Algorithms at Scale

(Week 12)

k-Machine Models

Summary

Today: k-Machine

k-Machine Model

- Cluster computing

Some simple examples

- Luby's
- Bellman-Ford

Minimum Spanning Tree

- Basic algorithm
- Fully distributed algorithm
- Lower bound

Last Week: Map-Reduce

Map-Reduce Model

- Cluster computing

Some simple examples

- Word count
- Join

Algorithms

- Bellman-Ford
- PageRank

Announcements / Reminders

Today:

MiniProject presentation due today.

Next week:

Six groups (TBA) to present in class

Nov. 17:

Final report due

A few comments…

On writing a report:

1. Begin with an overview / introduction.

What is this report about?

What will I learn if I read it?

What are the “results” or conclusions?

(Maybe: why is this topic important?)

A few comments…

On writing a report:

2. Explain so that everyone can understand.

Anyone in this class should understand the algorithm.

Goal: more clear than a Wikipedia page!

A few comments...

On writing a report:

3. Give technical details of the algorithm.

From your description, can I implement the algorithm?

Did you include enough detail that I know how every step works?

A few comments…

On writing a report:

4. Give intuition.

From your description, do I understand WHY the algorithm works?

Which steps are important?

Which steps are just optimization?

Why do we do it this way?

A few comments…

On writing a report:

5. Draw pictures. Use examples

Illustrate how the algorithm works.

Draw a picture of the data structure.

Go through a step-by-step example.

A few comments…

On writing a report:

6. Cite properly

Did you invent the algorithm? If not, cite.

Did you invent this proof? If not, cite.

Do not simply copy proofs directly from existing sources. (Do cite sources you used.) Your goal is to give a *better* proof.

Don't plagiarize.

A few comments…

On dimensionality reduction:

1. Think about the trade-offs.

Cost of doing the dimensionality reduction vs. benefit of lower dimensions.

2. For non-linear methods especially, think about cost.

Is the method reusable (with a high one-time cost) or is each use expensive?

3. The final dimension is an important parameter.

Many techniques do better then the theory would predict on real-world data.

A few comments…

On discrete elements with windows:

1. It is interesting to adapt FM and HLL to generic windowed techniques.

For example, using smoothed histogram techniques.

2. If you look more closely, there is a simpler direct technique.

You don’t need histograms.

3. Interesting variants?

Queries on different window lengths? Other types of sketches?

A few comments…

On write-optimized data structure:

1. LSM is used a lot in practice. COLA is not.

Why? Is that a correct evaluation?

2. Are there hybrid LSM/COLA algorithms that might be good?

Imagine using the COLA for x levels and the LSM for levels > x.

3. Can you speed up the COLA with LSM-optimizations?

For example, a LSM often uses a Bloom filter to speed up queries. A COLA?

Summary

Today: k-Machine

k-Machine Model

- Cluster computing

Some simple examples

- Luby’s
- Bellman-Ford

Minimum Spanning Tree

- Basic algorithm
- Fully distributed algorithm
- Lower bound

Last Week: Map-Reduce

Map-Reduce Model

- Cluster computing

Some simple examples

- Word count
- Join

Algorithms

- Bellman-Ford
- PageRank

Fork-Join algorithms

Assumptions:

- Tightly synchronized
- Shared memory

Good model for multicore / multithreaded CPUs.

Advantages:

- Simple algorithm design
- Focus on parallelism (*computational*)
- Easy analysis: work and span is enough!
- Minimizes race conditions, deadlocks, etc.

High Performance Clusters

Fork/Join is not a good model for clusters.

Assumptions:

- Loosely synchronized
- No shared memory
- Data exchanged over fast interconnect

Issues:

- Communication cost?
- Coordination among cores?
- Fine-grained parallelism?

Map-Reduce Model

Basic round:

1. Map: process each (key, value) pair
2. Shuffle: group items by key
3. Reduce: process items with same key together

Key goals:

Target: high-performance clusters.

Focus: data (not computation)

Map-Reduce

Advantages:

- Based on real working systems (e.g., Hadoop)
- Focus on data processing
- Simple programming model: Map and Reduce
- Scales well in practice

Disadvantages:

- Bandwidth issues are invisible
- Expensive sorting operation is hidden
- Hard to coordinate data movement
- Stateless model is tricky

Map-Reduce

Advantages:

- Based on real work
- Focus on data proc
- Simple programmi
- Scales well in pract

Today's Goal:

A more abstract model.

Stateful.

Easier to design algorithms.

Easier to get a realistic sense of algorithm performance.

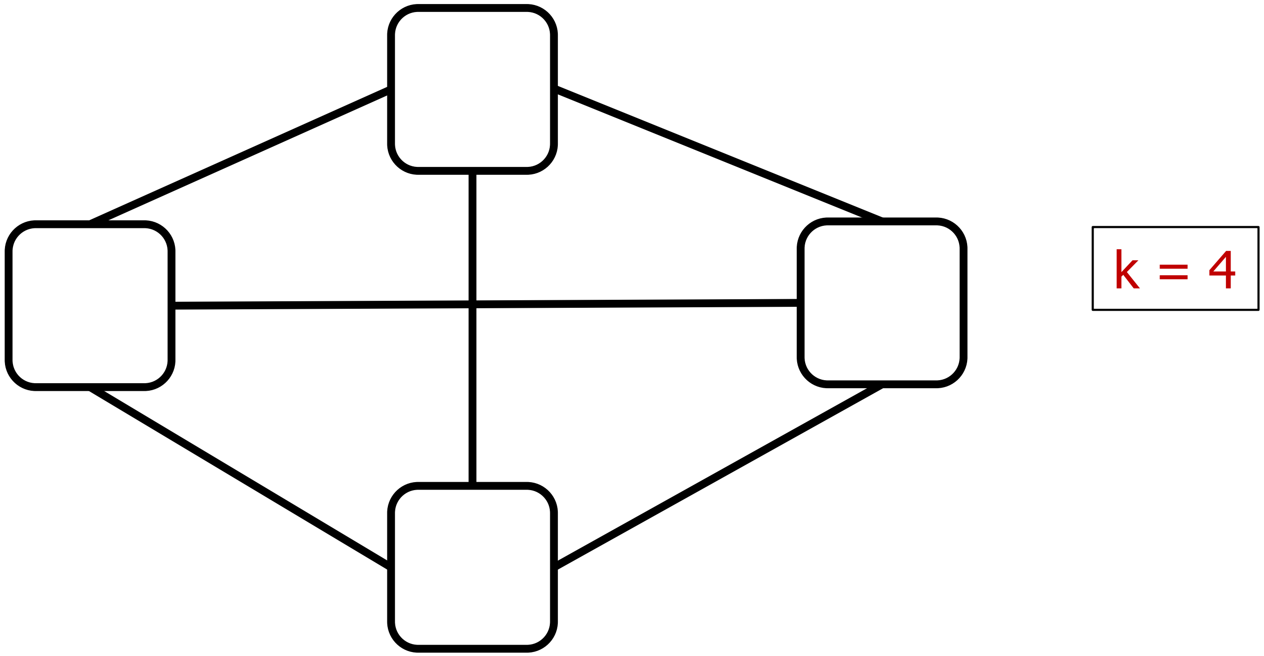
Disadvantages:

- Bandwidth issues are invisible
- Expensive sorting operation is hidden
- Hard to coordinate data movement
- Stateless model is tricky

k-Machine Model

Basic assumptions:

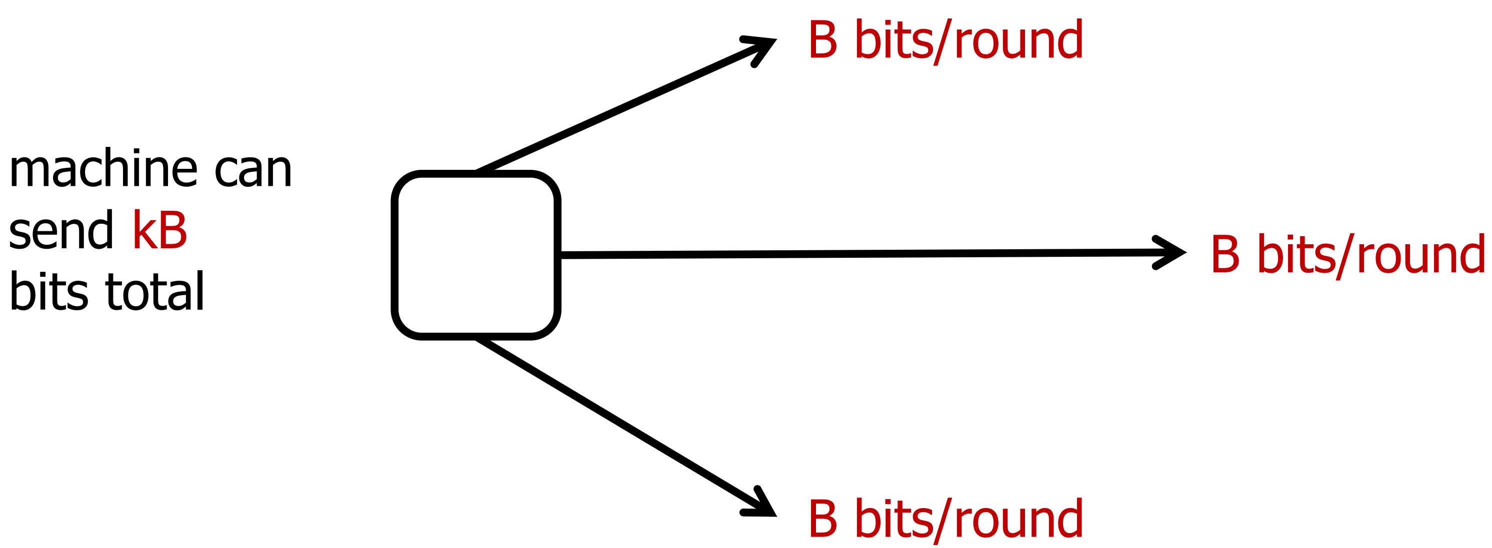
- k servers: system is a collection of cores/CPUs/etc.
- all-to-all communication: communicate via messages
- bandwidth limit B: limited data transfer



k-Machine Model

Basic assumptions:

- k servers: system is a collection of cores/CPUs/etc.
- all-to-all communication: communicate via messages
- bandwidth limit B: limited data transfer

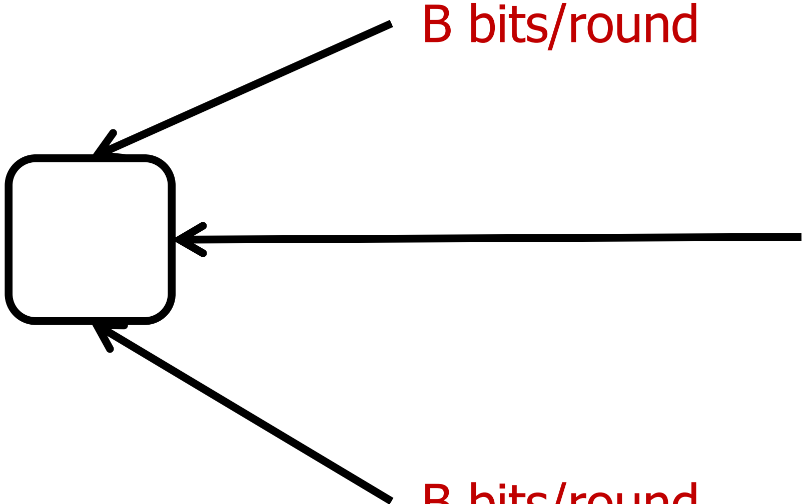


k-Machine Model

Basic assumptions:

- **k servers**: system is a collection of cores/CPUs/etc.
- **all-to-all communication**: communicate via messages
- **bandwidth limit B**: limited data transfer

machine can receive kB bits total



B bits/round

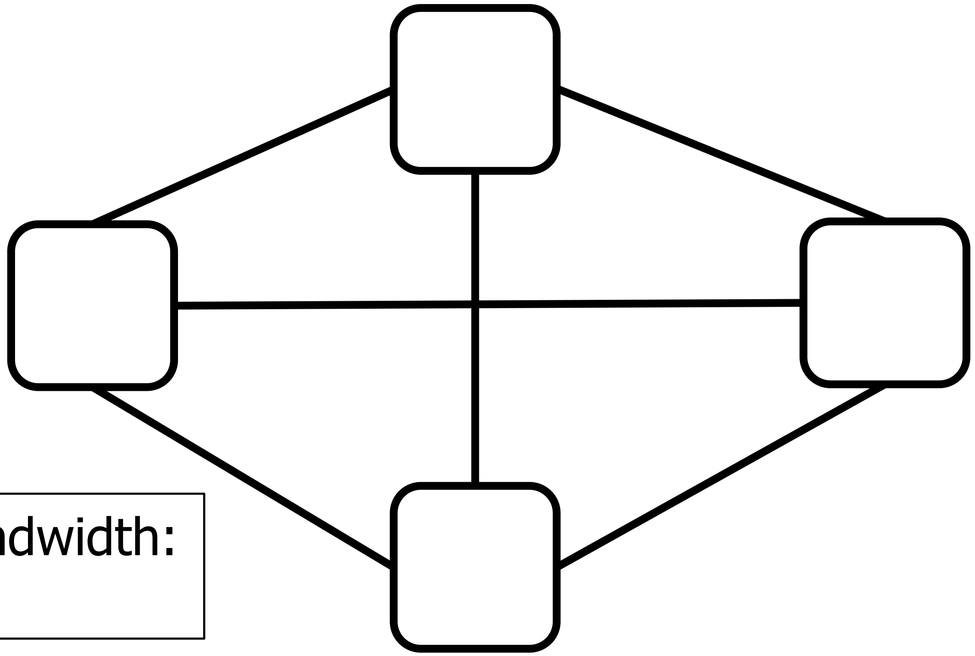
B bits/round

B bits/round

k-Machine Model

Basic assumptions:

- k servers: system is a collection of cores/CPUs/etc.
- all-to-all communication: communicate via messages
- bandwidth limit B: limited data transfer



k = 4

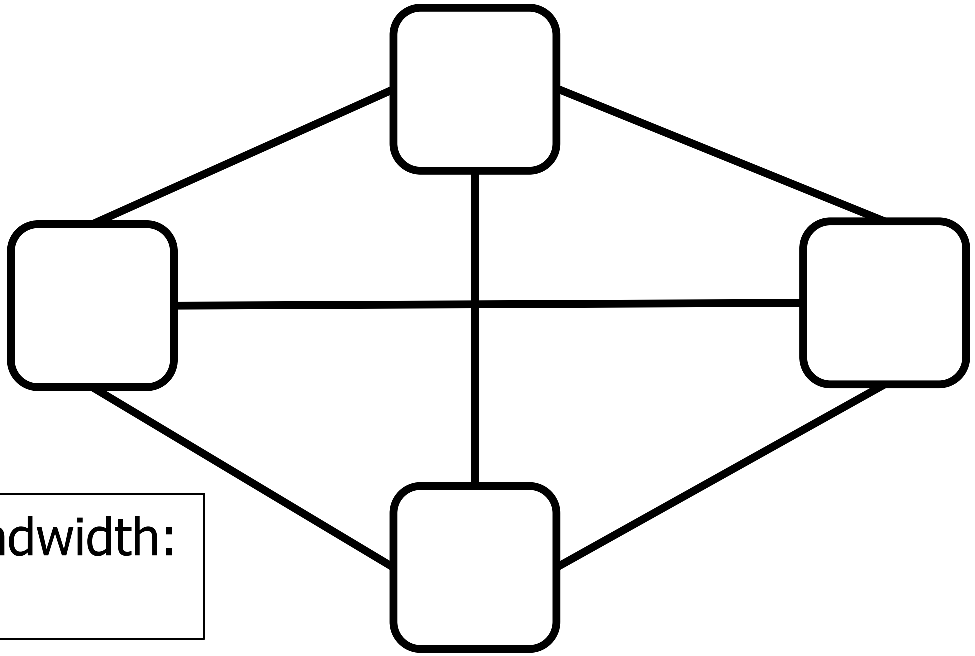
Total "switch" bandwidth:
$Bk(k-1) \cong k^2B$

k-Machine Model

Basic assumptions:

- k servers: system is a collection of cores/CPUs/etc.
- all-to-all communication: communicate via messages
- bandwidth limit B: limited data transfer

k = 4



Total "switch" bandwidth:
$Bk(k-1) \cong k^2B$

Example numbers:

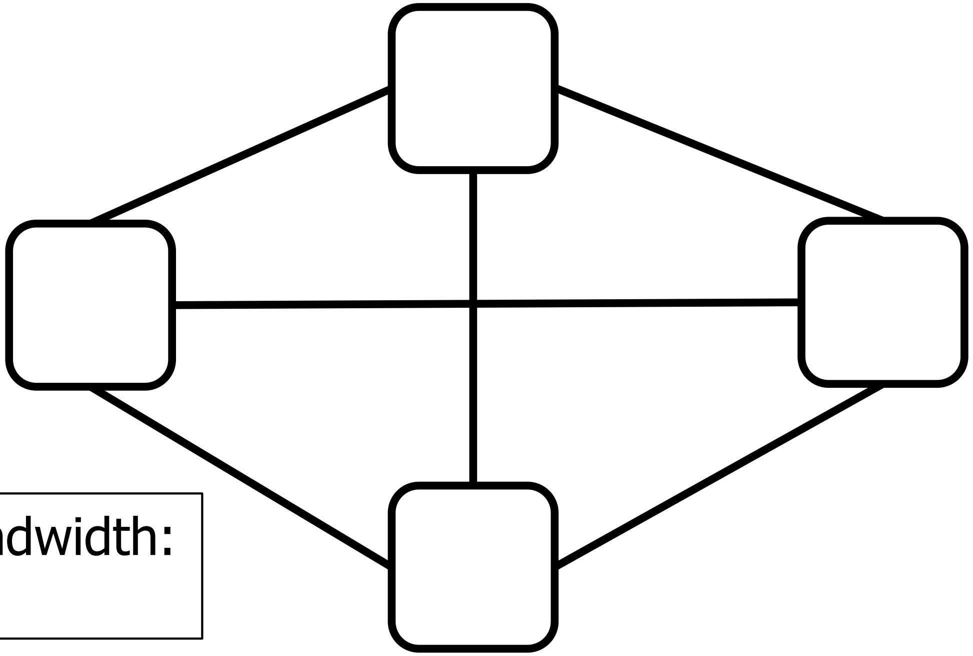
k = 5000,
10 Gbps switch
➔
B = 400 bits/sec

k-Machine Model

Basic assumptions:

- k servers: system is a collection of cores/CPUs/etc.
- all-to-all communication: communicate via messages
- bandwidth limit B: limited data transfer

$k = 4$



Total "switch" bandwidth:
$Bk(k-1) \cong k^2B$

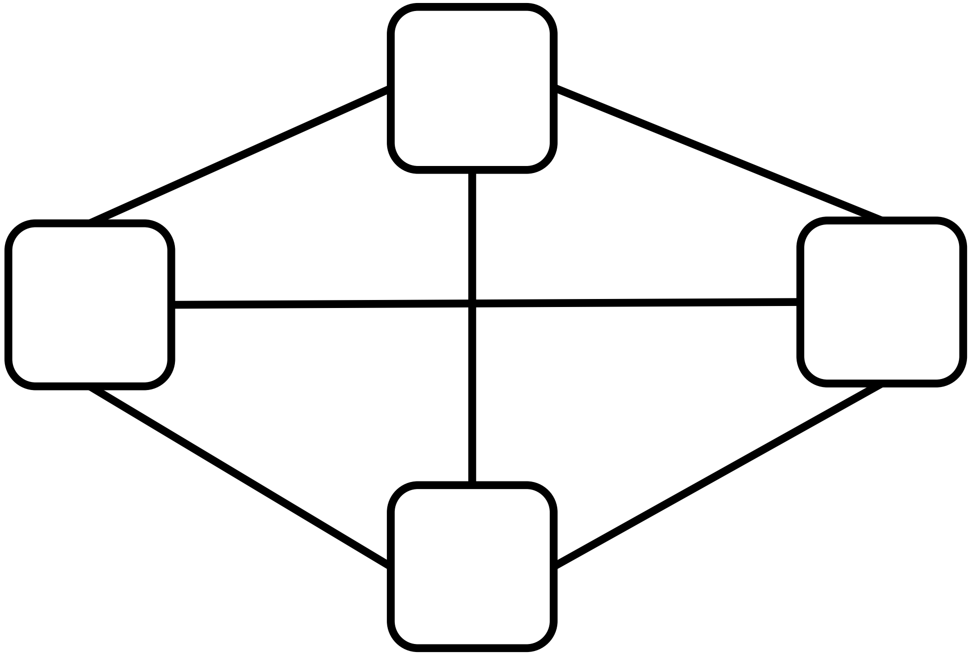
Example numbers:

$k = 5000$,
100 Gbps switch
➔
$B = 4000$ bits/sec

k-Machine Model

Space restriction:

- Problem size: assume size n
- Per server: approximately O(n/k)



k = 4

Example numbers:

k = 5000,
1 TB data
➔
space/core = 200MB

k-Machine Model

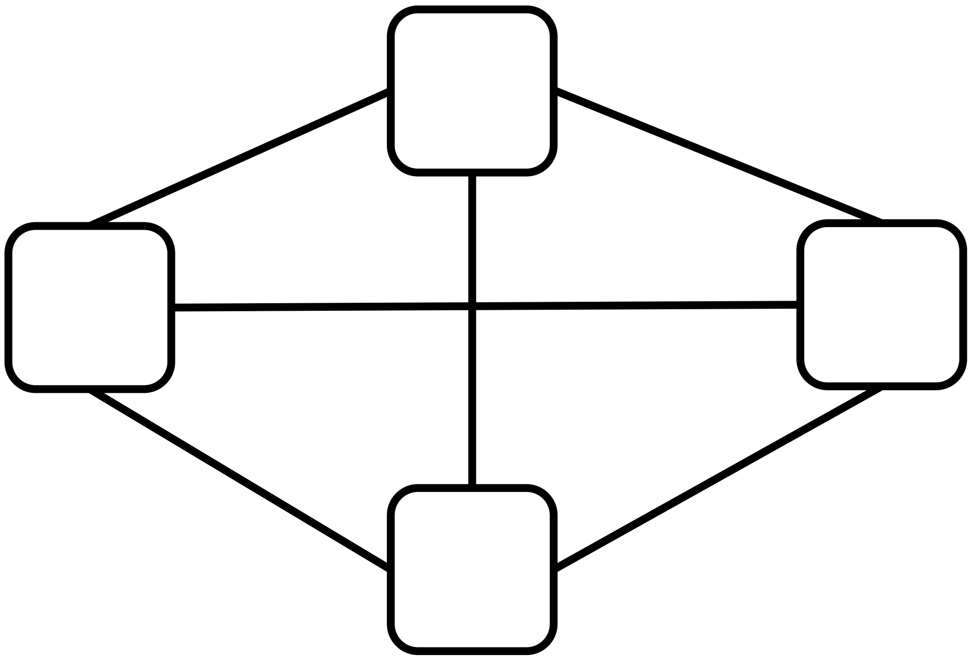
Space restriction:

- Problem size: ass
- Per server: approximately O(n/k)

Difference from Map-Reduce:

All the data always needs to be stored somewhere.

