! In fact, there is no obvious way to interpret the solution to this LP. What does it mean if we decide that $x_j = 0.4$?

Solving the relaxed ILP does tell us something: the solution to the linear program is *at least as good* as the optimal solution for the original ILP. In the case of weighted vertex cover, imagine we solve the relaxed ILP and the LP solver returns a set of variables $x_1, x_2, \ldots, x_n$ such that $\left(\sum_{j=1}^{n} w(v_j) \cdot x_j\right) = c$, for some value c. Then we know that $OPT(G) \geq c$, where $OPT(G)$ is the optimal (integral) solution for vertex cover.

Why? Imagine there were a better solution $x'_1, x'_2, \ldots, x'_n$ return by $OPT(G)$. In that case, this solution would also be a feasible solution for the relaxed linear program: each of these variables x'_j is either 0 or 1, and hence would be a valid choice for the relaxed case where $0 \leq x_j \leq 1$. Hence the LP solver would have found this better solution.

The general rule is that when you expand the space being optimized over, your solution can only improve. By relaxing an ILP, we are expanding the range of possible solutions, and hence we can only find a better solution.

Lemma 4 *Let I be an integer linear program, and let $L = relax(I)$ be the relaxed integer linear program. Then $OPT(I) \geq OPT(L)$.*

3.4 Solving Weighted Vertex Cover

Returning to vertex cover, we have defined an integer linear program I for solving vertex cover. We have relaxed this ILP and generated a linear program L. The first thing we must argue is that there is a feasible solution the linear program:

Lemma 5 *The relaxed ILP for the vertex cover problem has a feasible solution.*

Proof Consider the solution where each $x_j = 1$. This solution satisfies all the constraints. □

Assume we have now solved the LP L using an LP solver and discovered a solution $x_1, x_2, \ldots, x_n$ to the linear program L. Our goal is to use this (non-integral) solution to find an (integral) solution to the weighted vertex cover problem.

Here is a simple observation: if (u, v) is an edge in the graph, then either $x_u \geq 1/2$ or $x_v \geq 1/2$. Why? Well, the linear program guarantees that $x_u + x_v \geq 1$. The linear program may well choose non-integral values for x_u and x_v, but it will always ensure that all the (linear) constraints are met.

Consider, then, the following procedure for ***rounding*** our solution to the linear program:

- For every node $u \in V$: if $x_u \geq 1/2$, then add u to the vertex cover C.

We claim that the resulting set C is a vertex cover:

Lemma 6 *The set C constructed by rounding the variables $x_1, \ldots, x_n$ is a vertex cover.*

Proof Assume, for the sake of contradiction, that there is some edge (u, v) that is not covered by the set C. Since neither u nor v was added to the vertex cover, this implies that $x_u < 1/2$ and $x_v < 1/2$. In that case, $x_u + x_v < 1$, which violates the constraint for the LP, which is a contradiction. □

It remains to show that the rounded solution is a good approximation, i.e., that we have not increased the cost too much.

Lemma 7 *Let C be the set constructed by rounding the variables $x_1, \ldots, x_n$. Then $cost(C) \leq 2cost(OPT)$.*

Proof The proof relies on two inequalities. First, we relate the cost of OPT to the cost of the linear program solution:

$$cost(OPT) \geq \sum_{j=1}^{n} w(v_j) \cdot x_j \tag{1}$$

This follows because the x_j were calculated as the optimal solution to a relaxation of the original vertex cover problem. Recall, by relaxing the ILP to a linear program, we can only improve the solution (i.e., in this case, we can only get a solution that is $\leq$ the integral solution).

