CS4234: Optimization Algorithms

MiniProject Ideas

Mini-Project 5: Fair Division

Dividing resources is a classic problem in game theory/economics. Consider the following problem: You have just rented an apartment with three bedrooms. The total rent for the apartment is 2000SGD. You have two roommates who are going to share the apartment with you. Unfortunately, the rooms are all different: some are bigger than others; some have closets; some have more light. How should you assign the three rooms to the three roommates? And how much rent should each pay?

Ideally, your solution should be *envy-free*: no roommate should prefer the room/rent allocated to another. This will help to ensure roommate harmony!

There are many different solutions to this type of problem. One solution relies on an interesting application of Sperner's Lemma: imagine the hyperplane $x+y+z = 2000$, and focus on the triangle where this hyperplane intersects the three axes. Each point on this triangle represents a possible division of the rent among the three rooms (where x represents the rent of room one, y the rent of room two, and z the rent of room three). The algorithm proceeds as follows:

- Construct a fine triangulation of the bigger triangle, dividing it up into smaller triangles.
- Assign each point in the triangulation to one of the three roommates so that every small triangle has one roommate assigned to each vertex.
- Now, ask each roommate which room they prefer, as the specified prices.

The key claim is that for one of the small triangles, all three roommates will choose different rooms. For a fine enough triangulation, this will lead to a good division of the rent.

For a better description of this process read the following resources:

- `https://www.math.hmc.edu/~su/fairdivision/`
- `https://www.math.hmc.edu/~su/papers.dir/rent.pdf`

In a mini-project on fair division, you might explore this problem, first understanding the requirements and goals. (For example, what exactly do we mean by "fair" and "envy-free"?) Then there arise several questions you could could consider:

- Can you analyze the algorithm described above, e.g., proving the key lemma: given the triangle where the hyperplane $x + y + z = C$ intersects the three axes (where C is the total cost being divided), given a triangulation of the bigger triangle and a proper assignment of players and preferences to each point, there exists a triangle that assigns each room to a different player. Ideally, give your proof using graph theoretic tools (i.e., considering the graph structure).

- Imagine implmenenting the algorithm: the key problem is needing to ask the users repeatedly for their preferences. Assume that we are willing to be satisfied with an ϵ-allocation, i.e., one in which the total money sums to at least $(1-\epsilon)C$ and at most $(1+\epsilon)C$. The assumption is that the remaining ϵC dollars will be allocated in some other way, and is small enough that no one cares (e.g., if it is divided evenly among the three roommates).

  For a given ϵ, how many queries does each user need to answer (for a simple implementation of the algorithm)? Explain your answer. For this part, assume that the users are required to specify their preference for every point in the triangle. Give your answer as a function of ϵ and C.

- Can you devise an improved algorithm for minimizing the number of user queries that are needed. Explain your answer (and prove that your algorithm remains correct).

- Implement a fair division calculator that executes the algorithm you have divised above. Your algorithm should take as input the total cost C and an $\epsilon > 0$, and then query the three users for all the necessary points in the triangle. Finally, it should report an allocation of the three rooms.

- A harder challenge is to think about what happens if one of the players lies? How can one of the players cheat to get their room at a cheaper price? Consider the case where the cheating player knows exactly the preferences of the other players (i.e., their valuation of each of the three rooms). What is the best the cheating player can do? What if the cheating player only knows the preferences of one (or zero) of the other players? What if two of the players collude?

Alternatively, you might look at one of the other solutions to the fair division problems, e.g., based on cake cutting.