Size O(n/k) is optimal.



k = 4

Example numbers:

k = 5000,
1 TB data
➔
space/core = 200MB

k-Machine Model

Implement Map-Reduce:

- Map:
  1. Each server locally runs map function on every key-value pair, saving the new key-value pairs.
- Reduce:
  1. Use hash function h to map each key to a machine.
  2. Send (k, v) to machine h(k).
  3. Each machine execute reduce function locally.
- Repeat

k-Machine Model

Works correctly if bandwidth/space are sufficient to send/store key-values pairs during the reduce phase.

Implement Map-Reduce:

- Map:
  1. Each server locally runs map function on every key-value pair, saving the new key-value pairs.
- Reduce:
  1. Use hash function h to map each key to a machine.
  2. Send (k, v) to machine h(k).
  3. Each machine execute reduce function locally.
- Repeat

k-Machine Model

Works correctly if bandwidth/space are sufficient to send/store key-values pairs during the reduce phase.

Implement Map-Reduce:

- Map:
  1. Each server locally runs map function on every key-value pair, saving the new key-value pairs.
- Reduce:
  1. Use hash function h to map each key to a machine.
  2. Send (k, v) to machine h(k).
  3. Each machine execute reduce function locally.
- Repeat

ToDo: Implement associated reduce functions.

k-Machine Model

Conclusion:

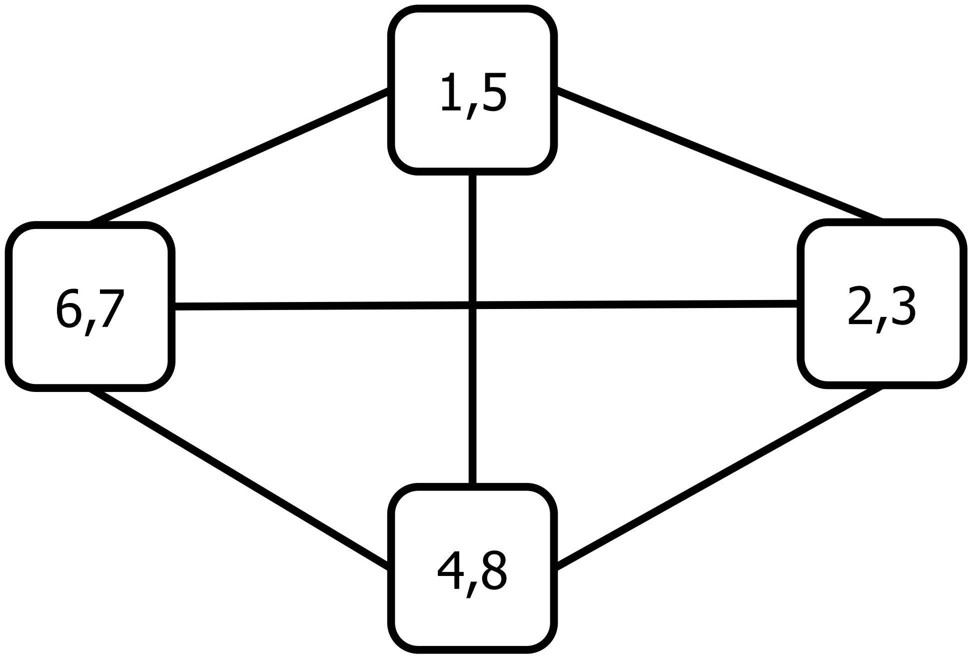
If you can solve the problem in T rounds of Map-Reduce, then you can solve it in the k-Machine model in T rounds.

k-Machine Model

Where is the data?

Random Partition Model:

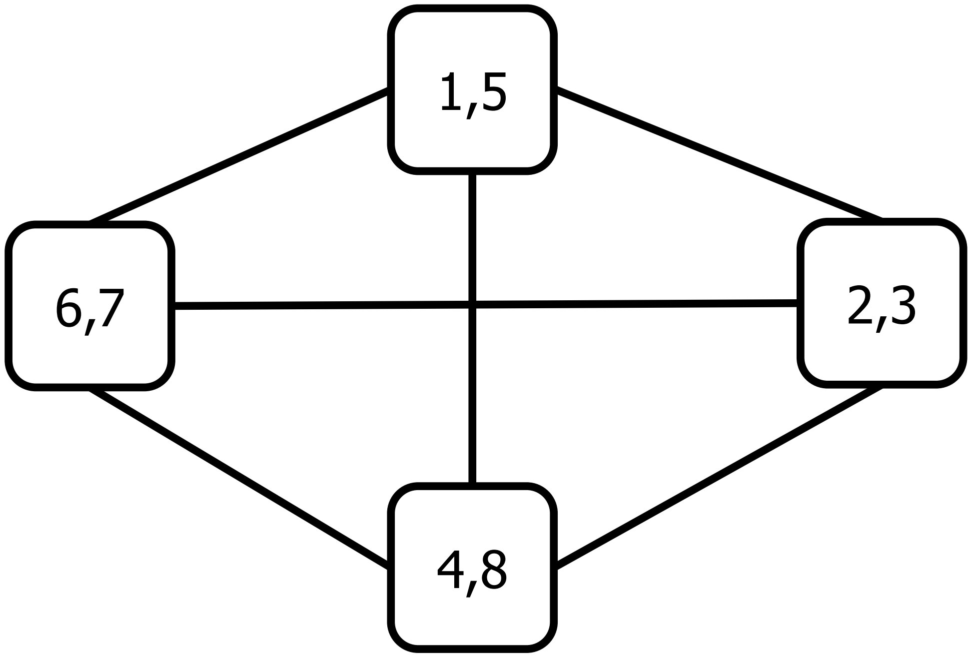
Initially, data is randomly divided among the machines.



k-Machine Model

Example: sorting n integers.

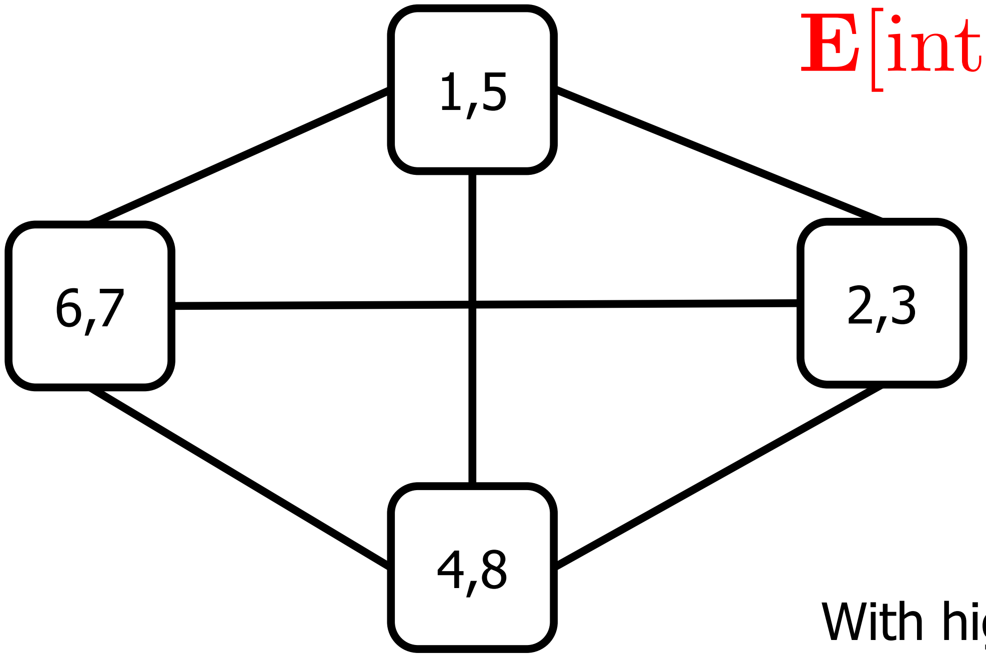
Each integer is assigned to a random machine.



k-Machine Model

Example: sorting n integers.

Each integer is assigned to a random machine.



$$\mathbf{E}[\text{ints per machine}] = \frac{n}{k}$$

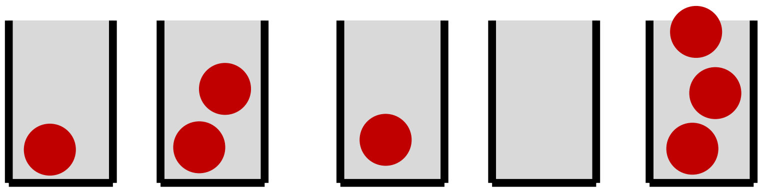
With high probability??

Detour: balls-in-bins

Random process:

– Take n balls and k < n bins.

– Put each ball in a random bin.



Theorem:

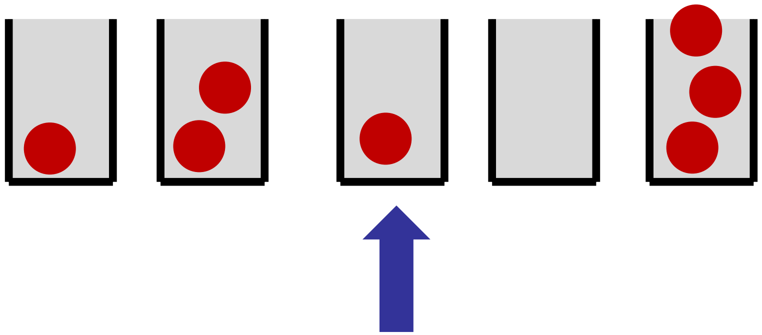
Each bin has $O\left(\frac{n}{k} + \log n\right)$ balls with high probability.

$\geq \left(1 - \frac{1}{n^c}\right)$

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.

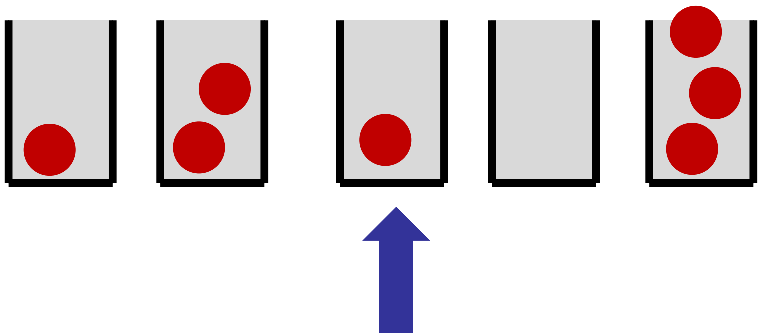


Proof:

Pick one bin.

Detour: balls-in-bins

Random process:



- Take n balls and k bins.
- Put each ball in a random bin.

Proof:

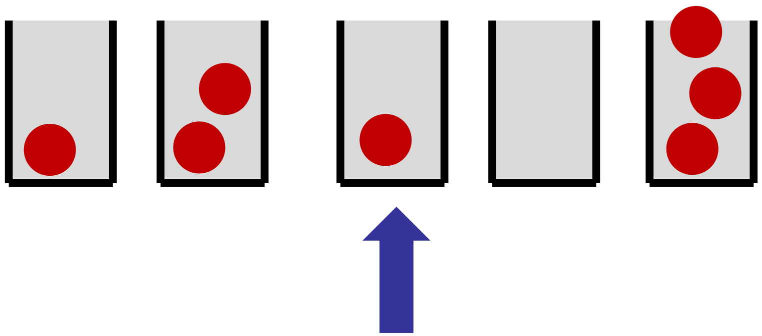
Define $x_i = 1$ if ball i is in the bin.

Define $x_i = 0$ if ball i is NOT in the bin.

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.



Proof:

Define $x_i = 1$ if ball i is in the bin.

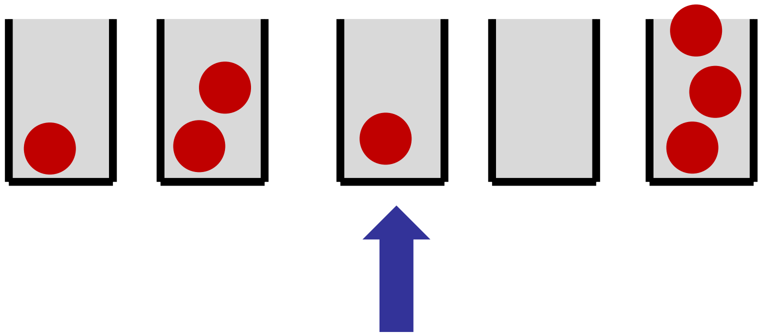
Define $x_i = 0$ if ball i is NOT in the bin.

$$\mathbf{E}[x_i] = \Pr(x_i = 1) = \frac{1}{k}$$

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.



Proof:

Define $x_i = 1$ if ball i is in the bin.

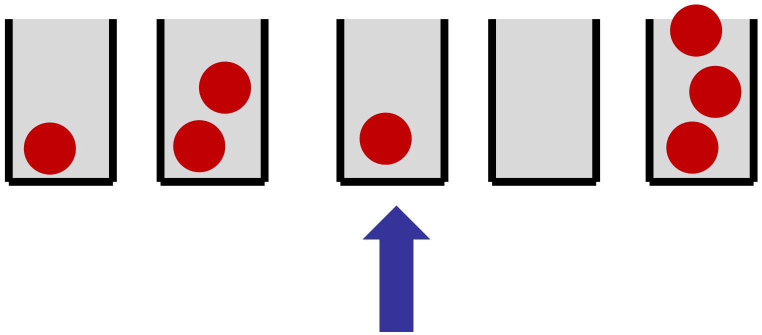
Define $x_i = 0$ if ball i is NOT in the bin.

$$\text{number of balls in bin} = X = \sum_{i=1}^{n} x_i$$

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.



Proof:

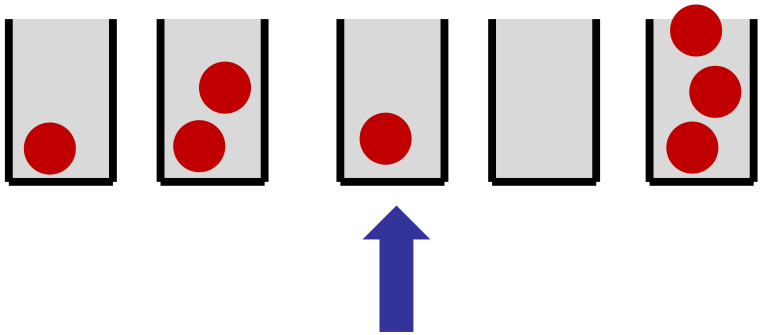
$$\text{number of balls in bin} = X = \sum_{i=1}^{n} x_i$$

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[x_i] = \frac{n}{k}$$

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.



$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[x_i] = \frac{n}{k}$$

Proof:

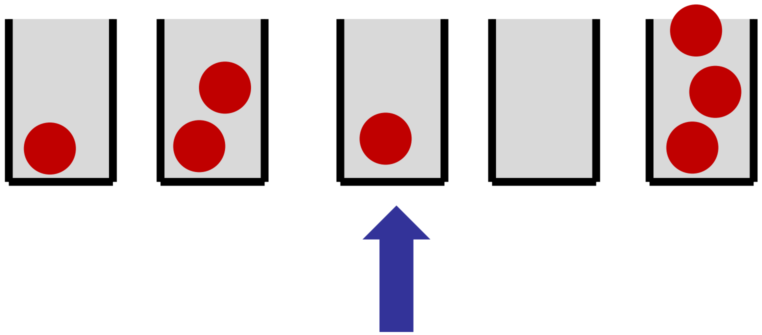
Chernoff Bound: $\delta > 1$

$$\Pr\left(X \geq (1+\delta)\frac{n}{k}\right) \leq e^{-\frac{n}{k}\frac{\delta}{3}}$$

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.



$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[x_i] = \frac{n}{k}$$

Proof:

Case 1: (n/k) > log(n)

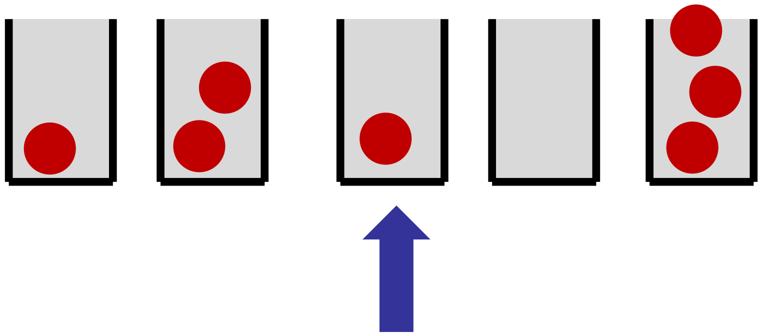
$\delta = 5$

$$\begin{aligned} \Pr\left(X \geq (1+5)\frac{n}{k}\right) &\leq e^{-\frac{n}{k}\frac{\delta}{3}} \\ &\leq e^{-2\log n} \\ &\leq 1/n^2 \end{aligned}$$

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.



Proof:

$$\delta = 6 \log n \frac{k}{n}$$

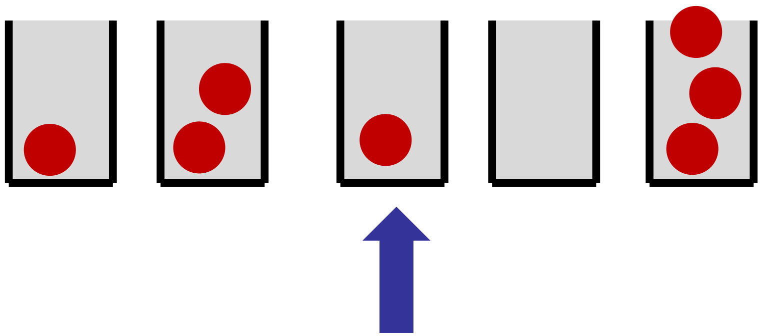
Case 2: (n/k) < log(n)

$$\begin{aligned}
\Pr\left(X \geq \left(1 + 6\log(n)\frac{k}{n}\right)\frac{n}{k}\right) &\leq e^{-\frac{n}{k}\frac{\delta}{3}} \\
&\leq e^{-6\log n \frac{k}{n}\frac{n}{k}\frac{1}{3}} \\
&\leq e^{-2\log n} \\
&\leq 1/n^2
\end{aligned}$$

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.



Proof:

$$\boxed{\delta = 6 \log n \frac{k}{n}}$$

Conclusion: w.p. > $(1 - 1/n^2)$

$$X \leq 6\frac{n}{k}$$

or

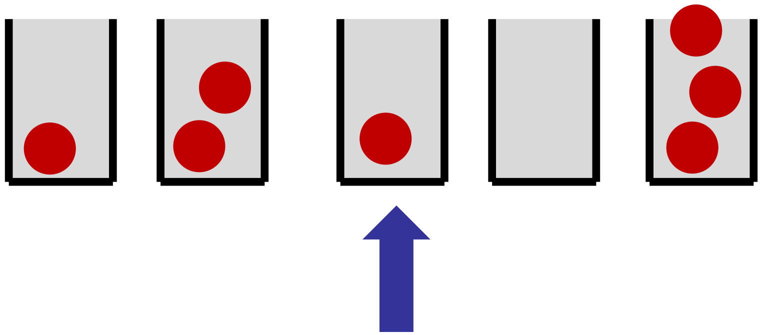
$$X \leq \left(1 + 6 \log n \frac{k}{n}\right) \frac{n}{k} \leq 7 \log n$$

n/k < log(n)

Detour: balls-in-bins

Random process:

- Take n balls and k bins.
- Put each ball in a random bin.



Proof:

$$\boxed{\delta = 6 \log n \frac{k}{n}}$$

Conclusion: w.p. > $(1 - 1/n^2)$

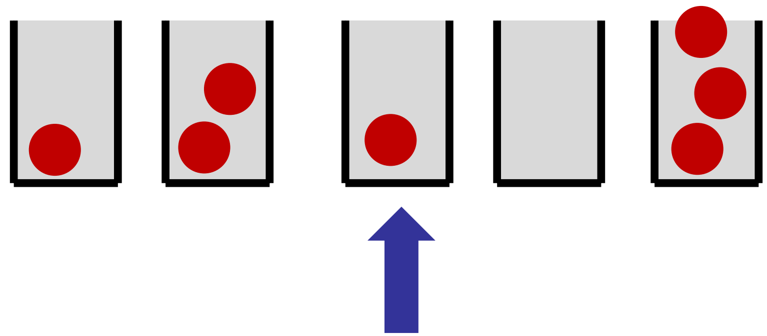
$$X \leq O\left(\frac{n}{k} + \log n\right)$$

Detour: balls-in-bins

Random process:

– Take n balls and k bins.

– Put each ball in a random bin.



Proof:

$$\boxed{\delta = 6 \log n \frac{k}{n}}$$

Conclusion: w.p. > (1 – 1/n^2)

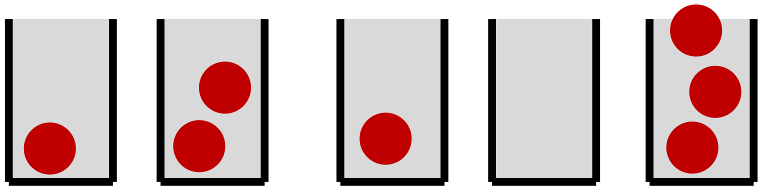
$$X \le O\left(\frac{n}{k} + \log n\right)$$

Union bound over all k<n bins…

Detour: balls-in-bins

Random process:

- Take n balls and k < n bins.
- Put each ball in a random bin.



Theorem:

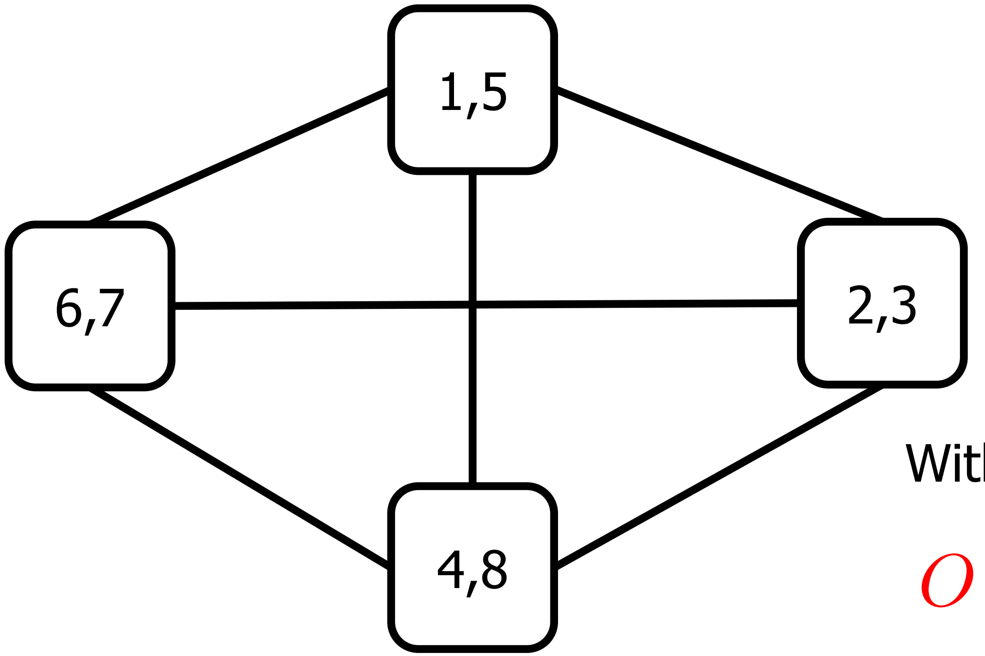
Each bin has $O\left(\frac{n}{k} + \log n\right)$ balls with high probability.

