CS4234: Optimization Algorithms	Lecture 3

Steiner Trees

Lecturer: Seth Gilbert — *August 25, 2015*

Abstract

Today we consider a new network construction problem where we are given a set of nodes in a graph to connect. A Steiner Tree is a subgraph that connects a set of required terminals, and we want to find the Steiner Tree with minimum cost. We see three variants: the Euclidean Steiner Tree, the Metric Steiner Tree, and the General Steiner Tree Problem, and show how to approximate each of these.

1 Steiner Trees

In this section, we will define the Steiner Tree problem. We begin with a well-known problem, i.e., that of a minimum spanning tree, and then generalize from there.

1.1 Minimum Spanning Trees

Imagine you were given a map containing a set of cities, and were asked to develop a plan for connecting these cities with roads. Building a road costs SGD1,000,000 per kilometer, and you want to minimize the length of the highways. Perhaps the map looks like this:

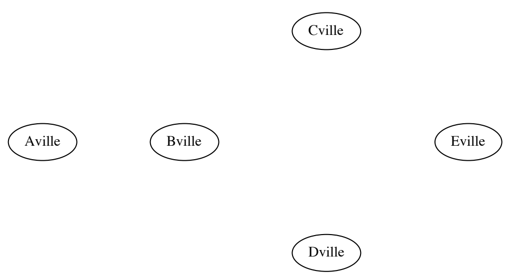


Figure 1: How do you connect the cities with a road network as cheaply as possible?

This is a standard network design question, and the solution is typically to find the minimum cost spanning tree. Recall that the problem of finding a minimum spanning tree is defined as follows:

Definition 1 *Given a graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$, find a subset of the edges $E' \subseteq E$ such that: (i) the subgraph (V, E') is a spanning tree, and (ii) the sum of edge weights $\sum_{e \in E'} w(e)$ is minimized.*

We can then solve the road network problem described above using the following general approach:

- For every pair of location (u, v) calculate the distance $d(u, v)$.
- Build a graph $G = (V, E)$ where V is the set of locations, and for every pair of nodes $u, v \in V$, define edge $e = (u, v)$ with weight $d(u, v)$. This results in a complete graph.

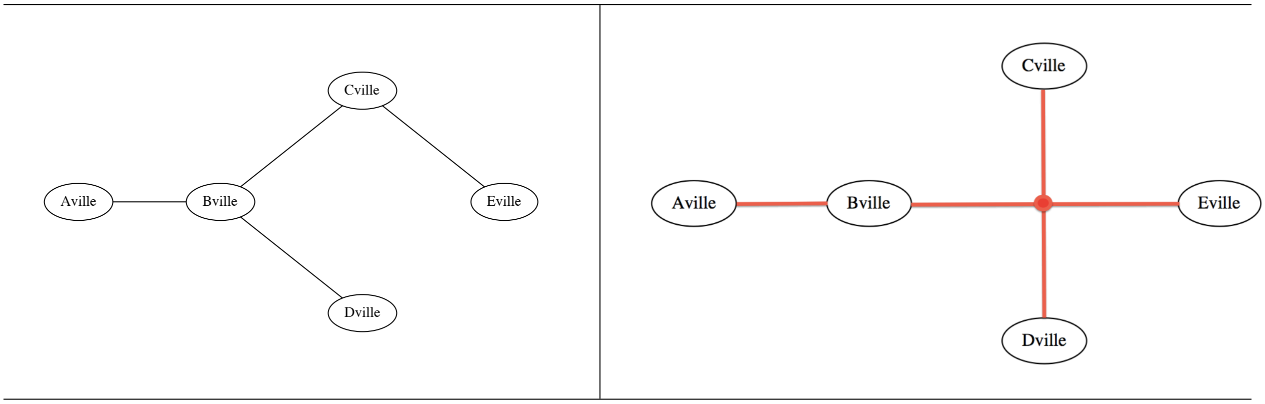


Figure 2: On the left is the graph that results from finding a minimum spanning tree. On the right is an alternate method of connecting the cities with a road network, by creating a new intersection in the middle.

- Find a minimum spanning tree of G using Kruskal's Algorithm (or Prim's Algorithm). (Recall, Kruskal's Algorithm iterates through the edges from lightest to heaviest, adding an edge if it does not create a cycle.)
- Return the spanning tree.