Second, we related the cost of the LP solution to the cost of the rounded solution. To represent the rounded solution, let $y_j = 1$ if $x_j \geq 1/2$, and let $y_j = 0$ otherwise. Now, the cost of the final solution, i.e., $cost(C)$, is equal to $\sum_{j=1}^{n} w(v_j) \cdot y_j$. Notice, however, that $y_j \leq 2x_j$, for all j. Therefore:

$$\begin{aligned} \sum_{j=1}^{n} w(v_j) \cdot y_j &\leq \sum_{j=1}^{n} w(v_j) \cdot (2x_j) \\ &\leq 2\left(\sum_{j=1}^{n} w(v_j) \cdot x_j\right) \\ &\leq 2OPT(G) \end{aligned}$$

Thus, the rounded solution has cost at most $2OPT(G)$. □

3.5 General Approach

We have thus discovered a polynomial time 2-approximation algorithm for the weighted vertex cover:

- Define the vertex cover problem as an integer linear program.
- Relax the integer linear program to a standard LP.
- Solve the LP using an LP solver.
- Round the solution, adding vertex v to the cover if and only if $x_j \geq 1/2$.

We will see later in the class how we can translate this an algorithm that does not require an LP solver.

The general approach we have defined here for vertex cover can also apply to a wide variety of combinatorial optimization problems. The basic idea is to find an ILP formulation, relax it to an LP, solve the LP, and then round the solution until it is integral. Vertex cover yields a solution that is particularly easy to round. For other problems, however, rounding the solution may be more difficult.

One question, perhaps, is when this approach works and when this approach fails. Sometimes, the non-integral solution (returned by the LP solver) will be much better than the optimal integral solution. In such a case, it will be very difficult to round the solution to find a good approximation. This ratio is defined as the ***integrality gap***. Assume I is some integer linear program:

$$\textbf{integrality gap} = \frac{OPT(relax(I))}{OPT(I)} \tag{2}$$

For a specific integer linear program, if the integrality gap is at least c, then the best approximation algorithm you can achieve by rounding the solution of the relaxed integer linear program is a c-approximation. This is a fundamental limit on this technique for developing approximation algorithms.

4 Linear Programming Terminology

Lastly, I want to review some basic linear programming terminology.

4.1 Definitions

In general, a linear program consists of:

- A set of variables: $x_1, x_2, \ldots, x_n$.
- A linear objective to maximize (or minimize): $c_1x_1 + c_2x_2 + \cdots + c_nx_n$. Thinking of c and x as vectors, this is often written more tersely as: $c^T x$ (where c^T represents the transpose of c, and multiplication here represents the dot product.)
- A set of linear constraints of the form: $a_{j,1}x_1 + a_{j,2}x_2 + \cdots + a_{j,n}x_n \le b_j$. (This version represents the jth constraint.) This is often abbreviated by the matrix equation: $Ax \le b$.

Putting these pieces together, a linear program is often presented in the following form:

$$\begin{array}{rcl} \max c^T x & & \text{where} \\ Ax & \le & b \\ x & \ge & 0 \end{array}$$

4.2 Terminology

Standard terminology for linear programs:

- A point x is ***feasible*** if it satisfies all the constraints.
- An LP is bounded if there is some value V such that $c^T x \le V$ for all points x.
- Given a point x and a constraint $a^T x \le b$, we say that the constraint is ***tight*** if $a^T x = b$; we say that the constraint is ***slack*** if $a^t x < b$.
- A ***halfspace*** is the set of points x that satisfy one constraint. For example, the constraint $a^t x \le b$ defines a halfspace containing all the points x for which this inequality is true. A halfspace is a convex set.
- The ***polytope*** of the LP is the set of points that satisfy all the constraints, i.e., the intersection of all the constraints. The polytope of an LP is convex, since it is the intersection of halfspaces (which are convex).
- A point x is a vertex for an n-dimensional LP if there are n linearly independent constraints for which it is tight.

For every linear program, we know that one of the following three cases holds:

- The LP is infeasible. There is no value of x that satisfies the constraints.
- The LP has an optimal solution.
- The LP is unbounded.