$\geq \left(1 - \frac{1}{n^c}\right)$

k-Machine Model

Example: sorting n integers.

Each integer is assigned to a random machine.



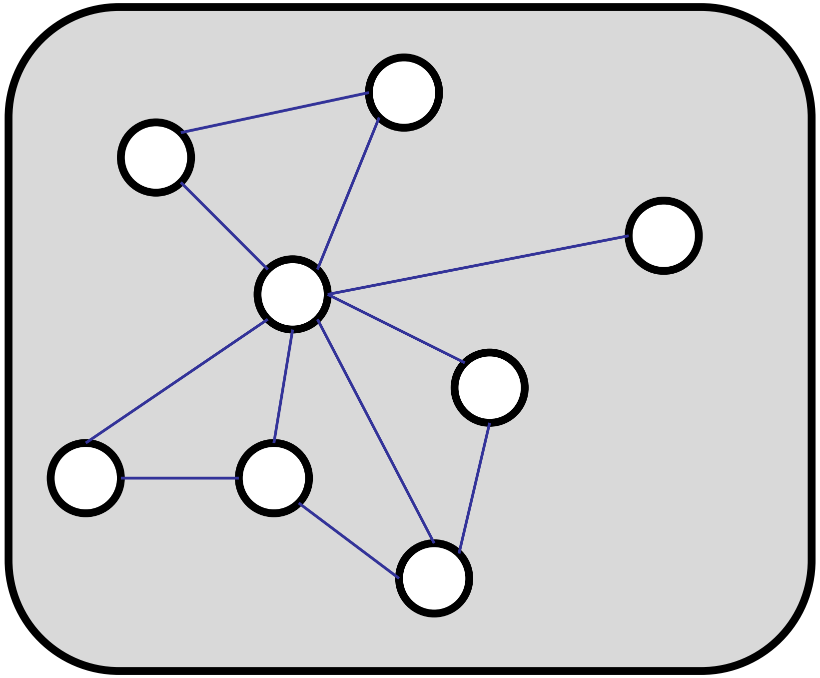
With high probability, per machine:

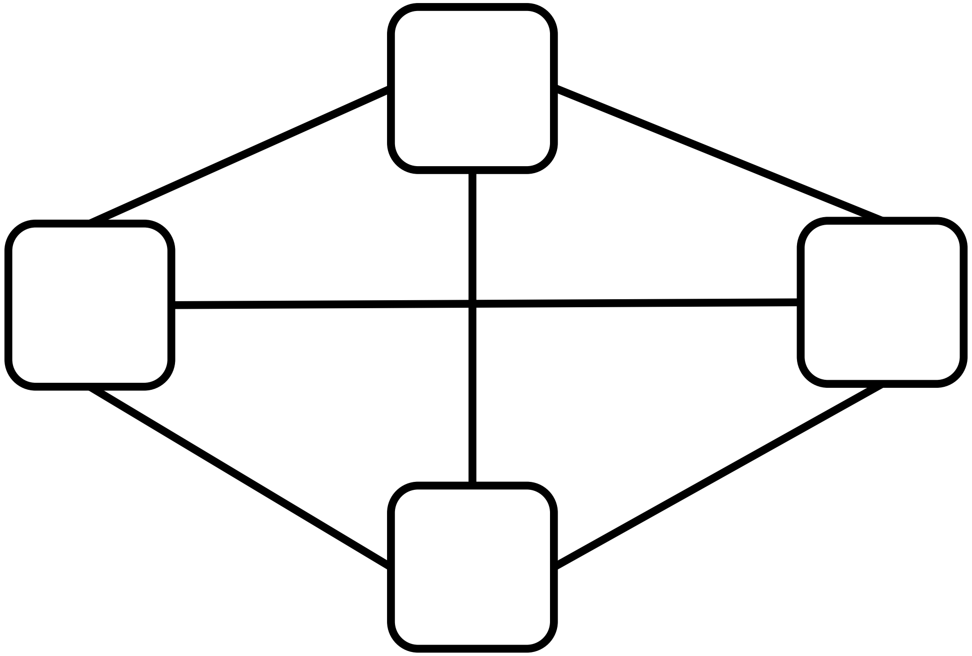
$$O\left(\frac{n}{k} + \log n\right)$$

Graph Algorithms

Assume $k < n^{½}$

Let $G = (V, E)$ be a graph with n nodes and m edges.



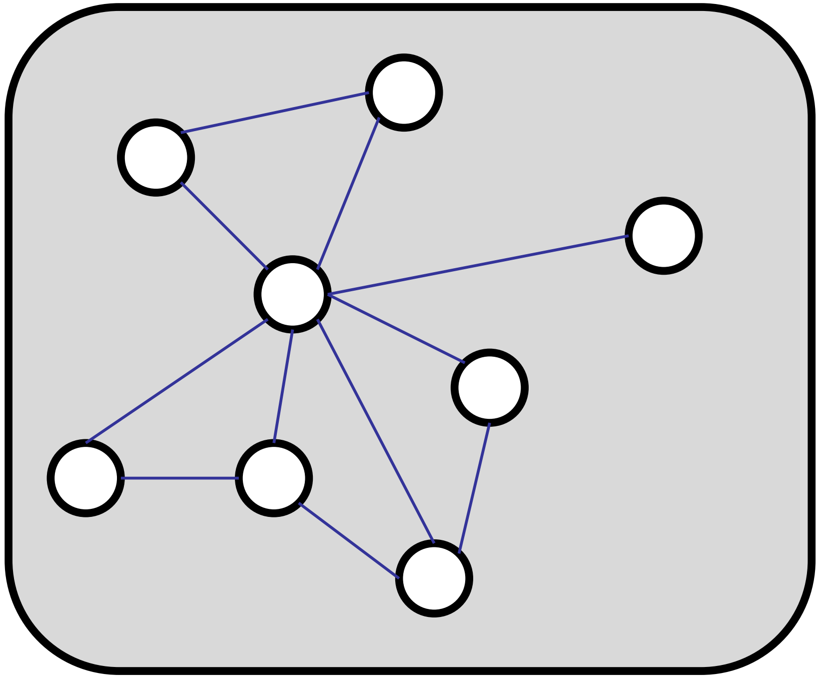


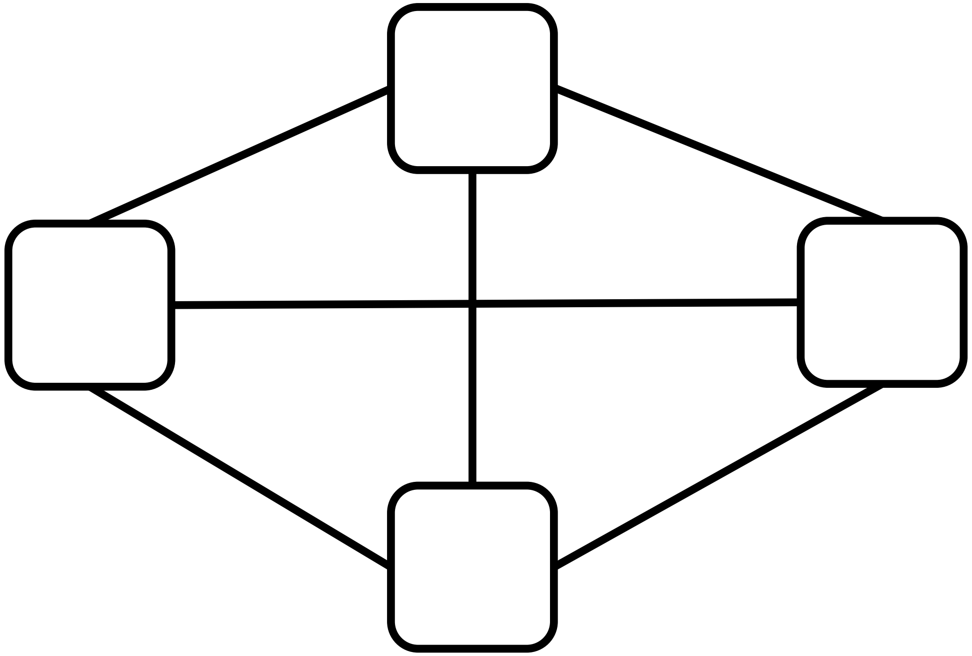
Graph Algorithms

Assume $k < n^{1/2}$

Let $G = (V, E)$ be a graph with n nodes and m edges.

Randomly assign nodes to machines.





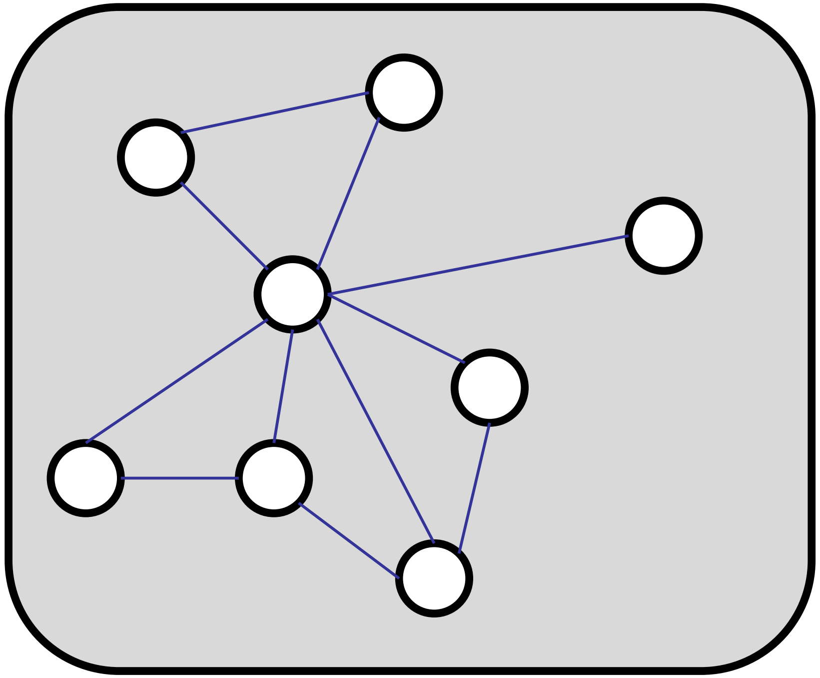
Graph Algorithms

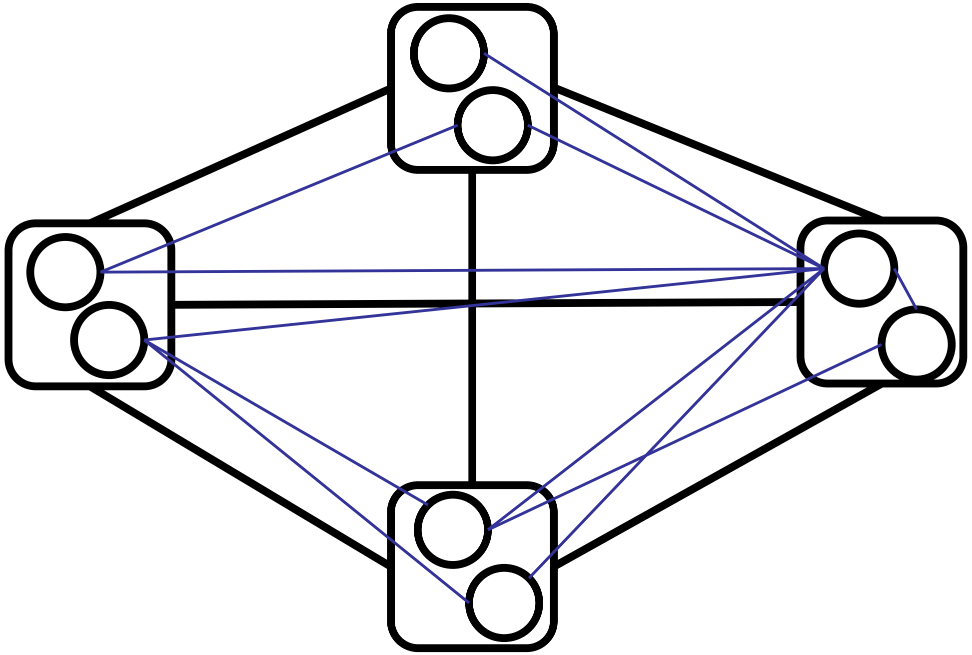
Assume $k < n^{1/2}$

Let $G = (V, E)$ be a graph

with n nodes and m edges.

Randomly assign nodes to machines.



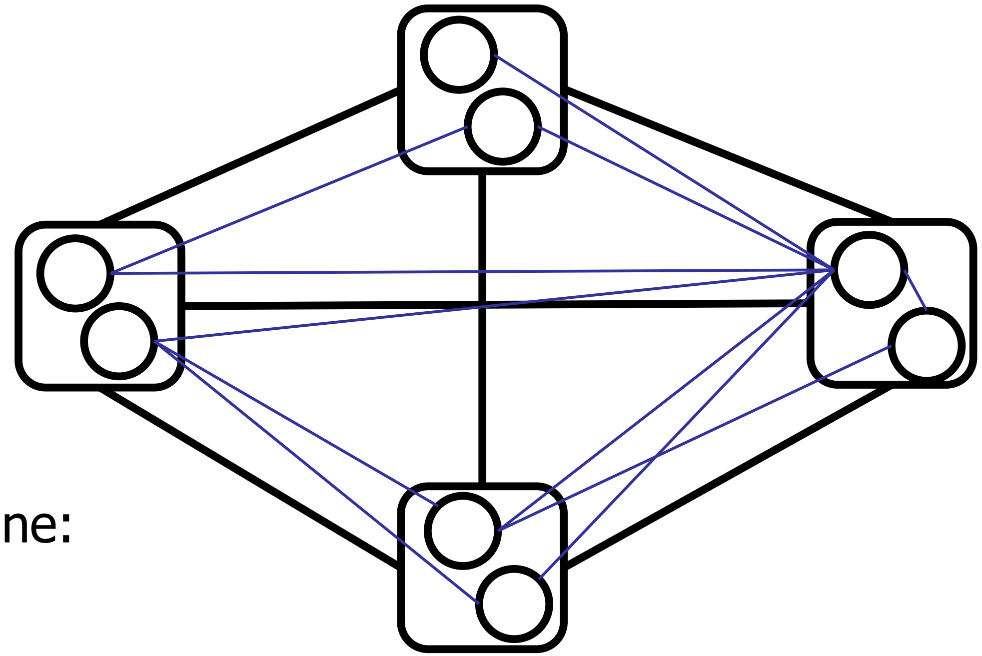


Graph Algorithms

Assume $k < n^{1/2}$

Let $G = (V, E)$ be a graph with n nodes and m edges.

Randomly assign nodes to machines.



With high probability, node per machine:

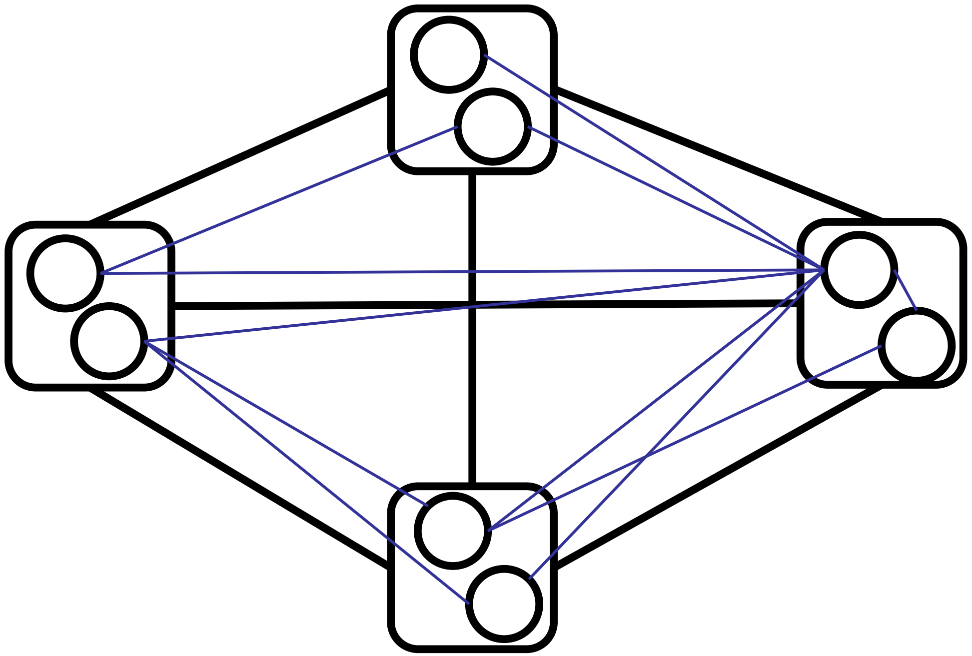
$$O\left(\frac{n}{k} + \log n\right) \leq O\left(\frac{n}{k}\right)$$

Graph Algorithms

Assume $k < n^{1/2}$

How many edges stored on each machine?

$$O\left(\frac{m}{k}\right)???$$



Graph Algorithms

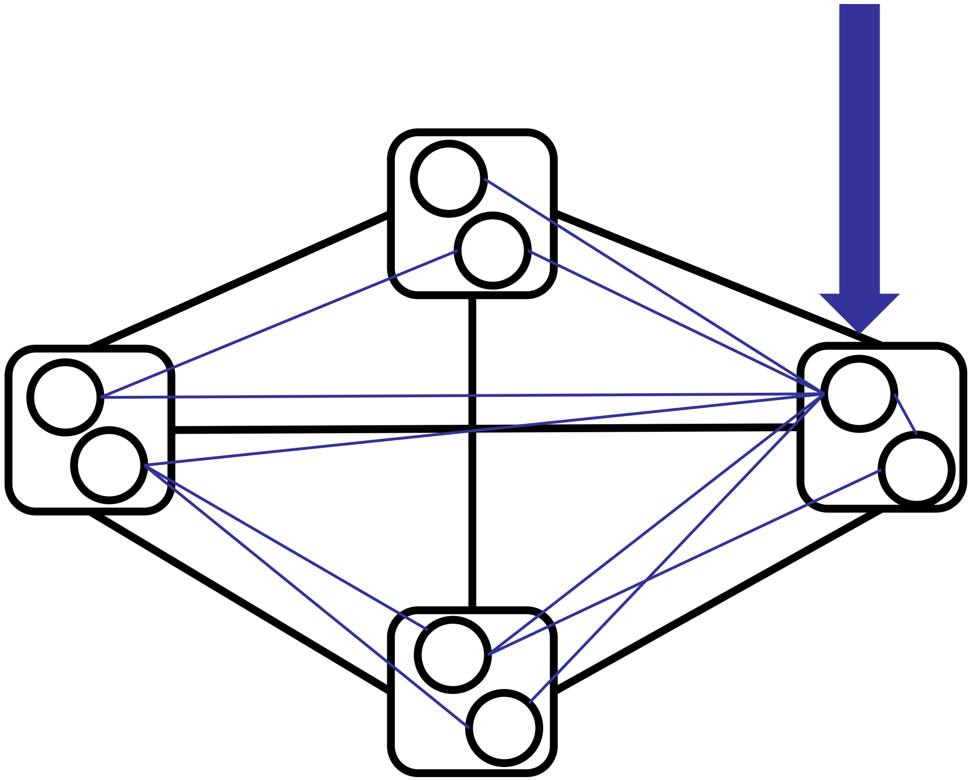
Assume $k < n^{1/2}$

How many edges stored on each machine?

$$O\left(\frac{m}{k}\right)???$$

one node ➔ 7 edges!

m = 11



Graph Algorithms

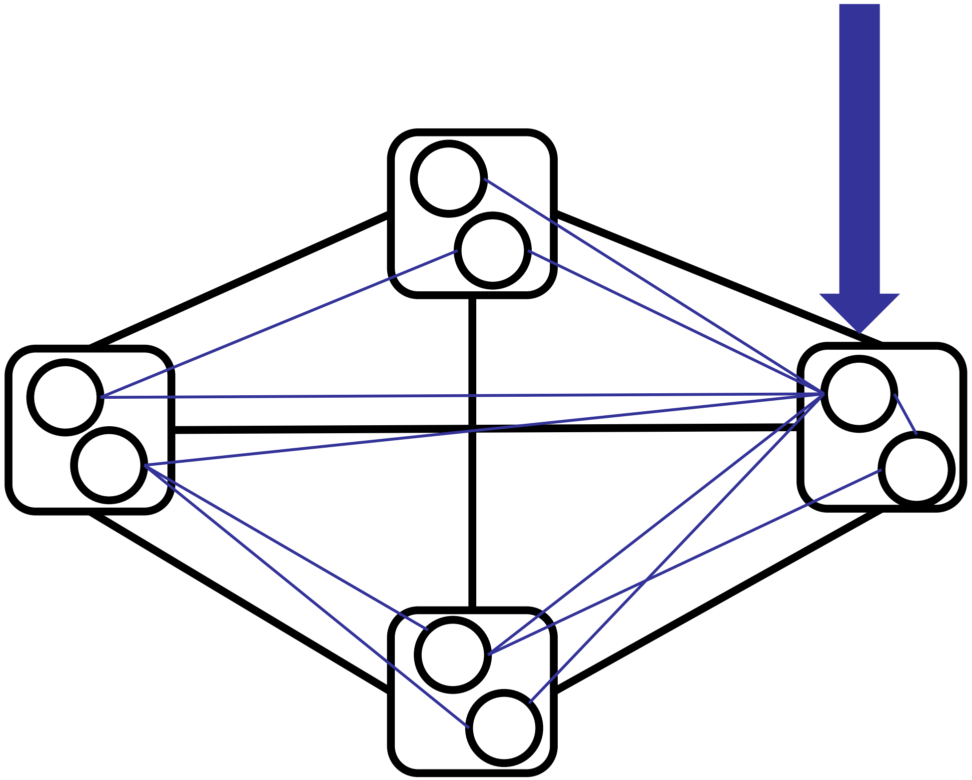
Assume $k < n^{1/2}$

How many edges stored on each machine?

$$O\left(\frac{m}{k}\right)???$$

Why doesn't

Chernoff Bound work?