In this case, you will get a road network that looks something like on the left side of Figure 2.

1.2 Euclidean Steiner Trees

Is this, however, really the best solution to the problem? What about the road network on the right side of Figure 2? Notice that this road network is not actually connected in the original graph! The two edges (B, E) and (C, D) do not share an endpoint. Instead, they intersect at some new point in the middle of nowhere. Thus the minimum spanning tree approach will never find this solution.

The goal, then, of the Euclidean Steiner Tree is to find the best way to connect the cities, even when you are allowed to add new nodes to the graph (e.g., the new intersection above). Formally, we define it as follows:

Definition 2 *Assume you are given a set R of n distinct points in the Euclidean (2-dimensional) plane. Find a set of points S and a spanning tree $T = (R \cup S, E)$ such that that the weight of the tree is minimized. The **weight** of the tree is defined as:*

$$\sum_{(u,v)\in E} |u - v| \ ,$$

*where $|u - v|$ refers to the Euclidean distance from u to v. The resulting tree is called a **Euclidean Steiner tree** and the points in S are called **Steiner** points.*

As you can see above, adding new points to the graph results in a spanning tree of lower cost! The goal of the Euclidean Steiner Tree problem is to determine how much we can reduce the cost.

Unlike the problem of finding a minimum spanning tree, finding a minimum Euclidean Steiner tree is NP-hard. We do, however, know some facts about the structure of any optimal Euclidean Steiner tree:

- Each Steiner point in an optimal soultion has degree 3.
- The three lines entering a Steiner point form 120 degree angles, in an optimal solution.
- An optimal solution has at most $n - 2$ Steiner points.

As an exercise, prove these facts to be true. (Hint: given a Steiner tree that does not satisfy these properties, show how to replace the Steiner points with a version that does satisfy these requirements.)

1.3 Metric Steiner Trees

It is natural, at this point, to generalize beyond the Euclidean plane. Assume you are given n points, as before. In addition you are given a distance function $d : V \times V \to \mathbb{R}$ which gives the pairwise distance between any two nodes (u, v). If the points are in the Euclidean plane, we can simply define $d(u, v)$ to be the Euclidean distance between u and v. However, the distance function d can be *any* metric function. Recall the definition of a metric:

Definition 3 *We say that function $d : V \times V \to \mathbb{R}$ is a **metric** if it satisfies the following properties:*

- Non-negativity*: for all $u, v \in V$, $d(u, v) \geq 0$.*
- Identity*: for all $u \in V$, $d(u, u) = 0$.*
- Symmetric*: for all $u, v \in V$, $d(u, v) = d(v, u)$.*
- Triangle inequality*: for all $u, v, w \in V$, $d(u, v) + d(v, w) \geq d(u, w)$.*

(Technically, this is often referred to as a pseudometric, since we allow distances $d(u, v)$ for $u \neq v$ to equal 0.) The key aspect of the distance function d is that it must satisfy the triangle inequality.

If we want to think of the input as a graph, we can define $G = (V, E)$ where V is the set of points, and E is the set of all $\binom{n}{2}$ pairs of edges, where the weight of edge (u, v) is equal to $d(u, v)$.

As in the Euclidean case, we can readily find a minimum spanning tree of G. However, the Steiner Tree problem is to find if there is any better network, if we are allowed to add additional points. Unlike in the Euclidean case, however, it is not immediately clear which points can be added. Therefore, we are also given a set S of possible Steiner points to add. The goal is to choose some subset of S to minimize the cost of the spanning tree. The Metric Steiner Tree problem is defined more precisely as follows:

Definition 4 *Assume we are given:*

- *a set of **required** nodes R,*
- *a set of **Steiner** nodes S,*
- *a distance function $d : (R \cup S) \times (R \cup S) \to \mathbb{R}$ that is a distance metric on the points in R and S.*

*The **Metric Steiner Tree** problem is to find a subset $S' \subset S$ of the Steiner points and a spanning tree $T = (R \cup S', E)$ of minimum weight. The **weight** of the tree $T = (R \cup S', E)$ is defined to be:*

$$\sum_{(u,v) \in E} d(u, v) \ .$$

1.4 General Steiner Tree

At this point, we can generalize even further to the case where d is not a distance metric. Instead, assume that we are simply given an arbitrary graph with edge weights, where some of the nodes are required and some of the nodes are Steiner points.