(Mathematically, this follows from the fact that if the LP is feasible and bounded, then it is a closed and bounded subset of $\mathbb{R}^n$ and hence has a maximum point.)

4.3 Standard Form

In general, we will think of linear programs as have a standard form. For a maximization problem, the standard form as:

$$\begin{aligned} \max c^T x & \quad \text{where} \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

For a minimization problem, we will think of the standard form as:

$$\begin{aligned} \min c^T x & \quad \text{where} \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

Notice that the standard forms obey the following rules:

- Inequalities: every constraint is an inequality. For a maximization problem, the inequalities are all $\leq$ (except for the $x \geq 0$ constraints). For a minimization problem, the inequalities are all $\geq$.
- Non-negative: every variable is constrained to be ≥ 0.

(Beware in other classes or other textbooks, there may be different preferred "standard forms.")

It is easy to translate linear programs in other forms into standard form. For example, we can translate a minimization problem into a maximization problem by negating the objective function. We can reverse the direction of an inequality by negating the constraint. We can translate an equality into an inequality by introducing two constraints. Etc.

Imagine, for example, that you are given the following linear program:

$$\begin{aligned} \min x_1 + 2x_2 - x_3 & \quad \text{where} \\ x_1 + x_2 &= 7 \\ x_2 - 2x_3 &\geq 4 \\ x_1 &\leq 2 \end{aligned}$$

First, we can translate this into a maximization problem by negating the objective function:

$$\begin{aligned} \max -(x_1 + 2x_2 - x_3) & \quad \text{where} \\ x_1 + x_2 &= 7 \\ x_2 - 2x_3 &\geq 4 \\ x_1 &\leq 2 \end{aligned}$$

Then, we can replace the equality with two inequalities:

$$\begin{aligned} \max -(x_1 + 2x_2 - x_3) & \quad \text{where} \\ x_1 + x_2 &\leq 7 \\ x_1 + x_2 &\geq 7 \\ x_2 - 2x_3 &\geq 4 \\ x_1 &\leq 2 \end{aligned}$$

Next, we can flip the direction on the inequalities that are in the incorrect direction:

$$\begin{array}{rcl} \max -(x_1 + 2x_2 - x_3) & & \text{where} \\ x_1 + x_2 & \leq & 7 \\ -(x_1 + x_2) & \leq & -7 \\ -(x_2 - 2x_3) & \leq & -4 \\ x_1 & \leq & 2 \end{array}$$

The last step in putting the LP in normal form is to constrain all the variables to be positive. To accomplish this, we create new variables to replace the variables here. Specifically, we define variables $x_1^+, x_1^-, x_2^+, x_2^-, x_3^+, x_3^-$. We define them as follows:

$$\begin{array}{rcl} x_1 & = & (x_1^+ - x_1^-) \\ x_2 & = & (x_2^+ - x_2^-) \\ x_3 & = & (x_3^+ - x_3^-) \end{array}$$

We can then constrain the new variables to all be positive. We end up with the following LP in normal form:

$$\begin{array}{rcl} \max -((x_1^+ - x_1^-) + 2(x_2^+ - x_2^-) - (x_3^+ - x_3^-)) & & \text{where} \\ (x_1^+ - x_1^-) + (x_2^+ - x_2^-) & \leq & 7 \\ -((x_1^+ - x_1^-) + (x_2^+ - x_2^-)) & \leq & -7 \\ -((x_2^+ - x_2^-) - 2(x_3^+ - x_3^-)) & \leq & -4 \\ (x_1^+ - x_1^-) & \leq & 2 \\ x_1^+ & \geq & 0 \\ x_1^- & \geq & 0 \\ x_2^+ & \geq & 0 \\ x_2^- & \geq & 0 \\ x_3^+ & \geq & 0 \\ x_3^- & \geq & 0 \end{array}$$

Solving this final LP (and translating back to the initial variables) gives us the correct optimal solution for the original LP. (Notice that it may increase the number of variables and constraints by a constant factor.)