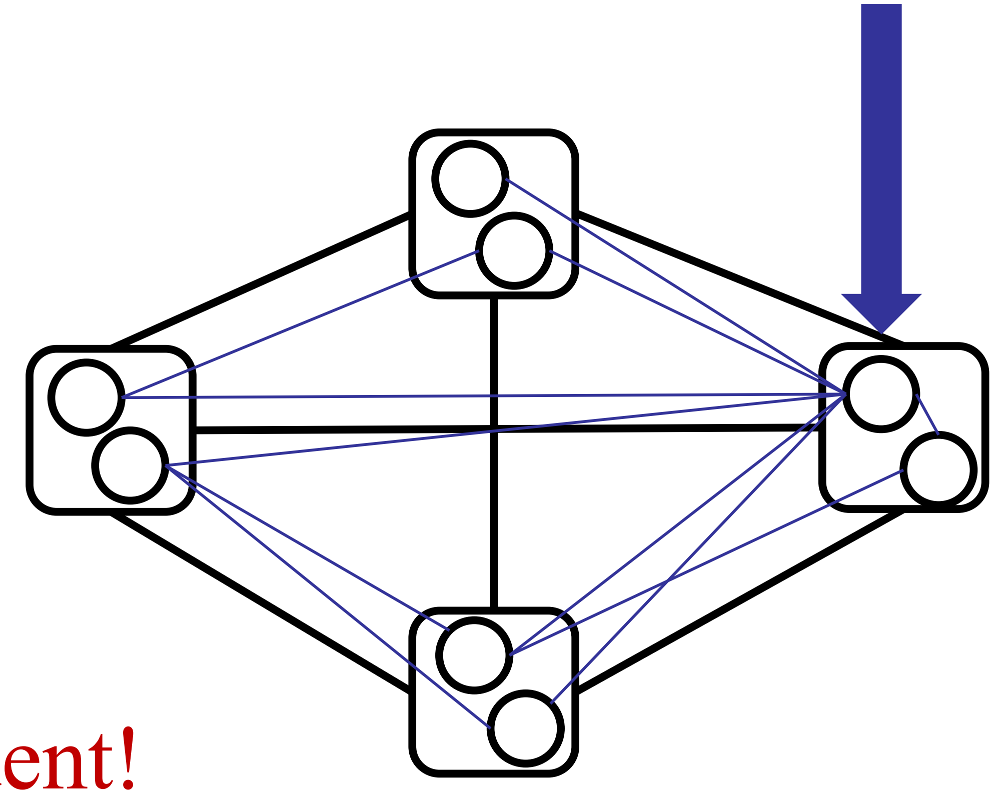
Graph Algorithms

Assume $k < n^{1/2}$

How many edges stored on each machine?

$$O\left(\frac{m}{k}\right) ???$$

Why doesn't Chernoff Bound work?



Edges are not independent!

Graph Algorithms

Assume $k < n^{1/2}$

Theorem:

With high probability, each machine has

$$O\left(\frac{m}{k} + \Delta \log n\right)$$

edges, where Δ = maximum degree of G.

Graph Algorithms

Assume $k < n^{1/2}$

Proof:

Let n_j = number of nodes with degree $[2^i, 2^{i+1})$

- N_1 = nodes with degree {1}
- N_2 = nodes with degree {2,3}
- N_3 = nodes with degree {4,5,6,7}

…

Graph Algorithms

Assume $k < n^{1/2}$

Proof:

Let n_i = number of nodes with degree $[2^i, 2^{i+1})$

Balls and bins:

Each machine has at most $O(n_i/k + \log n)$ nodes with degree $[2^i, 2^{i+1})$, w.h.p.

Graph Algorithms

Assume $k < n^{1/2}$

Proof:

degree

$$\text{edges} \leq \sum_{i=1}^{\log \Delta} \left(\frac{n_i}{k} 2^{i+1} + 2^{i+1} \log n \right)$$

balls and bins

Graph Algorithms

Assume k < $n^{½}$

Proof:

$$\text{edges} \leq \sum_{i=1}^{\log \Delta} \left(\frac{n_i}{k} 2^{i+1} + 2^{i+1} \log n \right)$$

$$= \frac{1}{k} \sum_{i=1}^{\log \Delta} (n_i 2^{i+1}) + \log n \sum_{i=1}^{\log \Delta} 2^{i+1}$$

Graph Algorithms

Assume $k < n^{1/2}$

Proof:

$$\text{edges} \leq \sum_{i=1}^{\log \Delta} \left(\frac{n_i}{k} 2^{i+1} + 2^{i+1} \log n \right)$$

$$= \frac{1}{k} \sum_{i=1}^{\log \Delta} (n_i 2^{i+1}) + \log n \sum_{i=1}^{\log \Delta} 2^{i+1}$$

$$= \frac{1}{k} \sum_{i=1}^{\log \Delta} (n_i 2^{i+1}) + 4\Delta \log n$$

sum:
$2\Delta + \Delta + \Delta/2 + \Delta/4 + \Delta/8 + \ldots$

Graph Algorithms

Assume k < n^{½}

Proof:

$$\text{edges} \leq \sum_{i=1}^{\log \Delta} \left(\frac{n_i}{k} 2^{i+1} + 2^{i+1} \log n \right)$$

$$= \frac{1}{k} \sum_{i=1}^{\log \Delta} (n_i 2^{i+1}) + \log n \sum_{i=1}^{\log \Delta} 2^{i+1}$$

$$= \frac{1}{k} \sum_{i=1}^{\log \Delta} (n_i 2^{i+1}) + 4\Delta \log n$$

$$= \frac{1}{k} (4m) + 4\Delta \log n$$

sum:
each edge in the graph, twice

Graph Algorithms

Assume $k < n^{1/2}$

Theorem:

With high probability, each machine has

$$O\left(\frac{m}{k} + \Delta \log n\right)$$

edges, where Δ = maximum degree of G.

Graph Algorithms

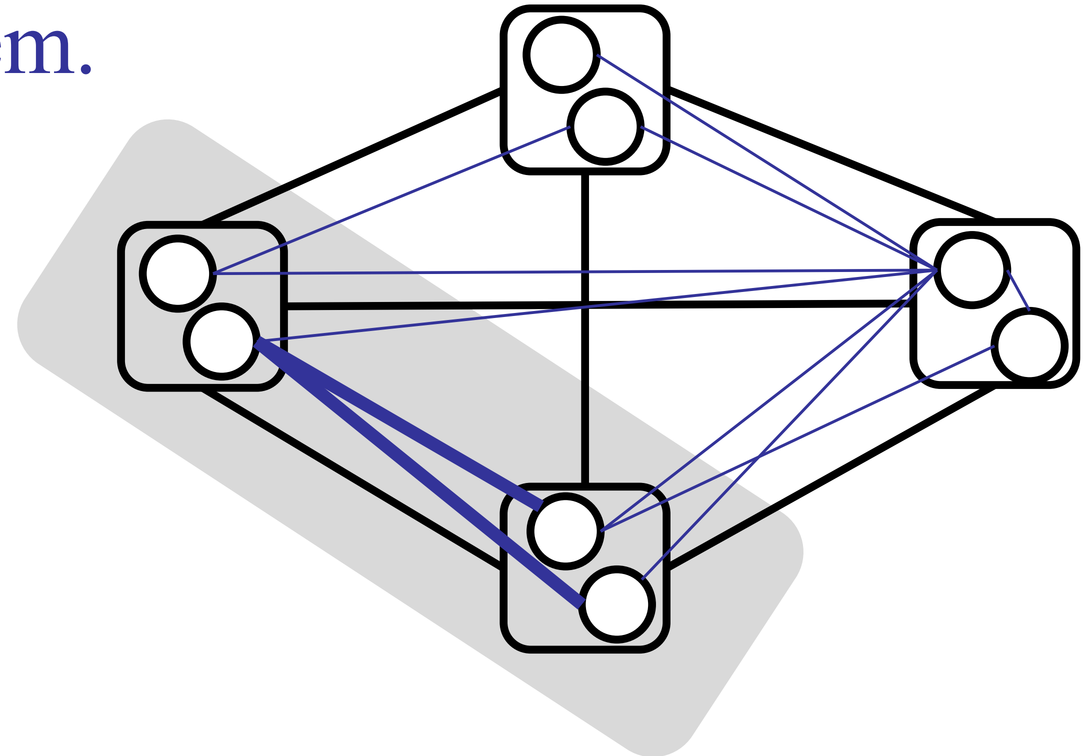
Assume $k < n^{1/2}$

Theorem:

With high probability, each pair of machines has

$$O\left(\frac{m}{k^2} + \frac{\Delta}{k}\log n\right)$$

edges connecting them.



Graph Algorithms

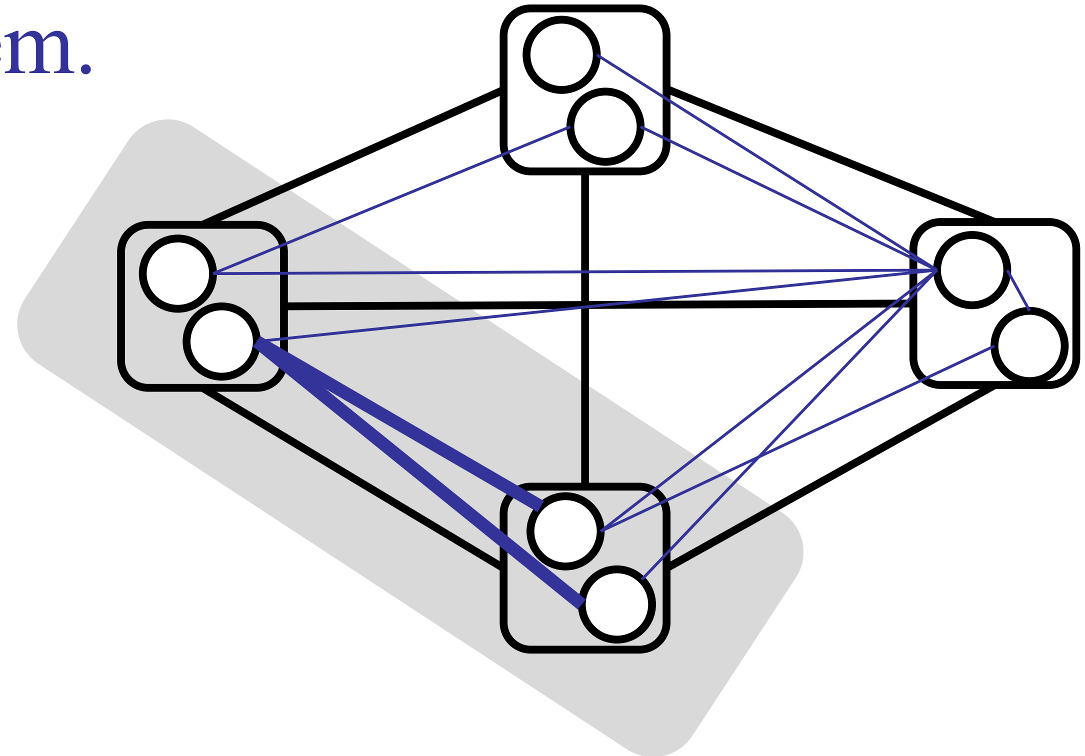
Assume k < $n^{1/2}$

Theorem:

With high probability, each pair of machines has

$$O\left(\frac{m}{k^2} + \frac{\Delta}{k}\log n\right)$$

edges connecting them.



Balls-and-bins?

Chernoff Bound?

Graph Algorithms

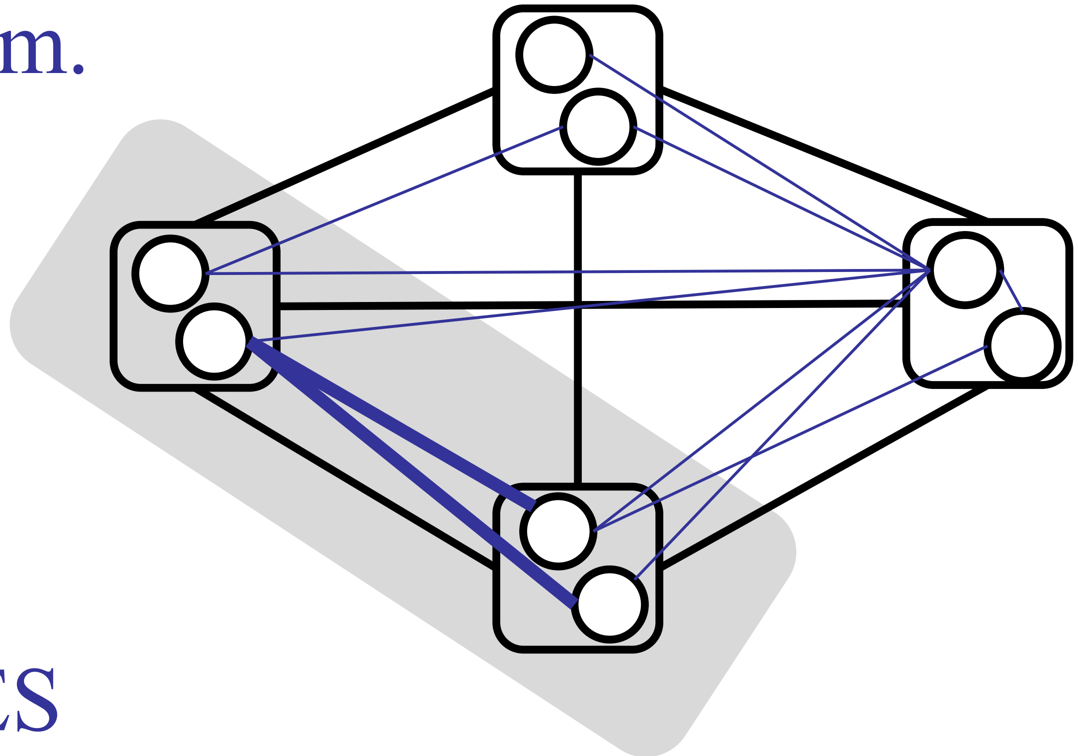
Assume $k < n^{1/2}$

Theorem:

With high probability, each pair of machines has

$$O\left(\frac{m}{k^2} + \frac{\Delta}{k}\log n\right)$$

edges connecting them.



Balls-and-bins?

Chernoff Bound? YES

Graph Algorithms

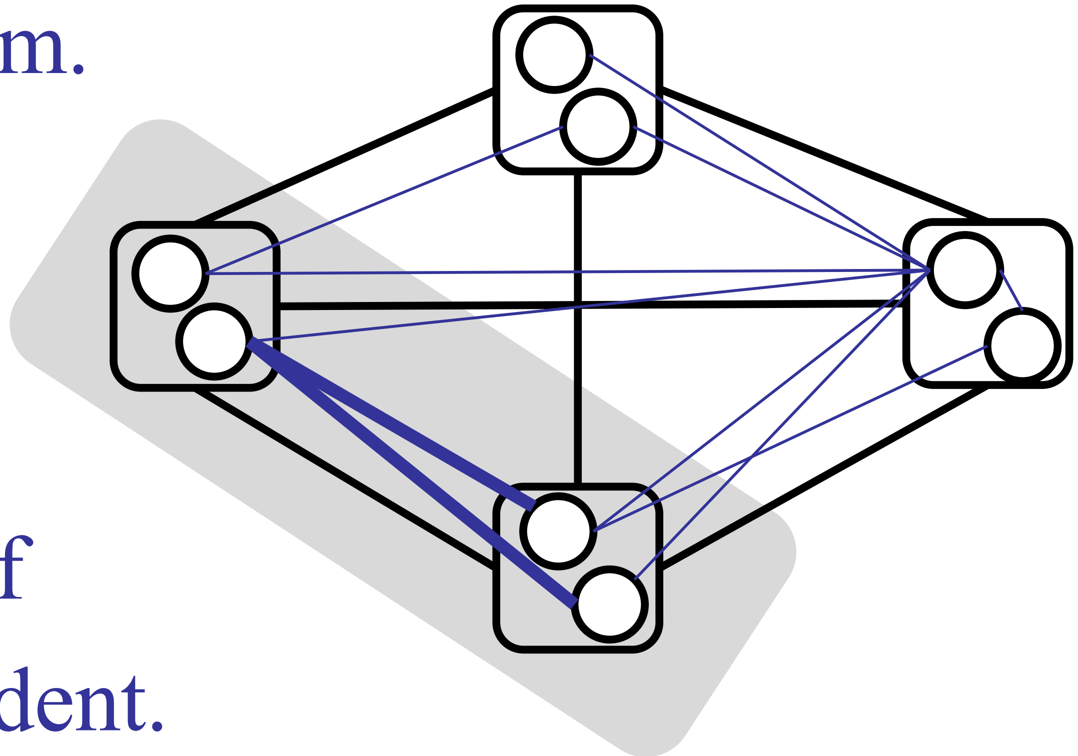
Assume k < $n^{1/2}$

Theorem:

With high probability, each pair of machines has

$$O\left(\frac{m}{k^2} + \frac{\Delta}{k}\log n\right)$$

edges connecting them.



Proof:

Fix a machine.

The other endpoint of each edge is independent.

Graph Algorithms

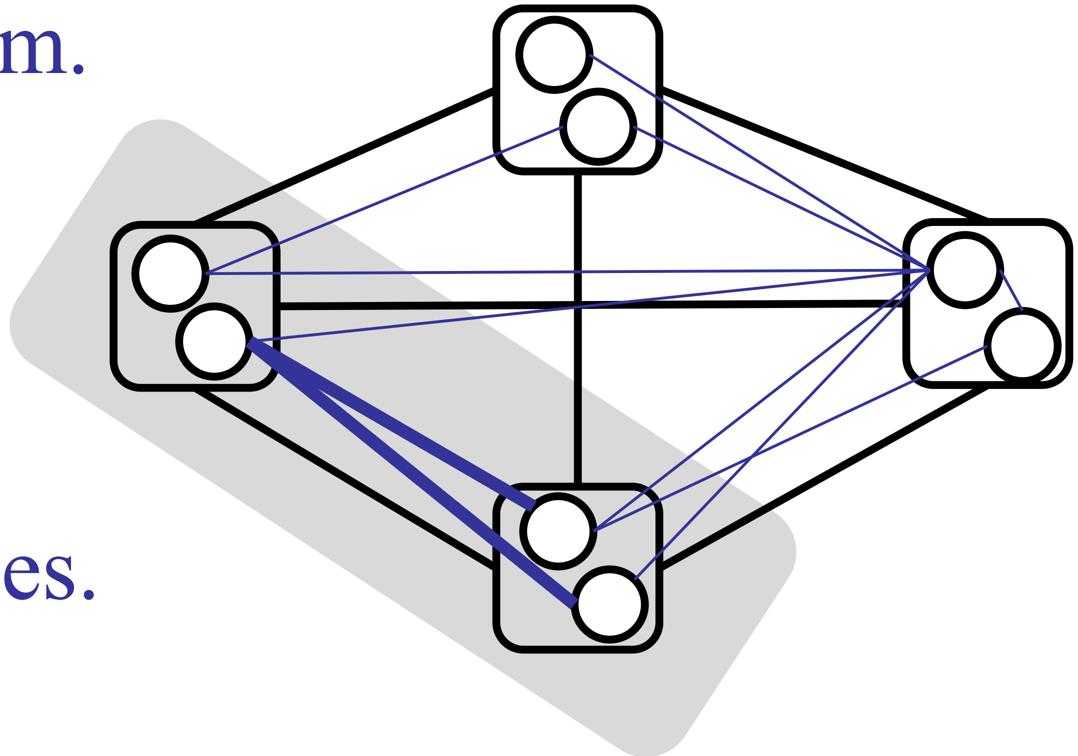
Assume $k < n^{1/2}$

Theorem:

With high probability, each pair of machines has

$$O\left(\frac{m}{k^2} + \frac{\Delta}{k}\log n\right)$$

edges connecting them.



Proof:

W.h.p., machine has

$O\left(\frac{m}{k} + \Delta \log n\right)$ edges.

Graph Algorithms

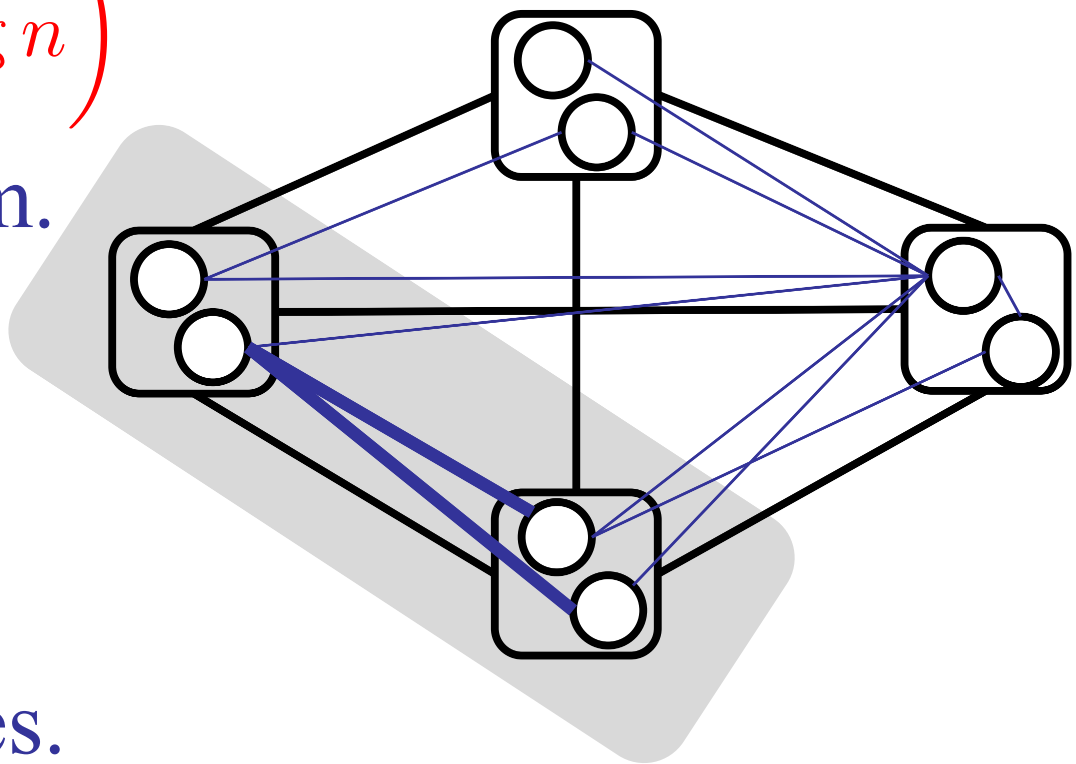
Assume k < $n^{1/2}$ / log n.

Theorem:

With high probability, each pair of machines has

$$O\left(\frac{m}{k^2} + \frac{\Delta}{k}\log n\right)$$

edges connecting them.



Proof:

W.h.p., machine has

$O\left(\frac{m}{k} + \Delta\log n\right)$ edges.

So w.h.p. 1/(k-1) got to each other machine.

Key Theorems

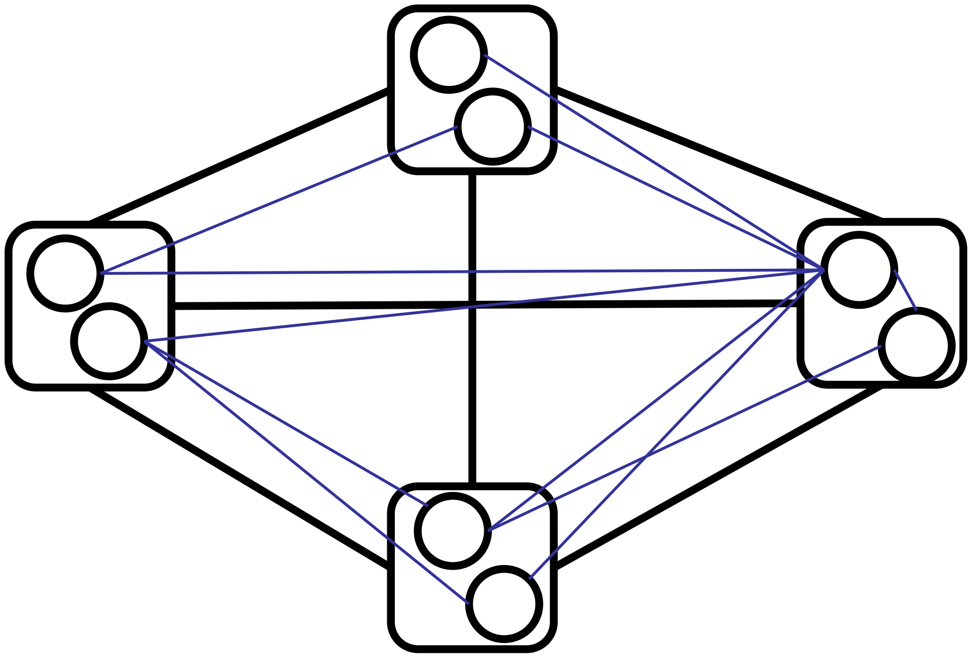
$O\left(\frac{n}{k}\right)$ nodes per machine, w.h.p.