Definition 5 *Assume we are given:*

- *a graph* $G = (V, E)$*,*
- *edge weights* $w : E \to \mathbb{R}$*,*
- *a set of* ***required*** *nodes* $R \subseteq V$*,*
- *a set of* ***Steiner*** *nodes* $S \subseteq V$ *.*

Assume that $V = R \cup S$*. The* ***General Steiner Tree*** *problem is to find a subset* $S' \subset S$ *of the Steiner points and a spanning tree* $T = (R \cup S', E)$ *of minimum weight. The* ***weight*** *of the tree* $T = (R \cup S', E)$ *is defined to be:*

$$\sum_{(u,v) \in E} d(u, v) \ .$$

1.5 Summarizing the different variants

So far, we have defined three variants of the Steiner Tree problem:

- *Euclidean:* The first variant assumes that we are considering points in the Euclidean plane.
- *Metric:* The second variant assumes that we have a distance metric.
- *General:* The third variant allows for an arbitrary graph.

Notice that the General Steiner Tree problem is clearly a generalization of the Metric Steiner Tree problem. On the other hand, if the set of Steiner points is retricted to be finite (or countable), then the Metric Steiner Tree problem is not simply a generalization of the Euclidean Steiner Tree problem—the Euclidean Steiner Tree problem allows any points in the plane to be a Steiner point!

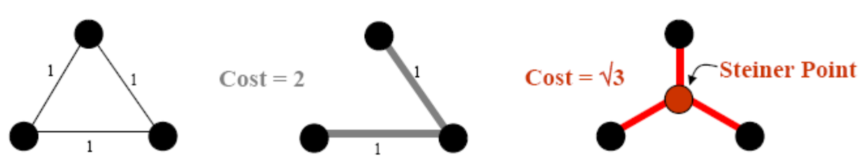
All the variants of the problem are relatively important in practice. In almost any network design problem, we are really interested in Steiner Trees, not simply minimum spanning trees. (One common example is VLSI layout, where we need to route wires between components on the chip.)

All three variants of the problem are NP-hard. As an exercise, prove that the General Steiner Tree problem is NP-hard by reduction from Set Cover. (Hint: think of the underlying elements as the required nodes and the sets as Steiner nodes.)

2 Steiner Tree Approximation: Bad Examples

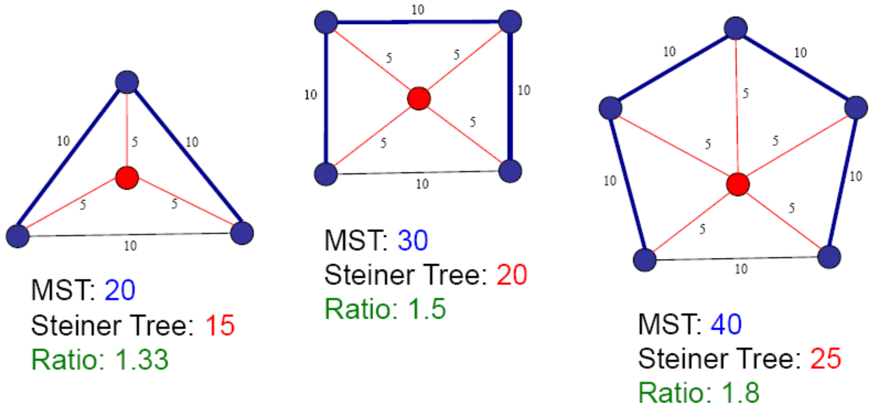
There is a simple and natural heuristic for solving the Steiner Tree problem: ignore all the Steiner nodes and simply find a minimum spanning tree for the required nodes R. Does this find a good approximation? We will see that for the Euclidean Steiner Tree problem and the Metric Steiner Tree problem, an MST is a good approximation of the optimal Steiner Tree. However, for the General Steiner Tree problem, an MST is *not* a good approximation.