$O\left(\frac{m}{k} + \Delta \log n\right)$ edges per machine, w.h.p.

$O\left(\frac{m}{k^2} + \frac{\Delta}{k} \log n\right)$ edges between two machines, w.h.p.

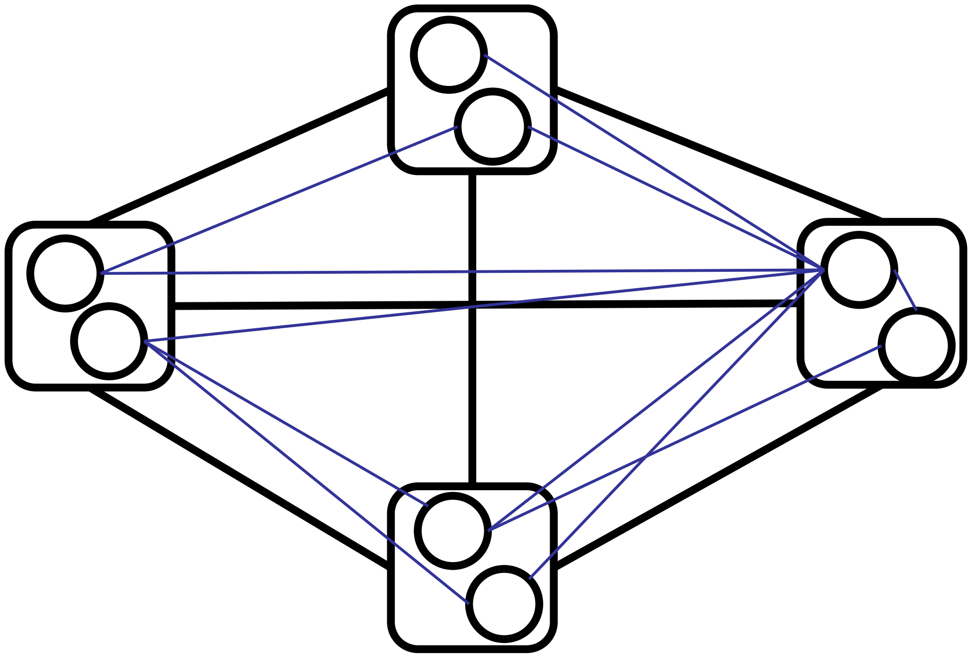
Example 1: Send information

Each node in the graph sends 1 bit to each of its neighbors.



Example 1: Send information

Each node in the graph sends 1 bit to each of its neighbors.



Time:

$$O\left(\frac{1}{B}\left[\frac{m}{k^2}+\frac{\Delta}{k}\log n\right]\right)$$

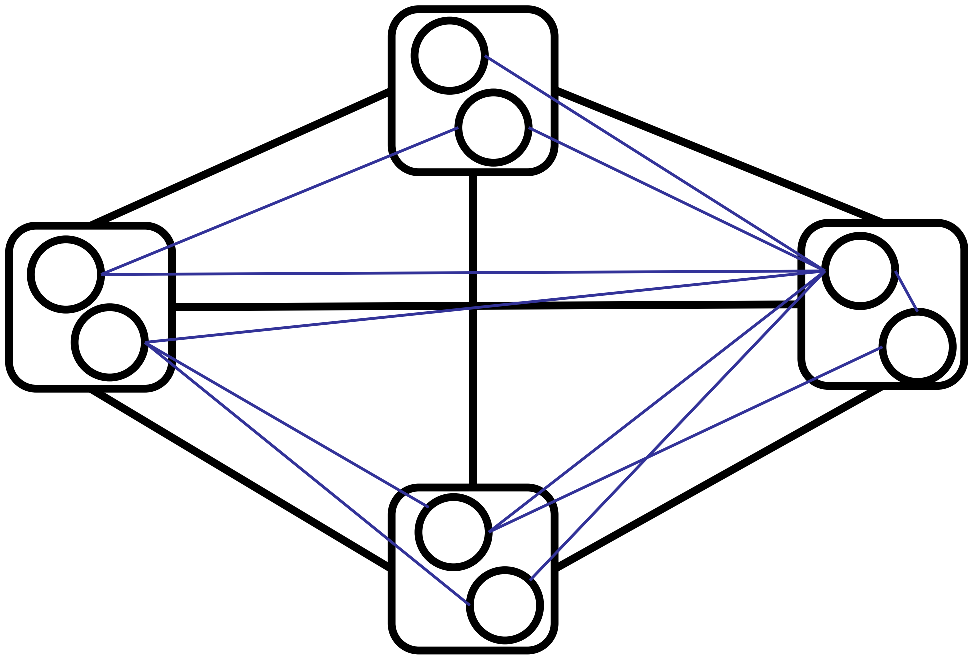
Example 2: Luby's Algorithm

Repeat log(n) times:

1. Mark and send to neighbors.
2. Unmark and send to neighbors.
3. Delete and send to neighbors.

Time:

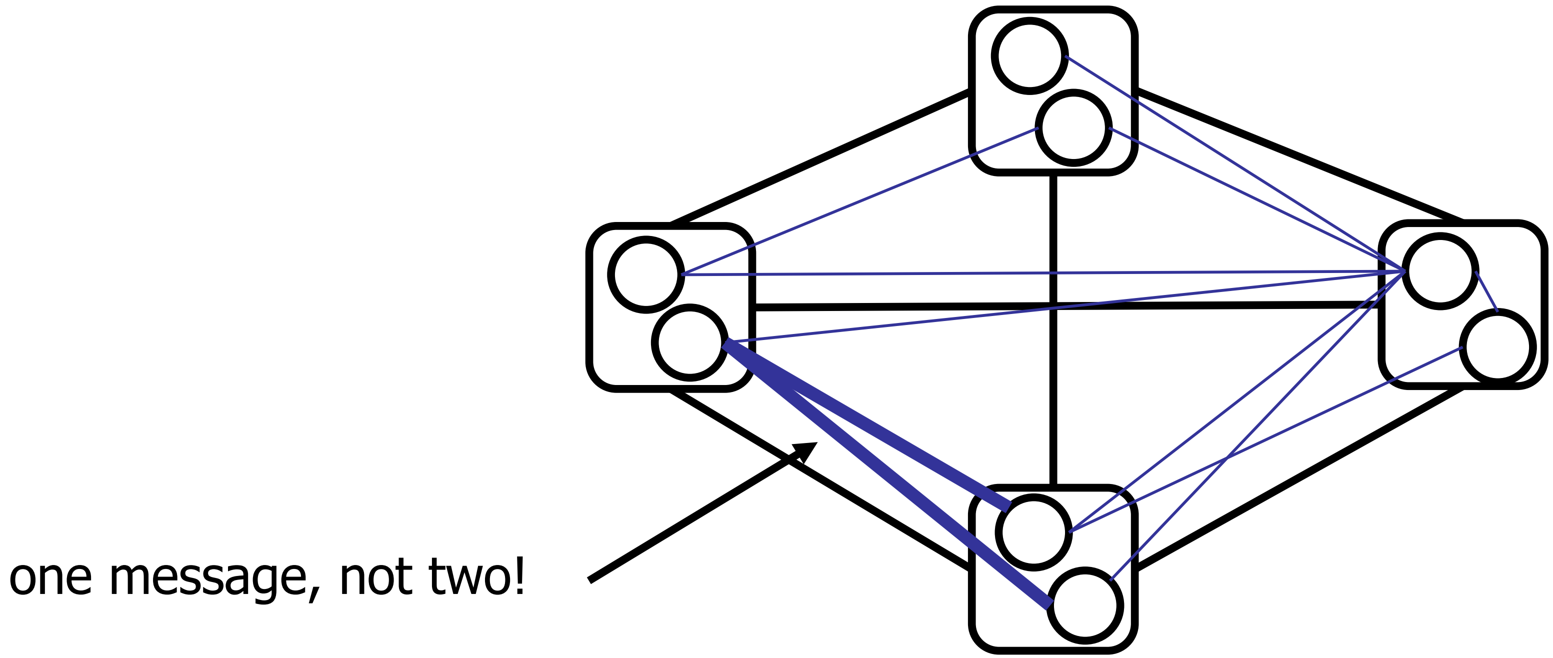
$$O\left(\frac{1}{B}\left[\frac{m}{k^2}+\frac{\Delta}{k}\log n\right]\right)$$



Example 2: Luby's Algorithm

Better analysis:

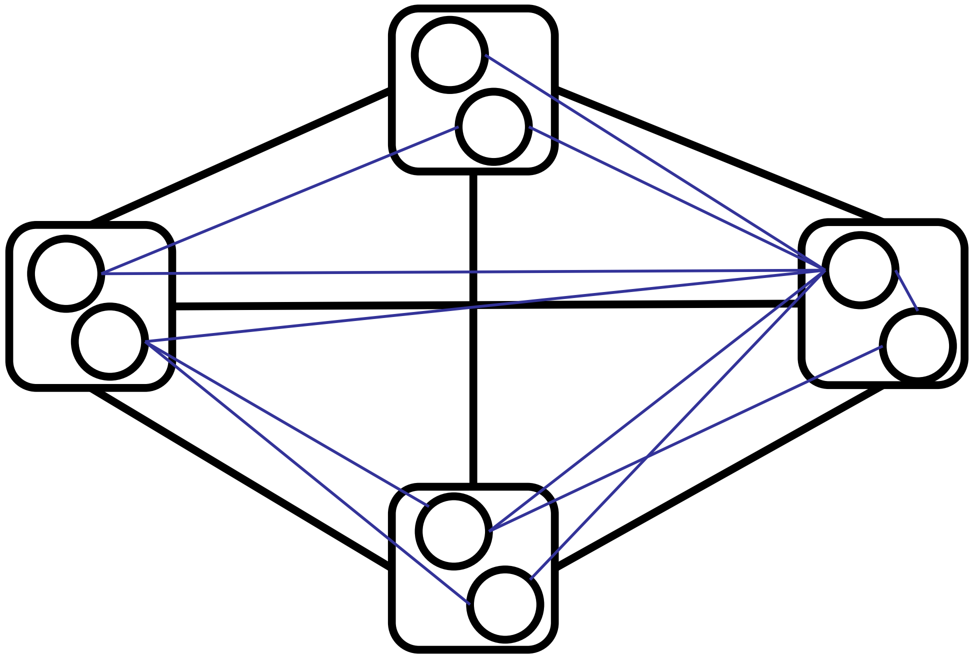
- Each node sends same message to all neighbors.
- Only need to send (n/k) messages per link.



Example 2: Luby's Algorithm

Repeat log(n) times:

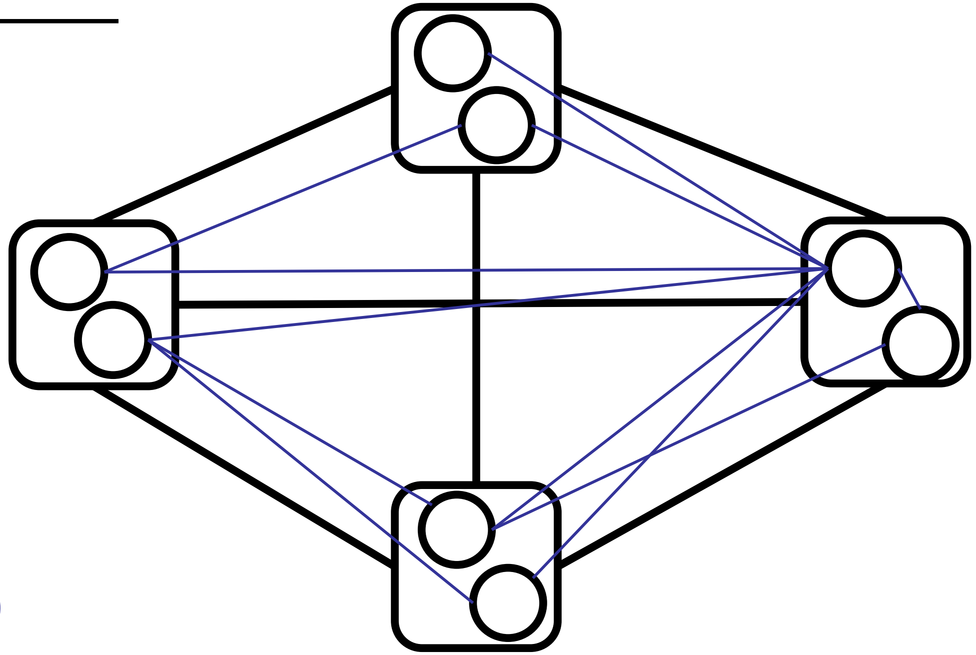
1. Mark and send to neighbors.
2. Unmark and send to neighbors.
3. Delete and send to neighbors.



Time:

$O\left(\frac{1}{B}\frac{n}{k}\log n\right)$

Some possible numbers:



Sparse graph:

- k = 5000
- n = 100,000
- m = 1,000,000
- B = 400 (10GBps switch)

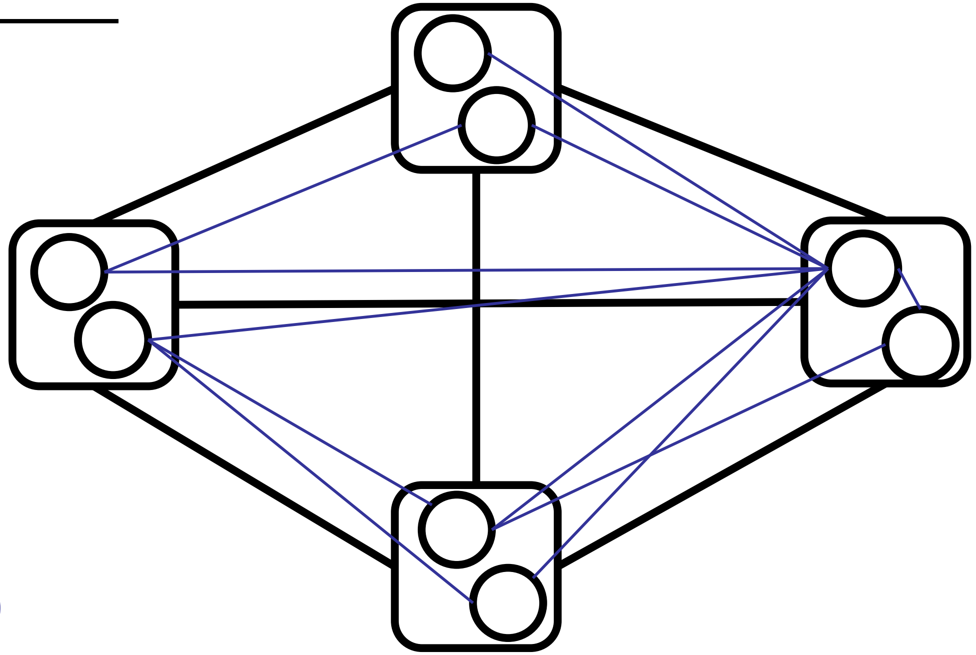
$$\frac{1}{B}\left(\frac{m}{k}\right) \approx 50s$$

$$\frac{1}{B}\left(\frac{n}{k}\right) \approx 50ms$$

fastest

$$\frac{1}{B}\left(\frac{m}{k^2}\right) \approx 10ms$$

Some possible numbers:



Dense graph:

- $k = 5000$
- $n = 100{,}000$
- $m = 3{,}000{,}000{,}000$
- $B = 400$ (10GBps switch)

$$\frac{1}{B}\left(\frac{m}{k}\right) \approx 25min$$

very slow

fastest

$$\frac{1}{B}\left(\frac{n}{k}\right) \approx 50ms$$

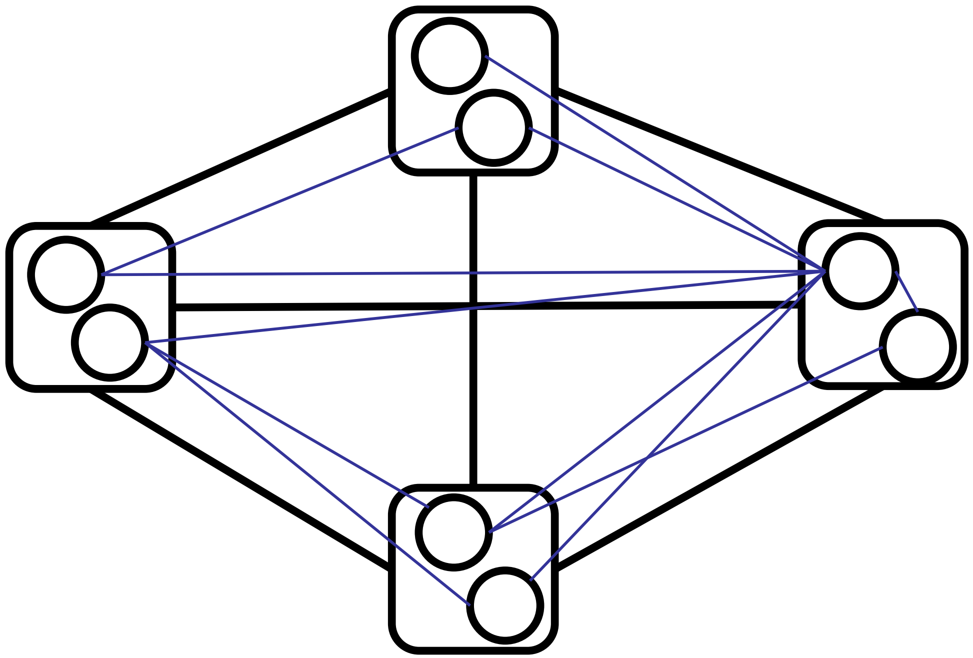
$$\frac{1}{B}\left(\frac{m}{k^2}\right) \approx 300ms$$

Example 2: Luby's Algorithm

Example:
Can use this model to predict running times.

Repeat log(n) times:

1. Mark and send to neighbors.
2. Unmark and send to neighbors.
3. Delete and send to neighbors.



Time:

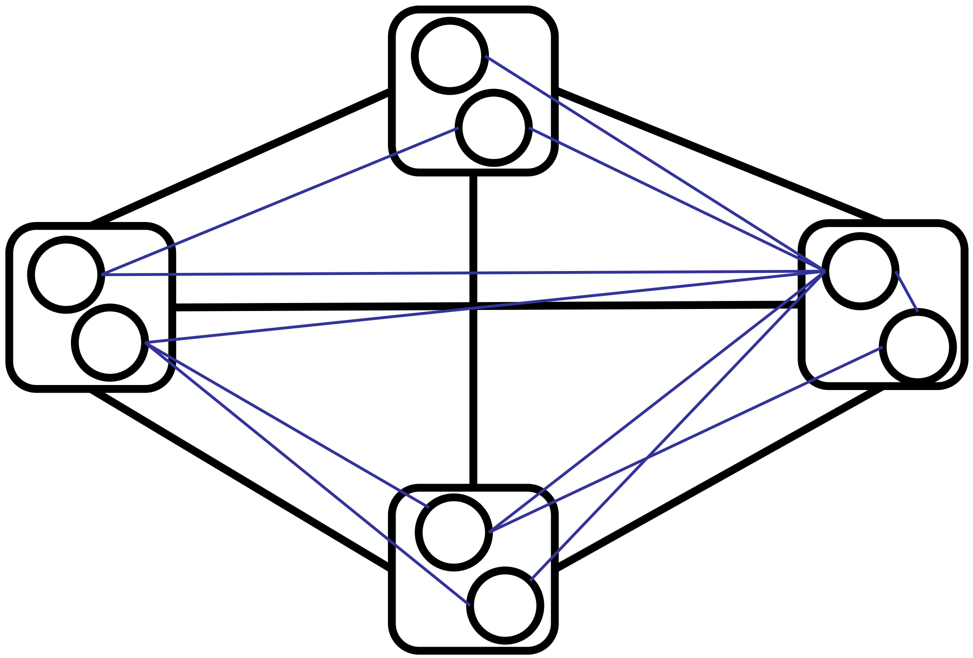
$$O\left(\frac{1}{B}\frac{n}{k}\log n\right)$$

➔ < 20 seconds?

Example 3: Bellman-Ford

Repeat D times:

1. Send your estimate to all your neighbors.
2. After receiving all neighbors estimates, relax all neighboring edges.



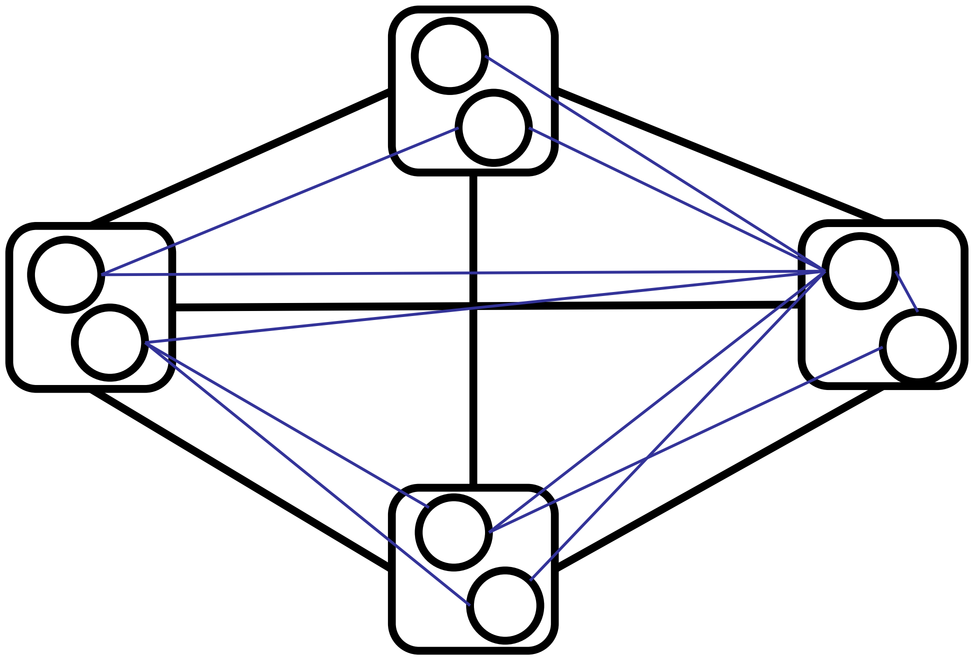
Example 3: Bellman-Ford

Repeat D times:

1. Send your estimate to all your neighbors.
2. After receiving all neighbors estimates, relax all neighboring edges.

Time:

$O\left(\frac{D}{B}\frac{n}{k}\right)$



Summary

Today: k-Machine

k-Machine Model

- Cluster computing

Some simple examples

- Luby’s
- Bellman-Ford

Minimum Spanning Tree

- Basic algorithm
- Fully distributed algorithm
- Lower bound

Last Week: Map-Reduce

Map-Reduce Model

- Cluster computing

Some simple examples

- Word count
- Join

Algorithms

- Bellman-Ford
- PageRank

Minimum Spanning Tree

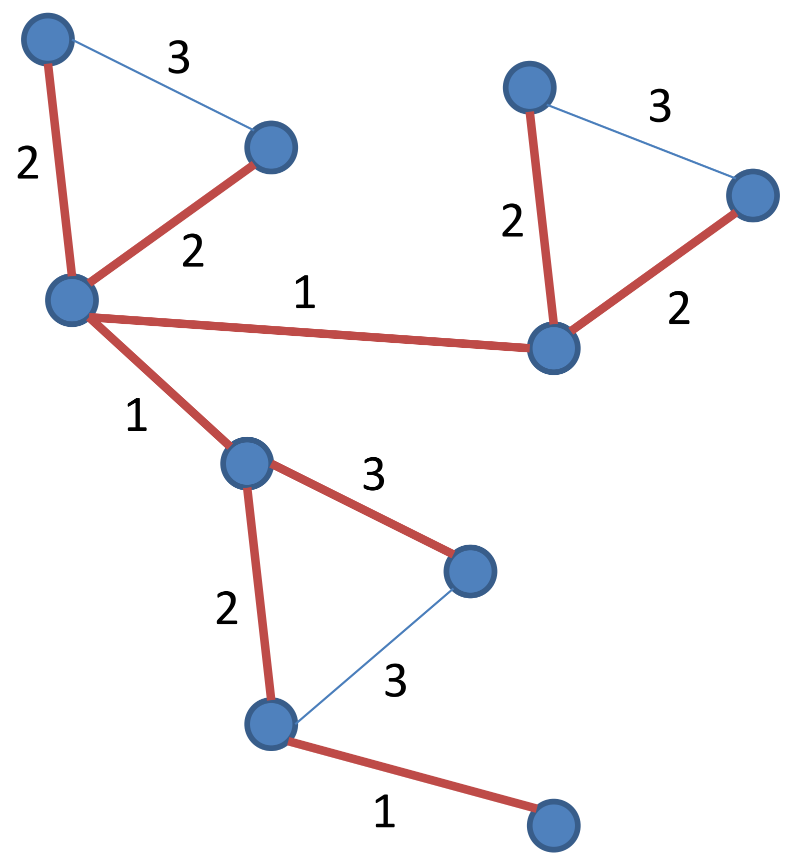
Assumptions:

Graph G = (V,E)

- Undirected
- Weighted
- Connected
- n nodes
- m edges

Output:

Each machine knows which edges adjacent to its nodes are in the MST.



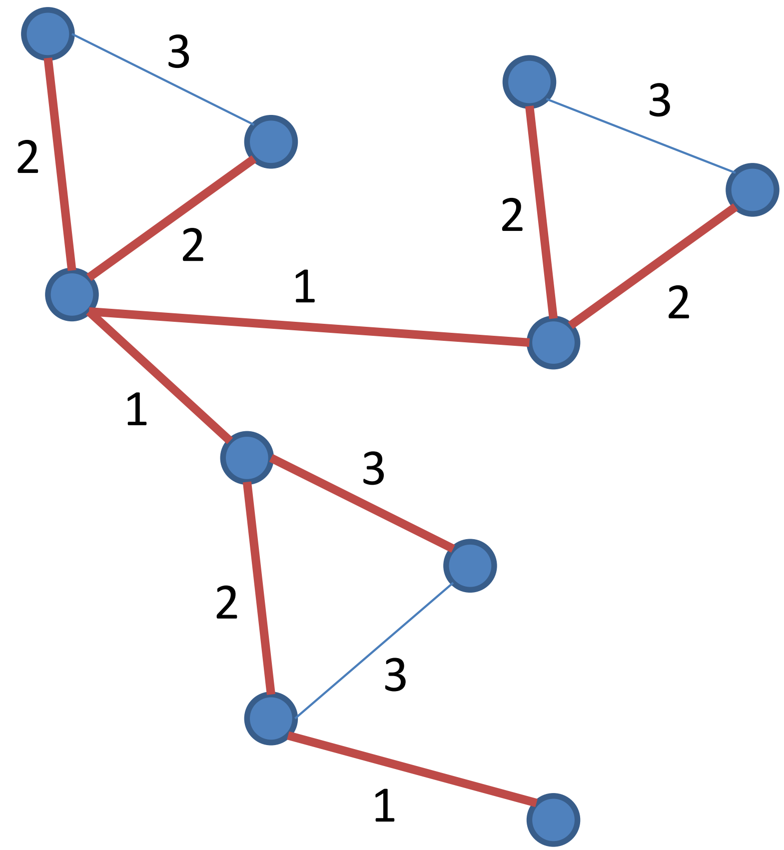
Example: output 16

Minimum Spanning Tree

Boruvka's Algorithm

Key idea:

For every cut in the graph, the minimum weight edge across the cut is in the MST.

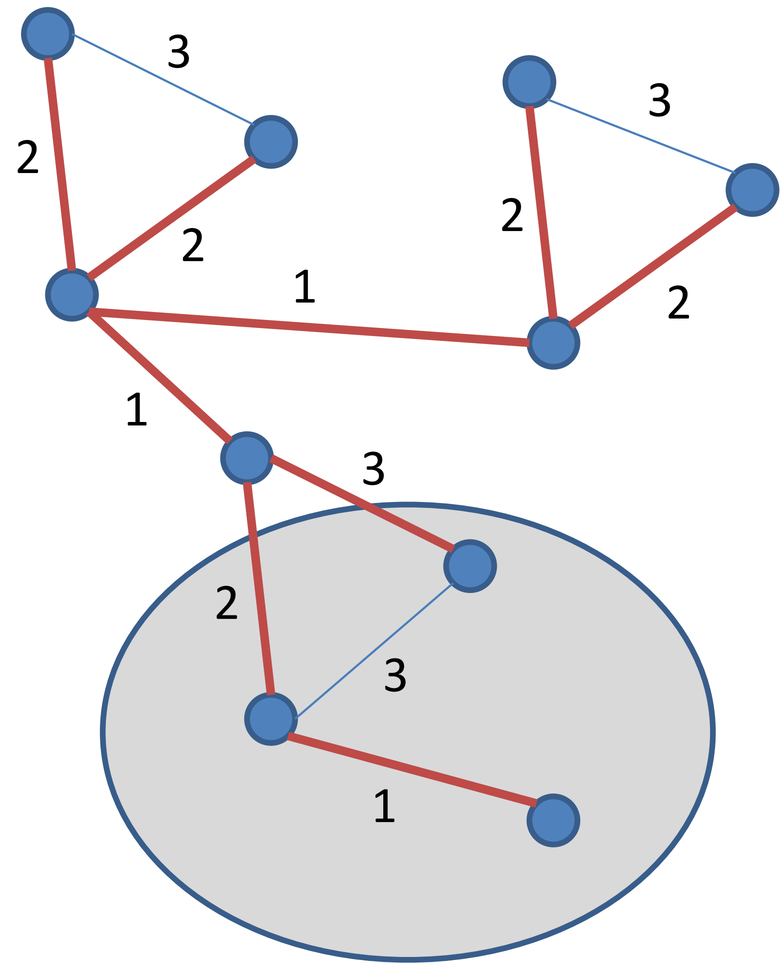


Minimum Spanning Tree

Boruvka's Algorithm

Key idea:

For every cut in the graph, the minimum weight edge across the cut is in the MST.

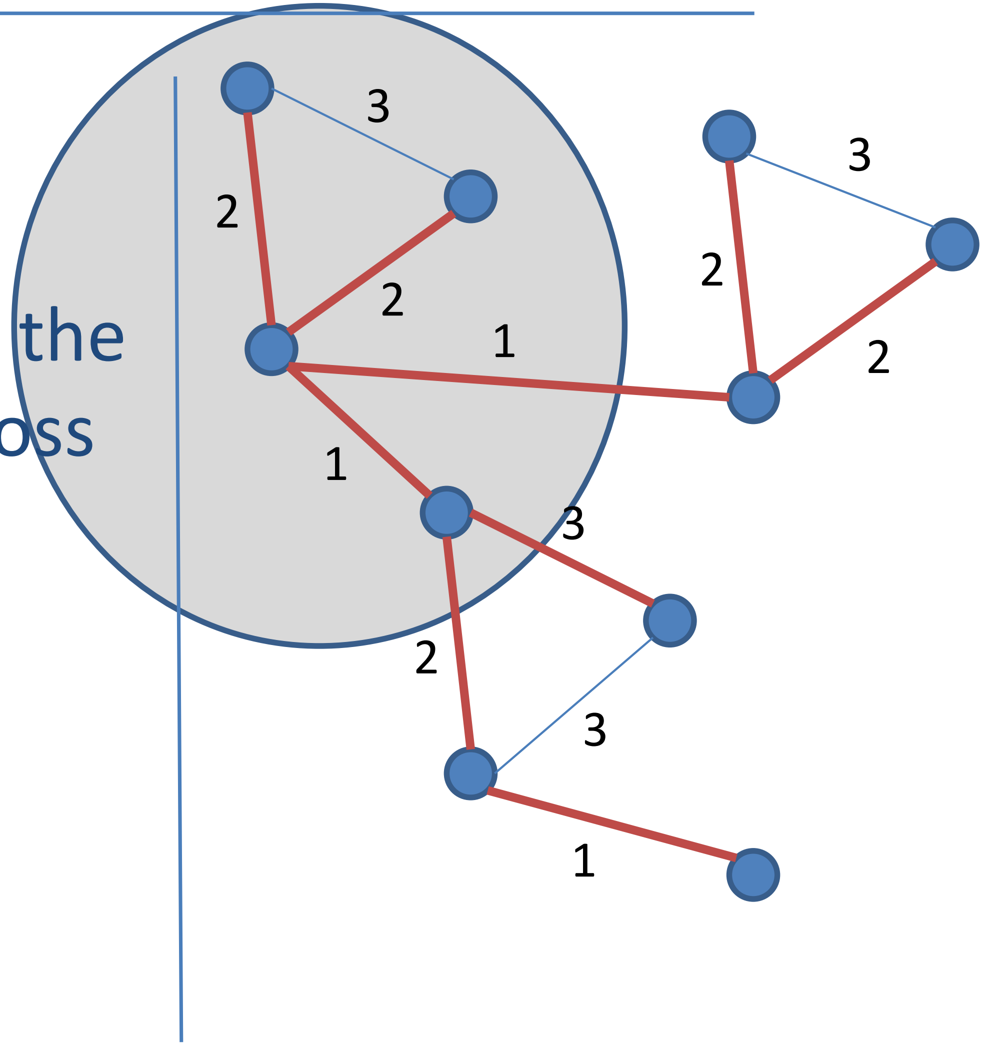


Minimum Spanning Tree

Boruvka's Algorithm

Key idea:

For every cut in the graph, the minimum weight edge across the cut is in the MST.

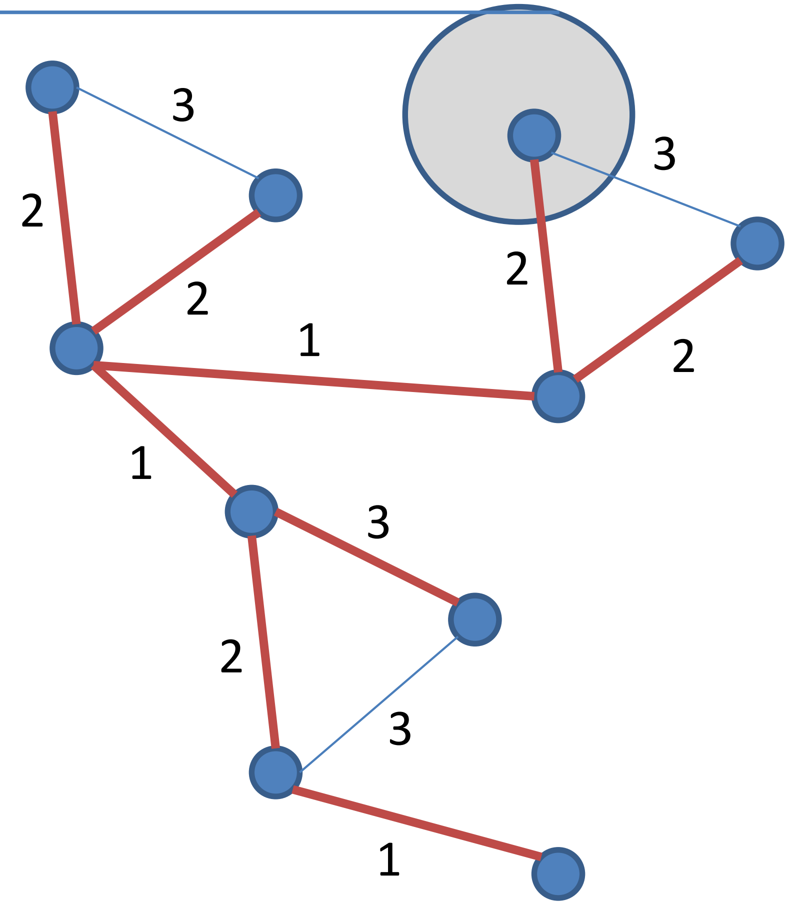


Minimum Spanning Tree

Boruvka's Algorithm

Key idea:

For every cut in the graph, the minimum weight edge across the cut is in the MST.

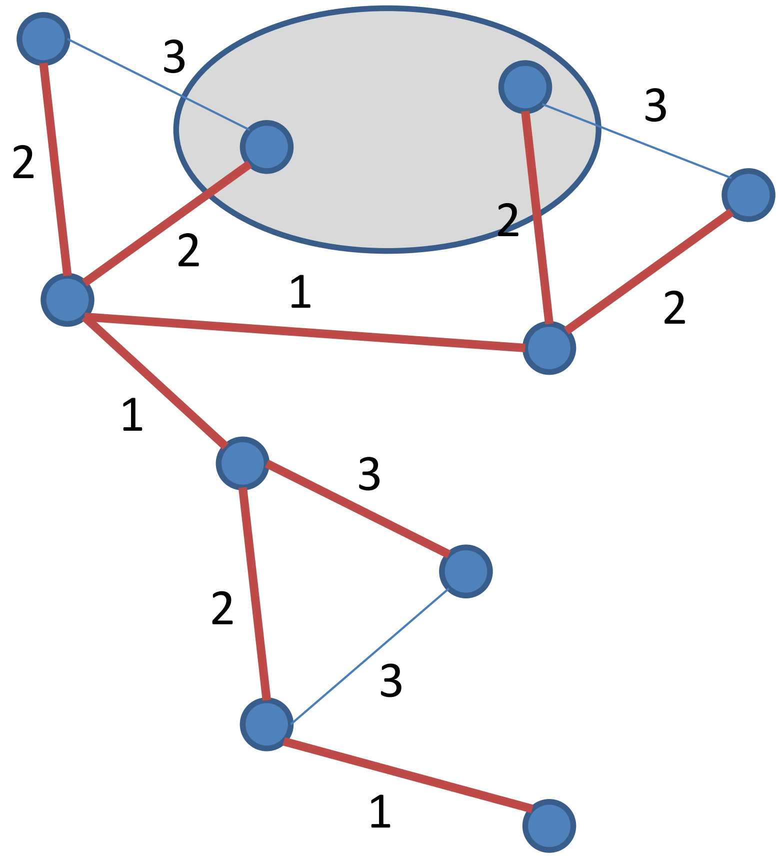


Minimum Spanning Tree

Boruvka's Algorithm

Key idea:

For every cut in the graph, the minimum weight edge across the cut is in the MST.



Minimum Spanning Tree

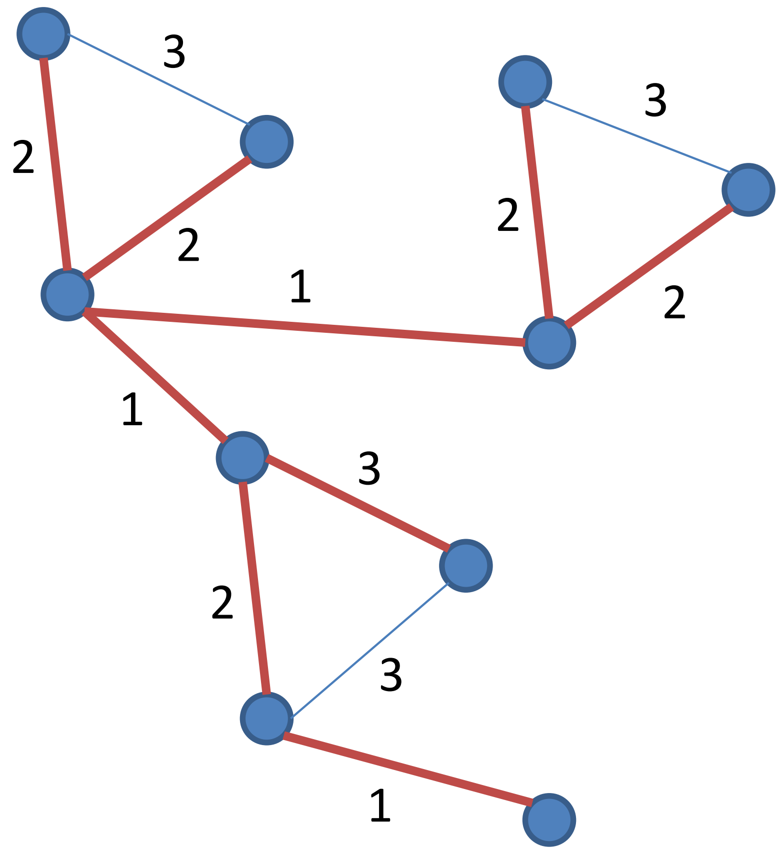
Boruvka's Algorithm

Key idea:

For every cut in the graph, the minimum weight edge across the cut is in the MST.

Proof (sketch):

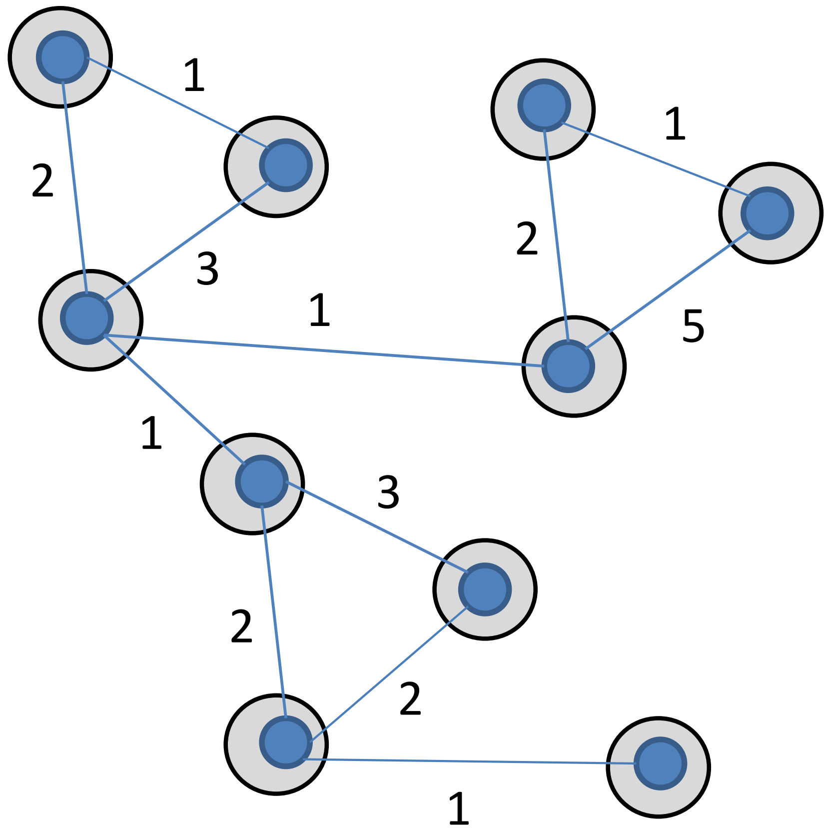
- Add the edge e, creating a cycle.
- Delete e', heaviest edge on cycle.
- Since e is smallest across cut, there is some heavier edge on cycle, i.e., e' ≠ e.



Minimum Spanning Tree

Boruvka's Algorithm

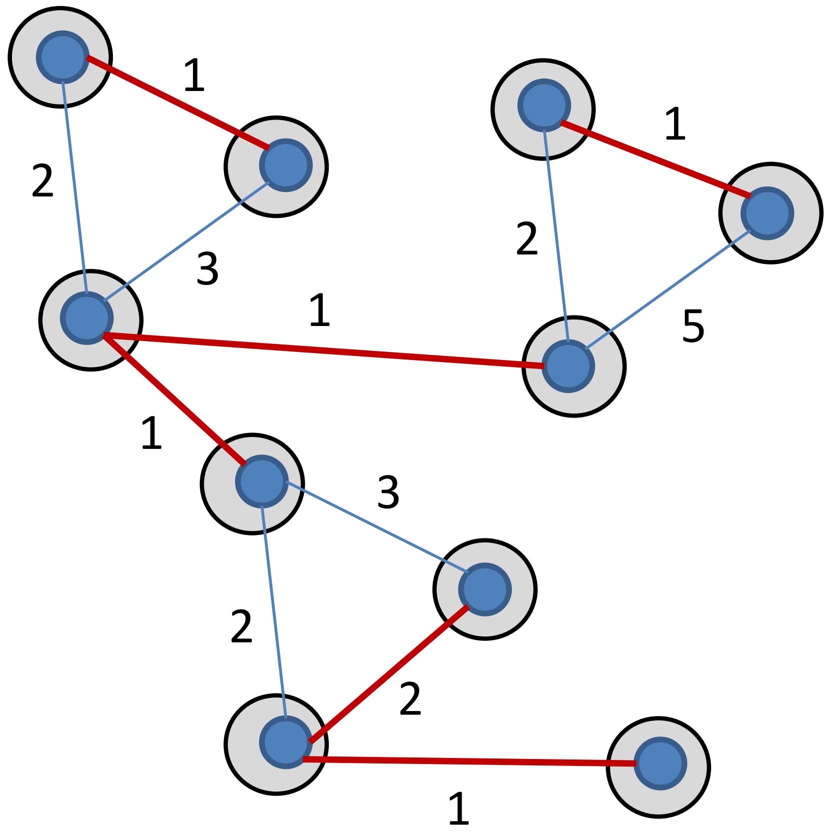
Initially: Every node is in its own component.