First, consider this example of the Euclidean Steiner Tree problem:



On the left, we are given an equilateral triangle with side-length 1. The minimum spanning tree for this triangle includes any two of the edges, and hence has length 2 (as in the middle). However, the minimum Steiner Tree for the triangle adds one Steiner point: the point in the middle of the triangle (as on the right). With this extra Steiner point, now the total cost is $\sqrt(3)$. (You can calculate this based on the fact that the angle in the middle is 120 degrees.) Thus, a minimum spanning tree is, at best, a $2/\sqrt(3)$-approximation of the optimal Euclidean Steiner tree. Is it always at least a $2/\sqrt(3)$-approximation of optimal? That remains an open conjecture. No one knows of any example that is worse than the triangle.

Now consider the Metric Steiner Tree problem. Obviously, a minimum spanning tree still cannot be better than a $2/\sqrt(3)$-approximation, since again we could consider the triangle with a single Steiner point in the middle. Now, however, we can construct a better example.

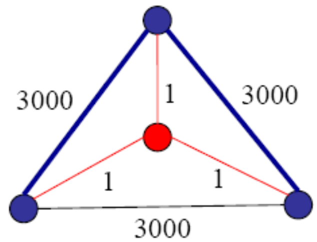


Assume in this example that the distance between every pair of required nodes is 10. (I only drew the edges between adjacent nodes, but the distance metric must define the distance between every pair of nodes.)

Here, we have three cases: a triangle, a square, and a pentagon. Each n-gon has n required nodes, where each pair of nodes is connected by edges of length 10. (In the picture, I have only drawn the outer edges.) Each has a single Steiner node in the middle. There is an edge from each required node to the Steiner node in the middle, and each of these edges has length 5. Notice that this is not Euclidean: for example, the diagonal of a square would have length $10\sqrt(2)$, but here has only length 10. However, all the distance satisfy the triangle inequality. (Verify that this is true!)

For an n-gon where all distances are 10, a minimum spanning tree has exactly $n-1$ edges, and hence has cost $10(n-1)$. On the other hand, the minimum Steiner tree uses the Steiner node in the middle and has n edges, one connecting the Steiner node to each required node. The total cost of the Steiner tree is $5n$. Thus, the minimum spanning tree is no better than a $2(n-1)/n$-approximation of the optimal spanning tree. As n gets large, this approaches a 2-approximation. We will later show that a minimum spanning tree is (at least) a 2-approximation of optimal.

Finally, it should now be clear that the minimum spanning tree is not a good approximation in general. Consider this example:



In this case, just considering the three outer nodes as required, the minimum spanning tree has cost 6000. However, the minimum Steiner tree, including the Steiner node in the middle, has cost 3. Moreover, this ratio can be made as large as desired. Notice that this depends critically on the fact that the triangle inequality does not hold. That is, the distances here are not a metric.

3 Metric Steiner Tree Approximation Algorithm

We are now going to show that, in a metric space, a minimum spanning tree is a good approximation of a Steiner tree. This yields a simple algorithm for finding an approximately optimal Steiner tree, in the metric case: simply ignore the Steiner points and return the minimum spanning tree!

Theorem 6 *For a set of required nodes R, a set of Steiner nodes S, and a metric distance function d, the minimum spanning tree of (R, d) is a 2-approximation of the optimal Steiner Tree for (R, S, d).*

Proof Throughout the proof, we will use as an example the graph in Figure 3. Here we have a graph with six required nodes (i.e., the blue ones) and two Steiner nodes (i.e., the red ones). All the edges drawn have distance 1; all the other edges (undrawn) have distance 2.

Let $T = (V, E)$ be the optimal Steiner tree, for some $V \subseteq R \cup S$. For our example, we have drawn this optimal Steiner tree in Figure 4.

The first step of the proof is to transform the tree T into a cycle C of at most twice the cost. We accomplish this by performing a DFS of the tree T, adding each edge to the cycle as it is traversed (both down and up). Notice that the cycle begins and ends at the root of the tree, and each edge appears in the cycle exactly twice: once traversing down from a parent to a child, and once traversing up from a child to a parent. (Also notice that this cycle visits some nodes multiple times: a node with x children in the tree will appear $2(x + 1)$ times, twice for each of the $x + 1$ adjacent edges.)