Minimum Spanning Tree

Boruvka's Algorithm

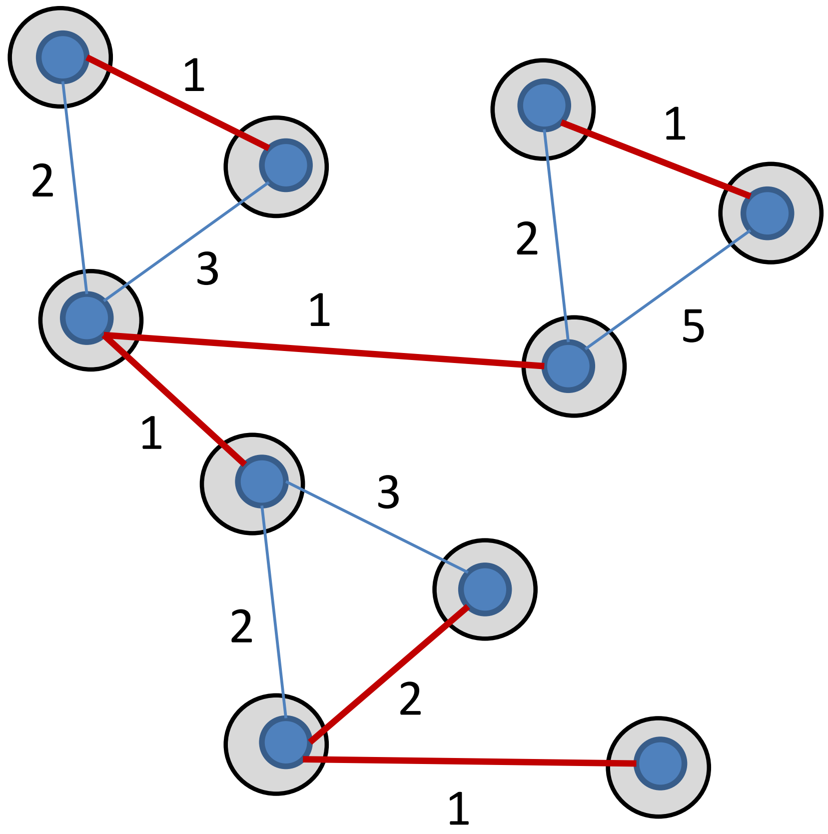
Every component finds its min weight outgoing edge.



Minimum Spanning Tree

Boruvka's Algorithm

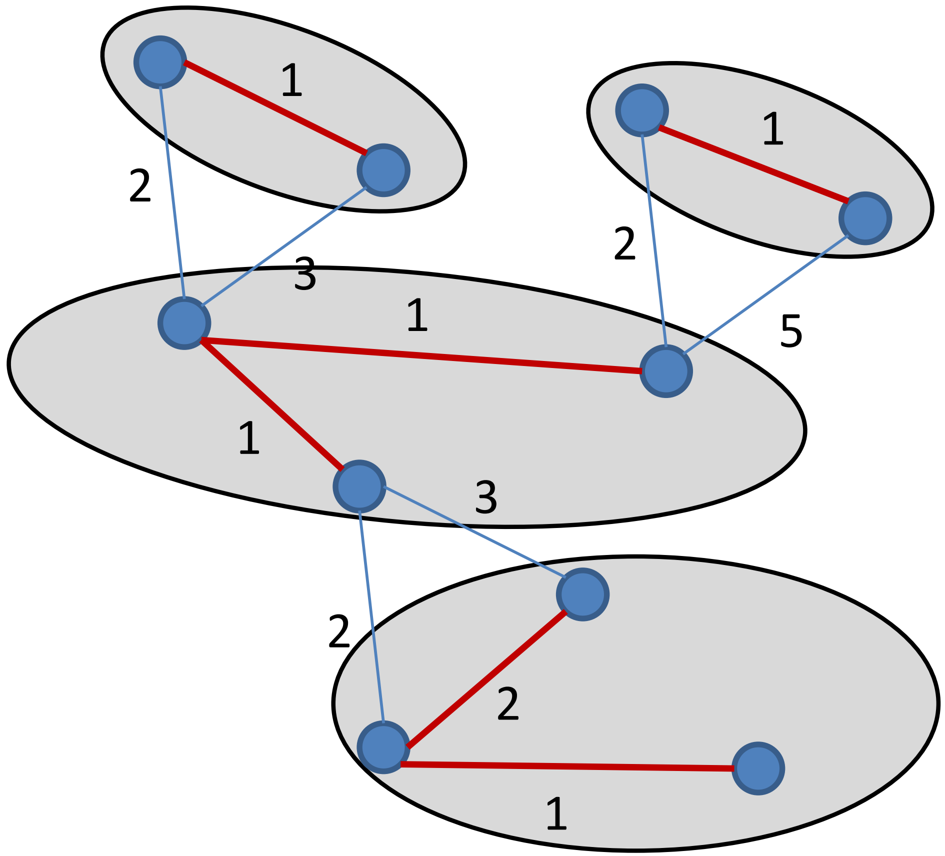
Add min weight outgoing edges to MST.



Minimum Spanning Tree

Boruvka's Algorithm

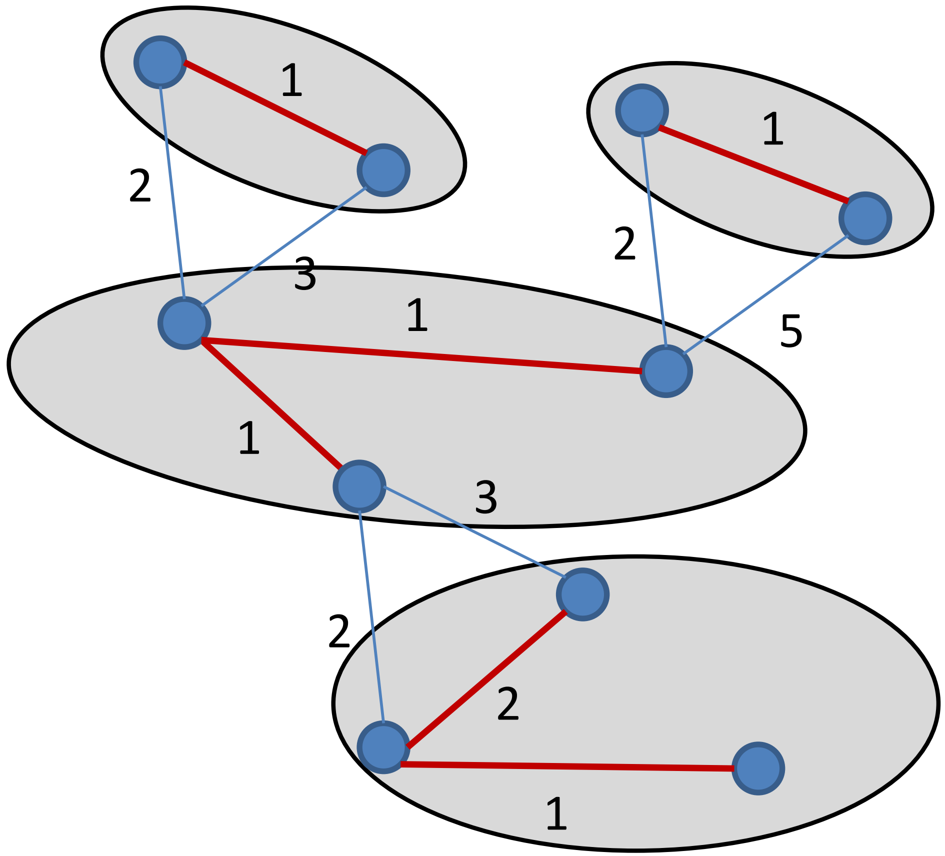
Merge components connected by MWOE.



Minimum Spanning Tree

Boruvka's Algorithm

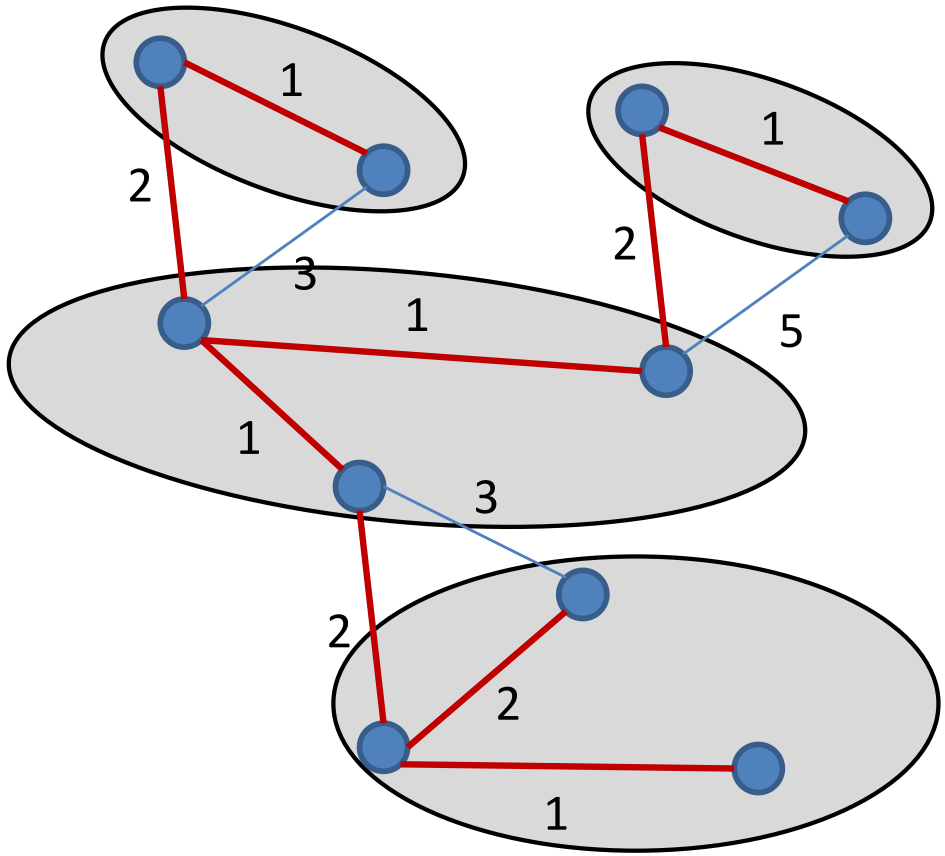
Repeat



Minimum Spanning Tree

Boruvka's Algorithm

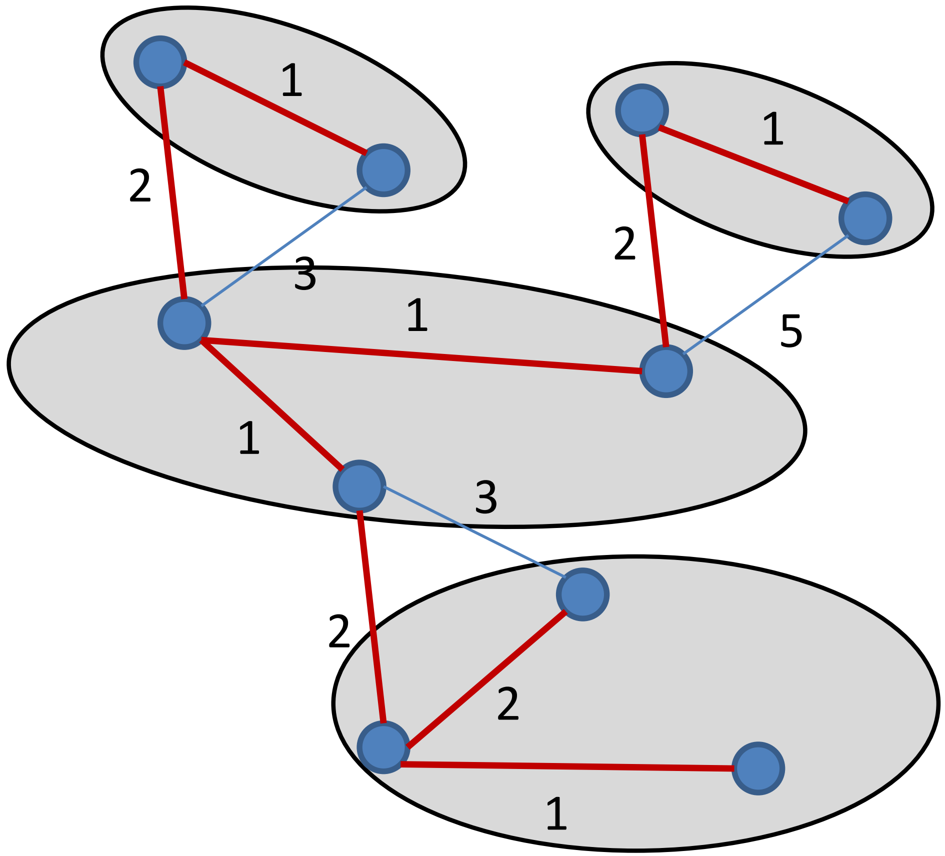
Find and add MWOE



Minimum Spanning Tree

Boruvka's Algorithm

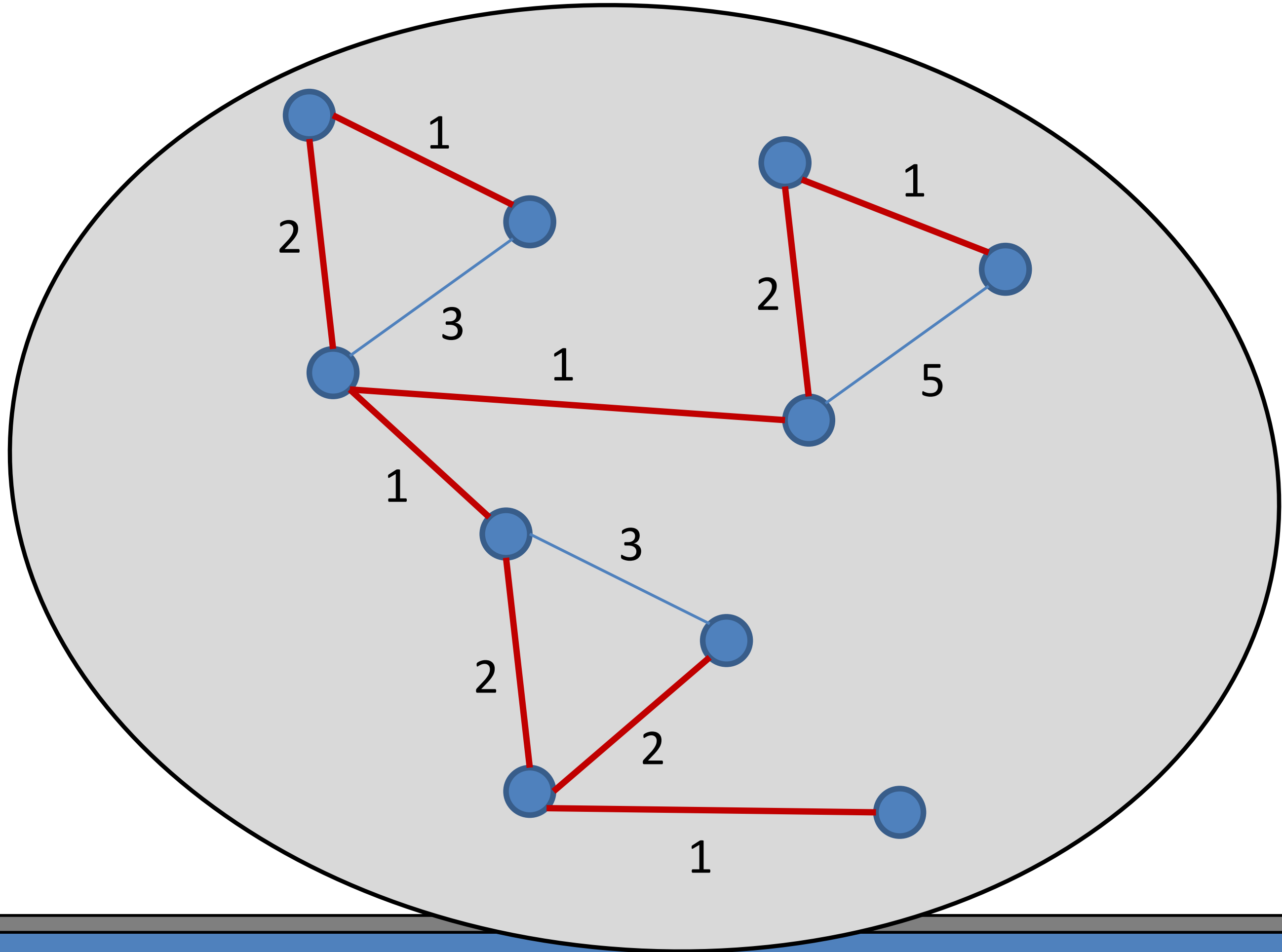
Merge components connected by MWOE.



Minimum Spanning Tree

Boruvka's Algorithm

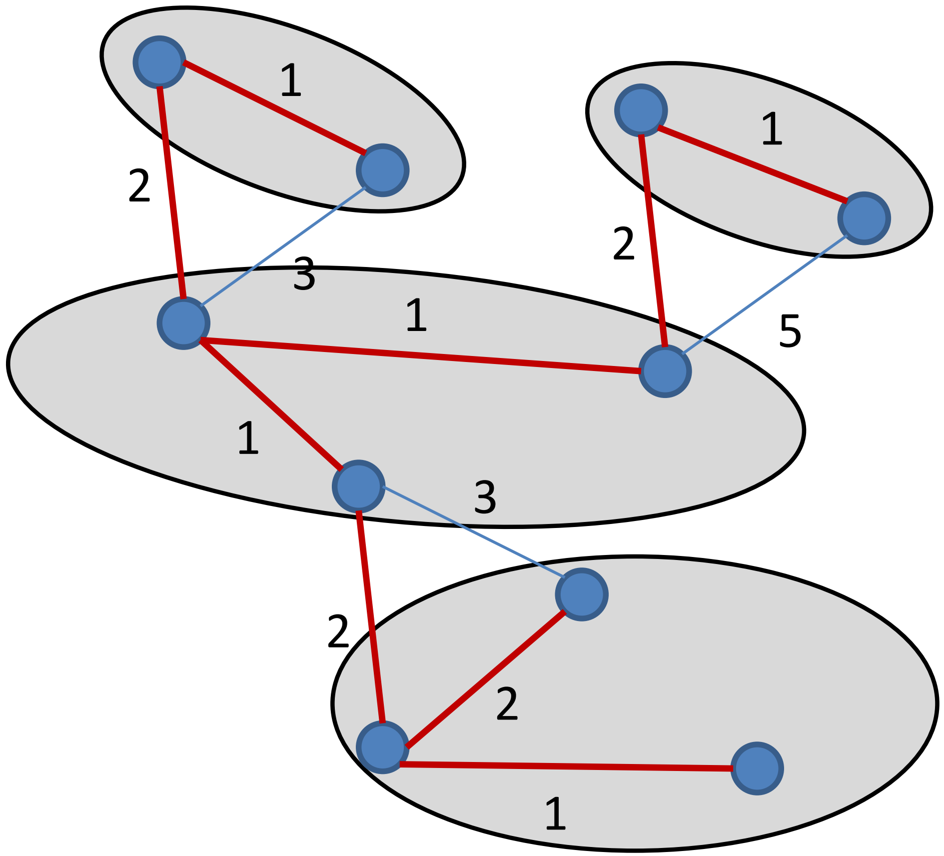
Merge components connected by MWOE.



Minimum Spanning Tree

Boruvka's Algorithm

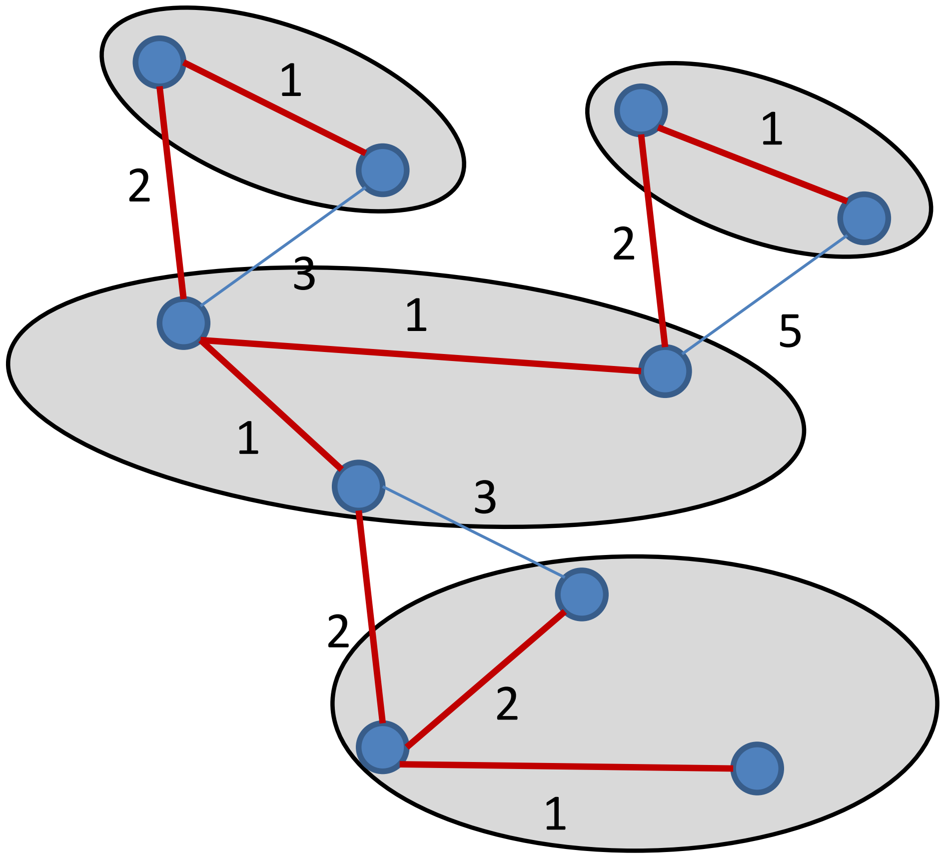
Claim: in each step, the number of components at least divides by 2.



Minimum Spanning Tree

Boruvka's Algorithm

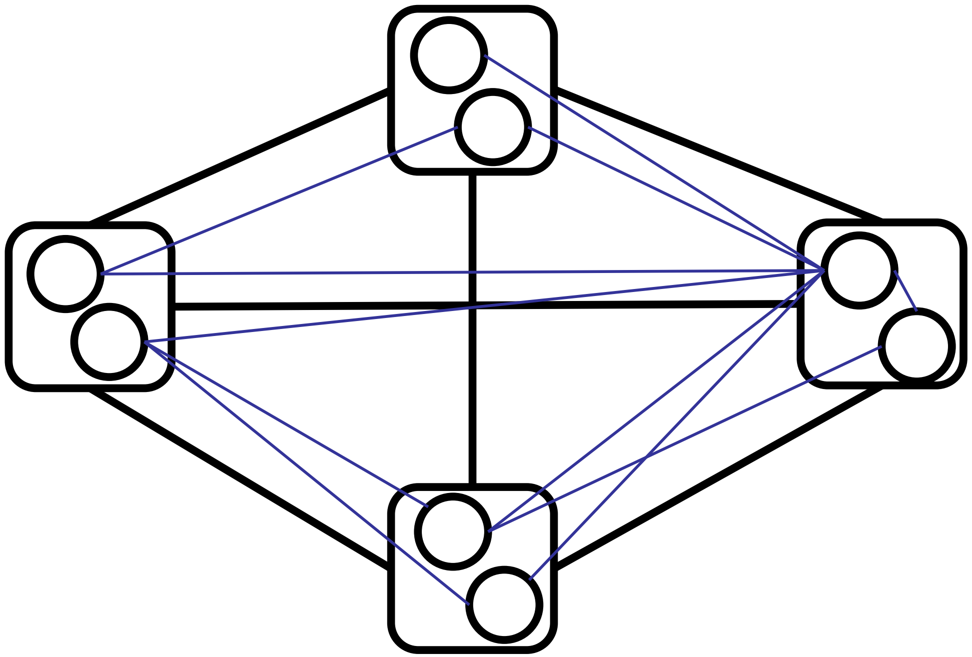
Claim: Terminates in O(log n) iterations.