In our example, the cycle C is as follows:

$$(a, g) \to (g, d) \to (d, g) \to (g, f) \to (f, g) \to (g, a) \to (a, h) \to (h, c) \to (c, h) \to (h, b) \to (b, h) \to (h, a) \to (a, e) \to (e, a)$$

Notice that this cycle has 14 edges, whereas the original tree has 7 edges and 8 nodes.

We have already defined the $cost(T) = \sum_{e \in E} d(e)$. Similarly, the cost of cycle C is defined as $cost(C) = \sum_{e \in C} d(e)$. Since every edge in the tree T appears exactly twice in the cycle C, we know that $cost(C) = 2cost(T)$. In our example, we see that the cost of the original tree T is 8 and the cost of the cycle C is 16.

The cycle C contains both required nodes in R and Steiner nodes in S. We now want to remove all the Steiner nodes from C, without increasing the cost of the cycle C. Find any two consecutive edges in the cycle (u, v) and (v, w) where the intermediate node v is a Steiner node. (At this point, it does not matter whether u and w are required or Steiner). Replace the two edges (u, v) and $(v, w$ with a single edge (u, w), thus deleting the Steiner node v. We refer

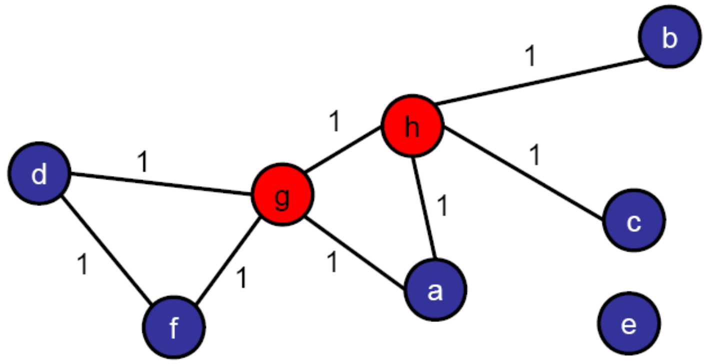


Figure 3: Example for showing that a minimum spanning tree is a 2-approximation of the optimal Steiner tree, if the distances are a metric. Assume that all edges *not* drawn have distnace 2. Verify that these distances satisfy the triangle inquality.

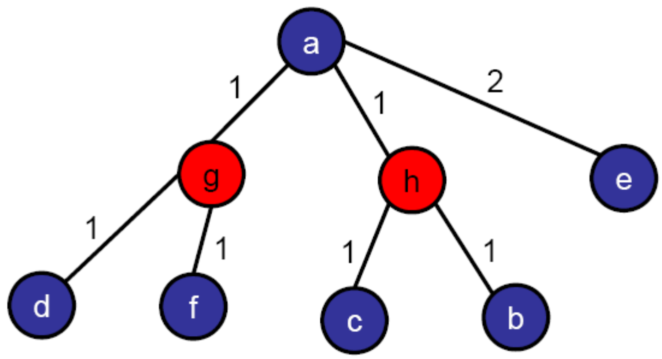


Figure 4: Example for showing that a minimum spanning tree is a 2-approximation of the optimal Steiner tree, if the distances are a metric. Here we have drawn the optimal Steiner tree T.

to this procedure as "short-cutting v." Notice that this replacement does not increase the cost of the cycle C, since $d(u, w) \leq d(u, v) + d(v, w)$—by the triangle inequality. *(Hint: always pay attention to where we use the assumption; here is where the proof depends on the triangle inequality.)* Continue short-cutting Steiner nodes until all the Steiner nodes have been deleted from C.

In our example, the Steiner nodes to be removed are g and h. Thus, we update the cycle C as follows:

$$(a, d) \to (d, f) \to (f, a) \to (a, c) \to (c, b) \to (b, a) \to (a, e) \to (e, a)$$

This revised cycle now has cost: $2 + 1 + 2 + 2 + 2 + 2 + 2 + 2 = 15$, which is no greater than the original cost of the cycle C (which was 16).

We now have a cycle C where $cost(C) \leq 2cost(T)$. (Notice that the cost may have decreased during the short-cutting process, but it could not have increased.) Moreover, this cycle visits every required node in R at least twice (and there are no Steiner nodes in the cycle). The next step is to remove duplicates. Beginning at the root, traverse the cycle labeling each node: the first time a node is visited, mark it *new*; mark every other instance of that node in the cycle as *old*. (Mark the root node visisted at the beginning and end as *new*, also.) As with Steiner nodes, we now short-cut all

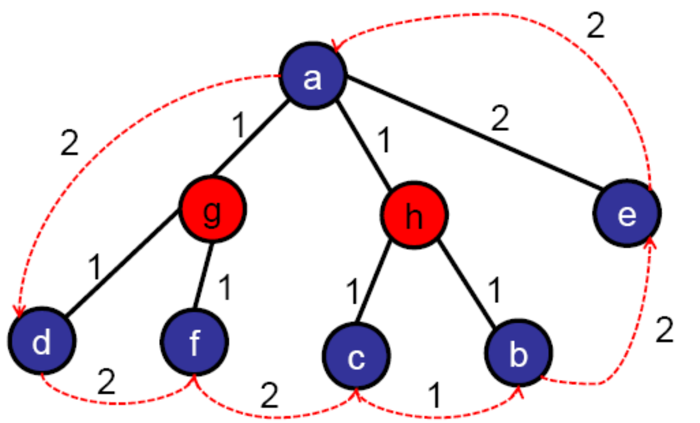


Figure 5: Example for showing that a minimum spanning tree is a 2-approximation of the optimal Steiner tree, if the distances are a metric. Here we have drawn the cycle C after the Steiner nodes have been short-cut and the repeated nodes have been deleted.

the *old* nodes: find two edges (u, v) and (v, w) where v is an old node and replace those two edges in the cycle with a single edge (u, w). As before, this does not increase the cost of the cycle.

In our example, the cycle C visits the nodes in the following order:

$$\mathbf{a} \to \mathbf{d} \to \mathbf{f} \to a \to \mathbf{c} \to \mathbf{b} \to a \to \mathbf{e} \to \mathbf{a}$$

I have bolded the new nodes and colored them blue, including node a at the beginning and the end. I have colored the old nodes red. This leaves two instances of node a to be shortcut. Once we shortcut past the old nodes, we are left with the following cycle C:

$$(a, d) \to (d, f) \to (f, c) \to (c, b) \to (b, e) \to (e, a)$$

This cycle has cost $2 + 1 + 2 + 2 + 2 + 2 = 11$, which is no greater than the original cost of the cycle (which was 16). This revised cycle C is depicted in Figure 5.

We now have a cycle C where $cost(C) \leq 2cost(T)$, and each required node in R appears exactly once. Finally, remove any one arbitrary edge from C. (Again, this cannot increase the cost of C.) At this point, C is a path which traverses each node in the graph exactly once. That is, C is a spanning tree with cost at most $2cost(T)$. In our example, the spanning tree C (where one arbitrary edge of the cycle has been deleted is):

$$(a, d) \to (d, f) \to (f, c) \to (c, b) \to (b, e) \; .$$

This is a spanning tree with cost 9.

Let M be the minimum spanning tree of the required nodes R. Since M is the spanning tree of minimum cost, clearly $cost(M) \leq cost(C) \leq 2cost(T)$. From this we conclude that M is a 2-approximation for the minimum cost Steiner tree of $R \cup S$. $\square$

To summarize, the proof goes through the following steps:

1. Begin with the optimal Steiner Tree T.

2. Use a DFS traversal to generate a cycle of twice the cost.
3. Eliminate the Steiner nodes and repeated required nodes, without increasing the cost. (Use the assumption that d satisfies the triangle inequality.)
4. Remove on edge from the cycle, yielding a spanning tree of cost at most twice the cost of T.
5. Observe that the minimum spanning tree can have cost no greater than the constructed spanning tree, and hence no greater than twice the cost of T.

This implies that the minimum spanning tree has cost at most twice the cost of T, the optimal spanning tree.

One question is whether this bound is tight. We know that the minimum spanning tree is no better than a $2(1 - 1/n)$ approximation, and we have proved here that it is at least a 2-approximation. On the problem set this week, you will find a matching bound.

4 General Steiner Tree Approximation Algorithm

In general, a minimum spanning tree is *not* a good approximation for the general Steiner Tree problem. Here we want to show how to find a good approximation in this case. Instead of developing an algorithm from scratch, we are going to use a reduction. Part of the goal is to demonstrate how to use reductions when we are talking about approximation algorithms.

The typical process, with a reduction, is something like as follows:

- Begin with an instance of the general Steiner Tree problem.
- Via the reduction, construct a new instance of the Metric Steiner tree problem.
- Solve the Metric Steiner tree problem using our existing algorithm.
- Convert the solution to the Metric Steiner tree instance back to a solution for the general Steiner Tree problem.

The key to the analysis would typically be a lemma that says something like, "If we have an algorithm for finding an optimal solution to the Metric Steiner tree problem, then our construction/conversion process yields an optimal solution to the General Steiner tree problem."

With approximation algorithms, however, you have to be a little more careful. Normally, it is sufficient to show that an optimal solution to the translated problem yields an optimal solution to the original problem. However, we are looking here at approximation algorithms. Hence we need to show that, even though we are only finding an approximate solution to the Metric Steiner tree problem, that still yields an approximate solution to the General Steiner tree problem. A reduction that preserves approximation ratios is known as a ***gap-preserving*** reduction.

Assume we are given a graph $G = (V, E)$ and non-negative edge weights $w : E \to \mathbb{R}^{\geq 0}$. The nodes V are divided into required nodes R and Steiner nodes S. Our goal is to find a minimum cost Steiner tree. There are no restrictions on the weights w (i.e., they do not necessarily satisfy the triangle inequality).

Defining a metric. In order to perform the reduction, we need to construct an instance of the Metric Steiner tree problem. In particular, we need to define a metric. The specific distance metric we are going to define is known as the ***metric completion*** of G.

Definition 7 *Given a graph $G = (V, E)$ and non-negative edge weights w, we define the **metric completion** of G to the be the distance function $d : V \times V \to \mathbb{R}$ constructed as follows: for every $u, v \in V$, define $d(u, v)$ to be the distance of the shortest path from u to v in G with respect to the weight function w.*

Notice that the metric completion d provides distances between every pair of nodes, not just the edges in E. Also notice that, computationally, d is relatively easy to calculate, e.g., via an All-Paris-Shortest-Paths algorithm (such as Floyd-Warshall, which runs in $O(n^3)$ time). Critically, d is a metric:

Lemma 8 *Given a graph $G = (V, E)$, the metric completion d is a metric with respect to V.*

Proof Since all the edge weights are non-negative, clearly $d(u, v) \geq 0$ for all $u, v \in V$. Similarly, $d(u, u) = 0$, by definition. Since the original graph is undirected, the shortest path from u to v is also the shortest path from v to u; hence $d(u, v) = d(v, u)$.

The most interesting property is the triangle inequality. We need to show that for all nodes $u, v, w \in V$, the distances $d(u, v) + d(v, w) \geq d(u, w)$. Assume, for the sake of contradiction, that this inequality does not hold, i.e., $d(u, v) + d(v, w) < d(u, w)$. Let $P_{u,v}$ be the shortest path from u to v in G (with respect to the weight function w), and let $P_{v,w}$ be the shortest path from v to w (with respect to the weight function w). Consider the path $P_{u,v} + P_{v,w}$, which is a path from u to w of length $d(u, v) + d(v, w)$. This path is of length less than $d(u, w)$. But that is a contradiction, since $d(u, w)$ was defined to be the length of the shortest path from u to w.

From this, we conclude that d satisfies the triangle inequality, and hence is a metric. $\square$

In the following, we will sometimes be calculating the cost with respect to the metric completion d, and sometimes with respect to the edge weights w. To be clear, we will use $cost_d$ to refer to the former and $cost_w$ to refer to the latter. Similarly, we will refer to graph G^g when talking about the General Steiner tree problem, and G^m when talking about the metric Steiner tree problem.

Converting from General to Metric. We can now reduce the original instance of the General Steiner tree problem to an instance of the Metric Steiner tree problem. Assume we have an algorithm A that finds an α-approximate minimum cost Metric Steiner tree.

- Given a graph $G^g = (V, E^g)$ with requires nodes R, Steiner nodes S, and a non-negative edge weight function w:
- Let d be the metric completion of G^g.
- Consider the Metric Steiner tree problem: (R, S, d).
- Let $T^m = A(R, S, d)$ be the α-approximate minimum cost Metric Steiner tree.

At this point, we have converted our General Steiner tree problem into a Metric Steiner tree problem, and solved it using our existing approximation algorithm.

Converting from Metric back to General. We are not yet done, however, since T^m is defined in terms of edges that may not exist in G^g, and in terms of a different set of costs. We need to convert the tree T^m back into a tree in G^g.

For every edge $e = (u, v)$ in the tree T^m, let p_e be the shortest path in G^g from u to v. Let $P = \bigcup_{e \in T} p_e$, i.e., the set of all paths that make up the tree T^m. Notice that some of these paths may overlap.

Define the cost of a path in G^g (with respect to w) to be the sum of the costs of the edge weights, i.e., $cost_w(p) = \sum_{e \in p} w(e)$. Notice that the cost of a path in G^g is with respect to edge weights, while the cost of the tree T^m is with respect to the metric completion d. (This difference is because we are converting back from the metric to the general problem.) If $e = (u, v)$ is an edge in the tree T^m, then $cost_w(p_e) = d(u, v)$.

Consider all the paths $p_e \in P$, i.e., for every edge e in the tree T^m: add every edge that appears in any path $p_e \in P$ to a set E'. Notice that the graph $G' = (V, E')$ is connected, since the original tree T^m was connected and if there was an edge (u, v) in T^m then there is a path connecting u to v in E'. Also, notice that the cost of all the edges in E' (with respect to w) is no greater than the cost of all the edges in T^m (with respect to d), since each path p_e costs the same amount as the edge e in T^m.

Finally, we need to remove any cycles from the graph (V, E') so that we have a spanning tree. Let T^g be the minimum spanning of (V, E'). The tree T^g in the graph $G^g = (V, E^g)$ with edge weights w has cost no greater than the tree T^m with edge weights d.

Analysis. To analyze this, we need to prove two key lemmas (left as an exercise). Notice that you need to prove two things: you need to relate the solution in the metric version to OPT in the general version, and you also need to relate the final solution in the general version to the solution found in the metric version.

Lemma 9 *Let OPT^g be the optimal minimum cost Steiner tree for G^g. Then $cost_d(T^m) \leq \alpha \cdot cost_w(OPT^g)$.*

Lemma 10 *Let T^g be the Steiner tree calculated by converting T^m back to graph G^g. Then $cost_w(T^g) \leq cost_d(T^m)$*

Putting these lemmas together, we get our final result:

Theorem 11 *Given an α-approximation algorithm for finding a Metric Steiner tree, we can find an α-approximation for a General Steiner tree.*

Proof Assume we have a graph $G^g = (V, E^g)$ with required nodes R, Steiner nodes S, and a non-negative edge weight function w. Let T^m be an α-approximate Steiner tree for (R, S, d), where d is the metric completion of G^g. Let T^g be the spanning tree constructed above by converting the edges in T^m into paths in $G^g = (V, E^g)$ and removing cycles. We will argue that T^g is an α-approximation of the minimum cost spanning tree for G.

First, we have shown that $cost_d(T^m) \leq \alpha \cdot cost_w(OPT^g)$. Second, we have shown that $cost_w(T^g) \leq cost_d(T^m)$. Putting the two pieces together, we conclude that $cost_w(T^g) \leq \alpha \cdot cost_w(OPT^g)$, and hence T^g is an α-approximation for the minimum cost Steiner tree for G with respect to w. $\square$

5 Questions

1. Prove that the General Steiner Tree problem is NP-hard. (Hint: reduce from set cover.)
2. Prove Lemma 9 and Lemma 10.
3. Prove that a minimum spanning tree is also a 2-approximation for the Euclidean Steiner Tree problem. How do the results extend to higher dimensional Euclidean spaces, e.g., 3-dimensional Euclidean Steiner Trees? What about d-dimensional Euclidean Steiner Trees?
4. Give an integer linear program for solving the minimum spanning tree problem. It is okay if your ILP has an exponential number of constraints.
5. Assume that edges are directed. How would you defined a directed version of the Steiner Tree problem? Do the approximation algorithms presented today still work?