Boruvka's Algorithm

Repeat log n times:

1. Find minimum weight outgoing edge (MWOE) for each component.
2. Merge components connected by MWOEs.



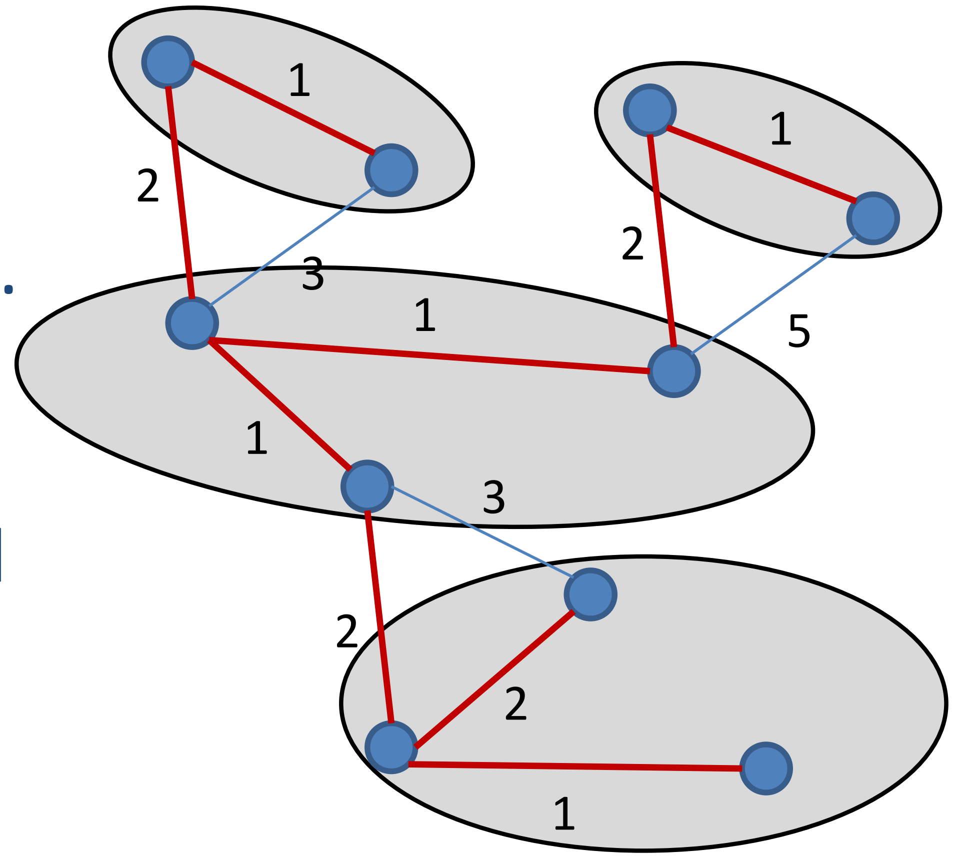
k-Machine Boruvka's Algorithm

Boruvka's Algorithm

Tag each node with its component identifier.

Initially, each node
is in its own component.

Component id = node id

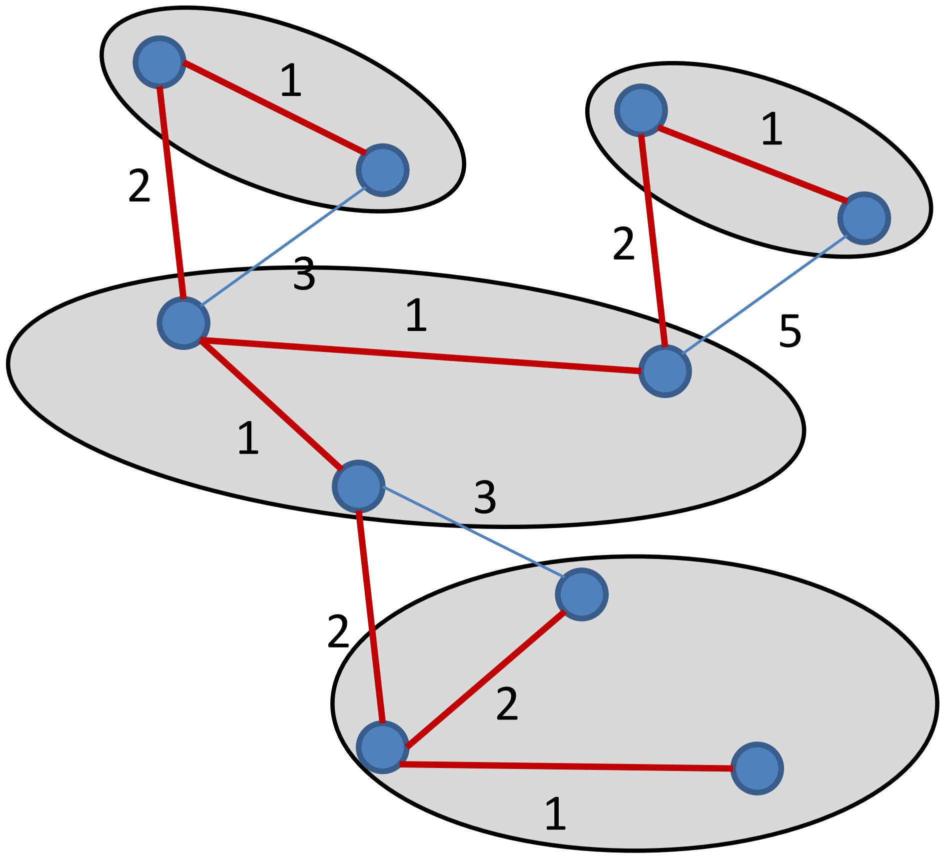


k-Machine Boruvka's Algorithm

Boruvka's Algorithm

Every node broadcasts *to everyone* its component id.

Each node now knows
the component
id of each neighbor
in the graph.

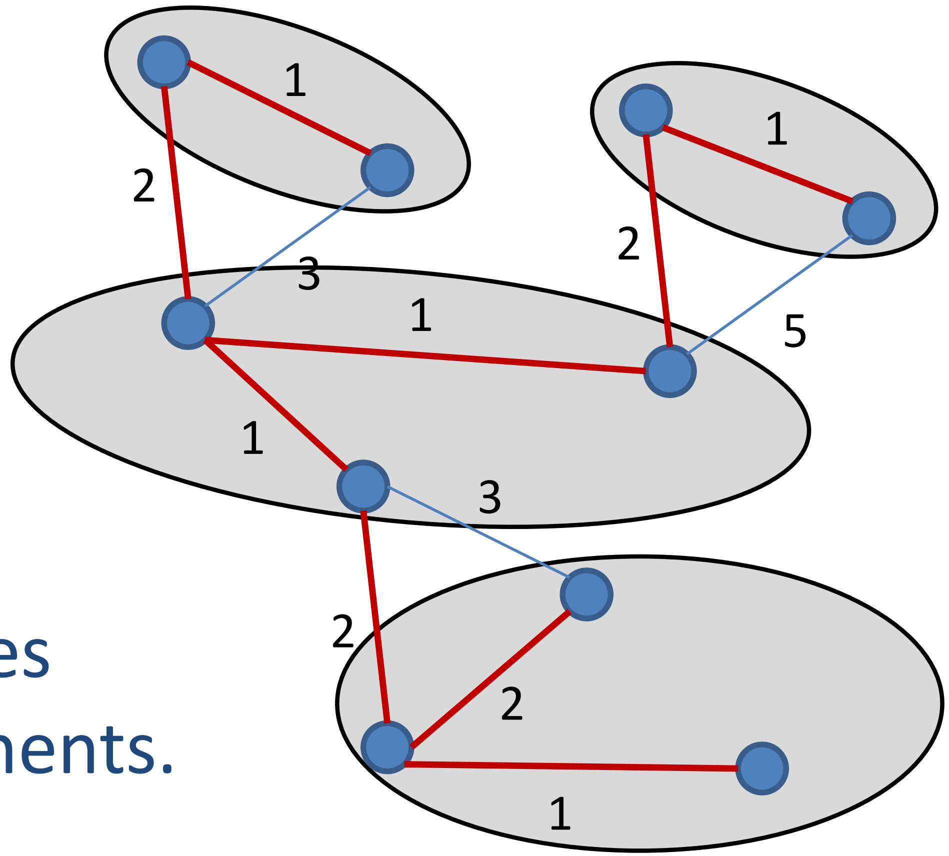


k-Machine Boruvka's Algorithm

Boruvka's Algorithm

Every node computes its MWOE.

Each node now knows
the component
id of each neighbor
in the graph.



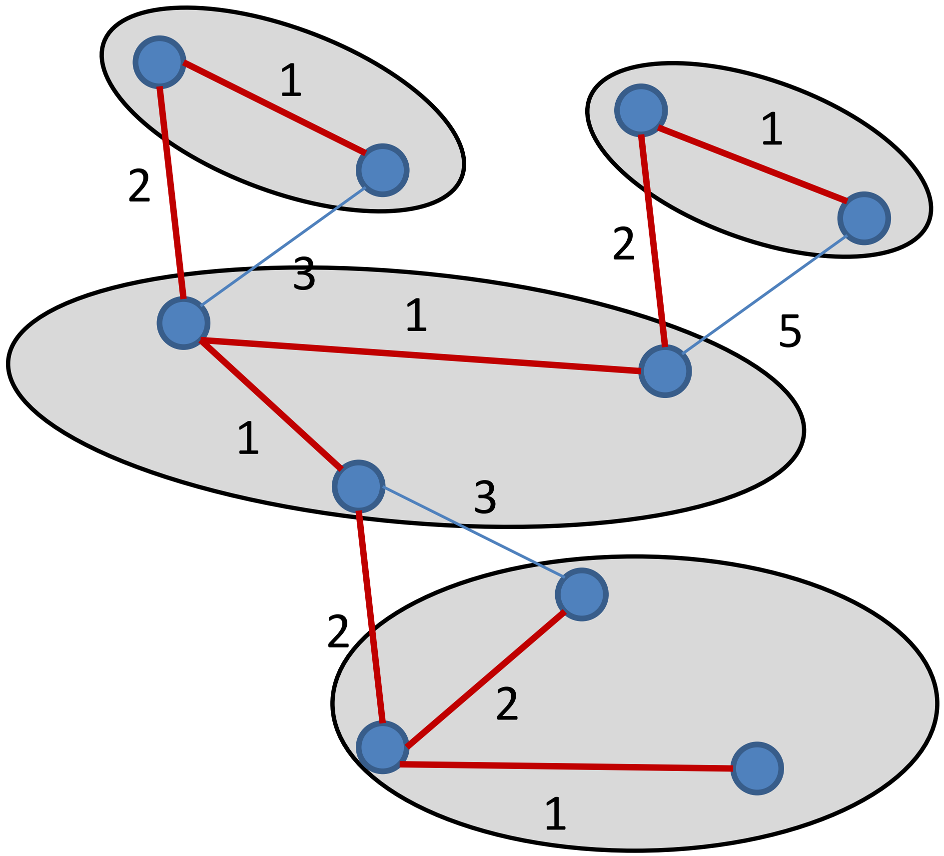
So it considers only edges
that go to other components.

k-Machine Boruvka's Algorithm

Boruvka's Algorithm

Every node broadcasts its MWOE *to everyone*.

Each node can compute MWOE for its component because it knows MWOE for every node in its component.



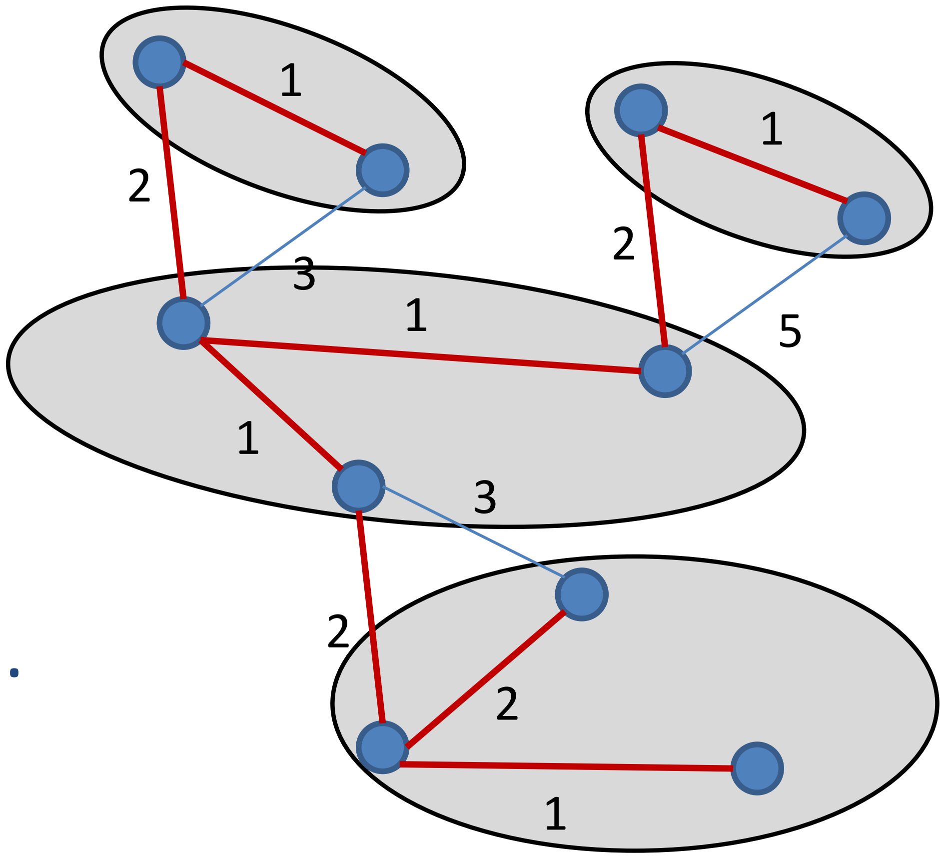
k-Machine Boruvka's Algorithm

Boruvka's Algorithm

Every node broadcasts its MWOE *to everyone*.

Each node can compute MWOE for all components!

Can find all components that you will merge with.

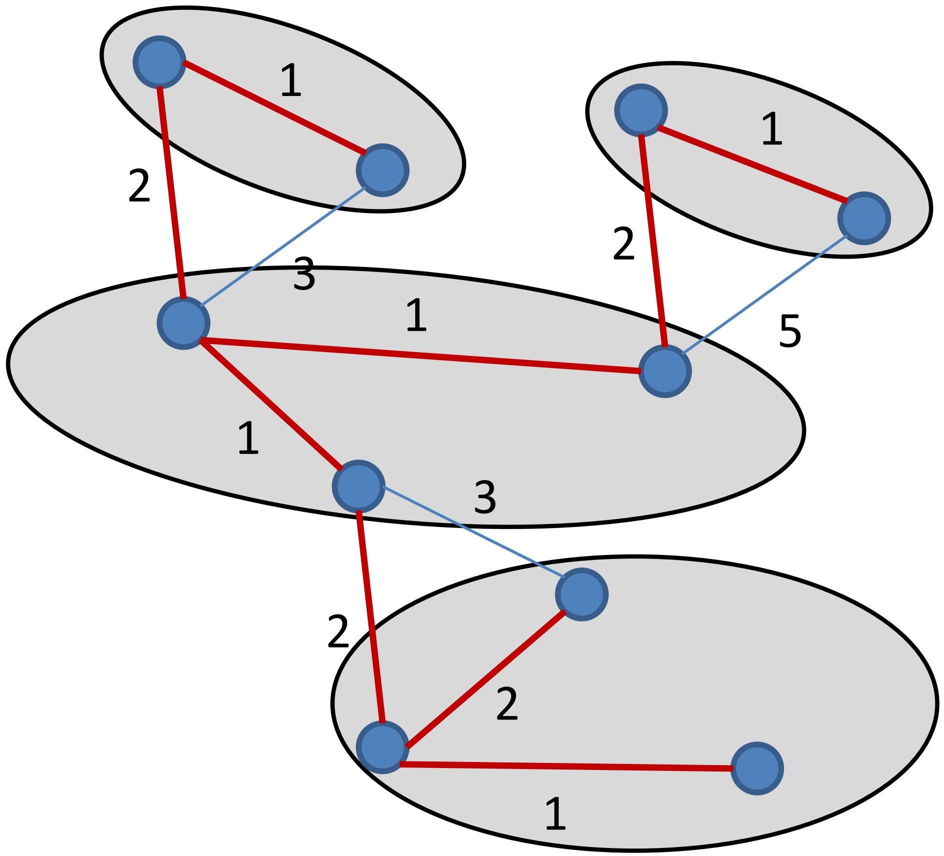


k-Machine Boruvka's Algorithm

Boruvka's Algorithm

Compute new component id.

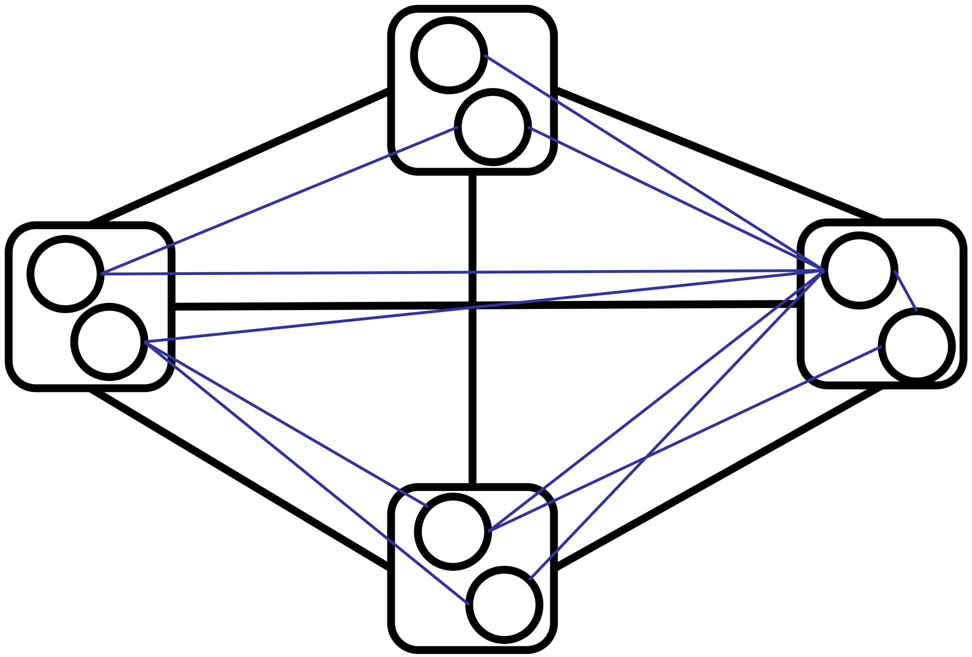
Find minimum
component id
of any component
that you merge with.



Boruvka's Algorithm

Repeat log n times:

1. Broadcast component id to all.
2. Broadcast MWOE to all.
3. Compute new component id.

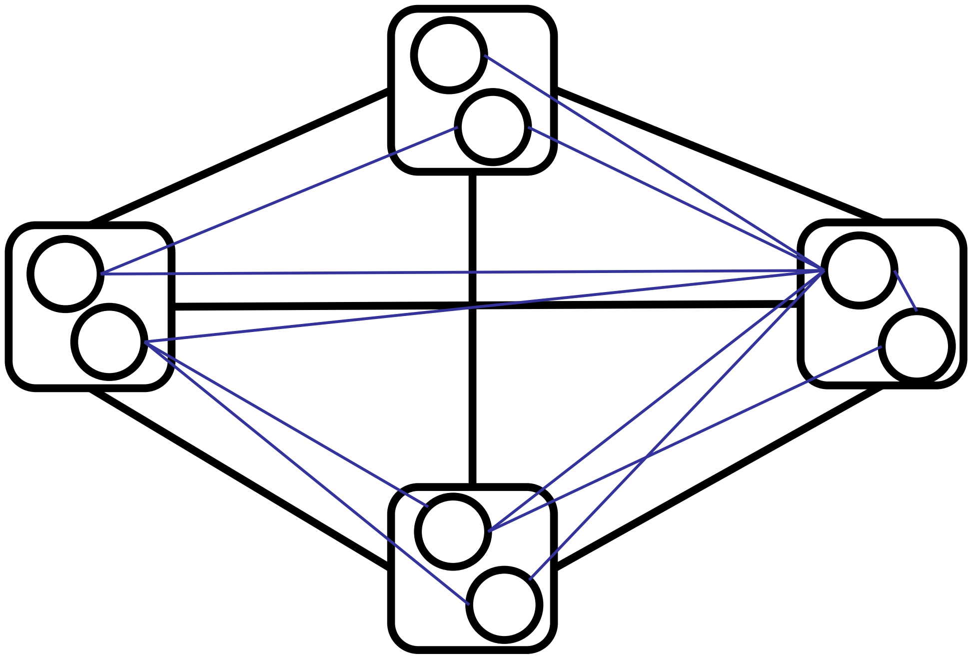


Boruvka's Algorithm

Repeat log n times:

1. Broadcast component id to all.
2. Broadcast MWOE to all.
3. Compute new component id.

What is the cost of broadcasting a message to "all" nodes in the graph?



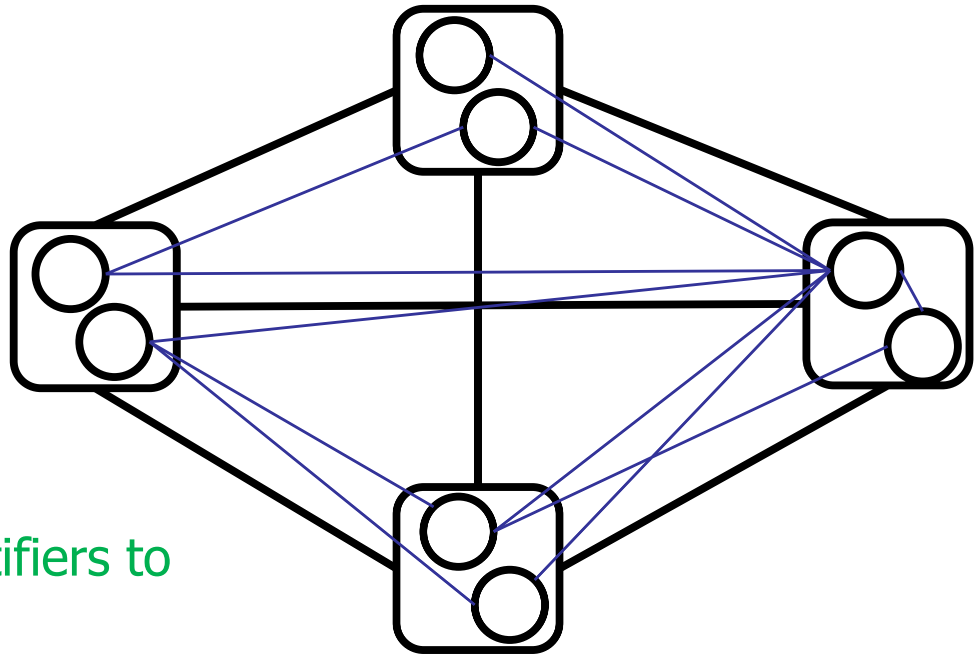
Boruvka's Algorithm

Repeat log n times:

1. Broadcast component id to all.
2. Broadcast MWOE to all.
3. Compute new component id.

What is the cost of broadcasting a message to "all" nodes in the graph?

$$O\left(\frac{1}{B}\frac{n}{k}\right)$$



Each machine needs to send n/k identifiers to all k other machines.

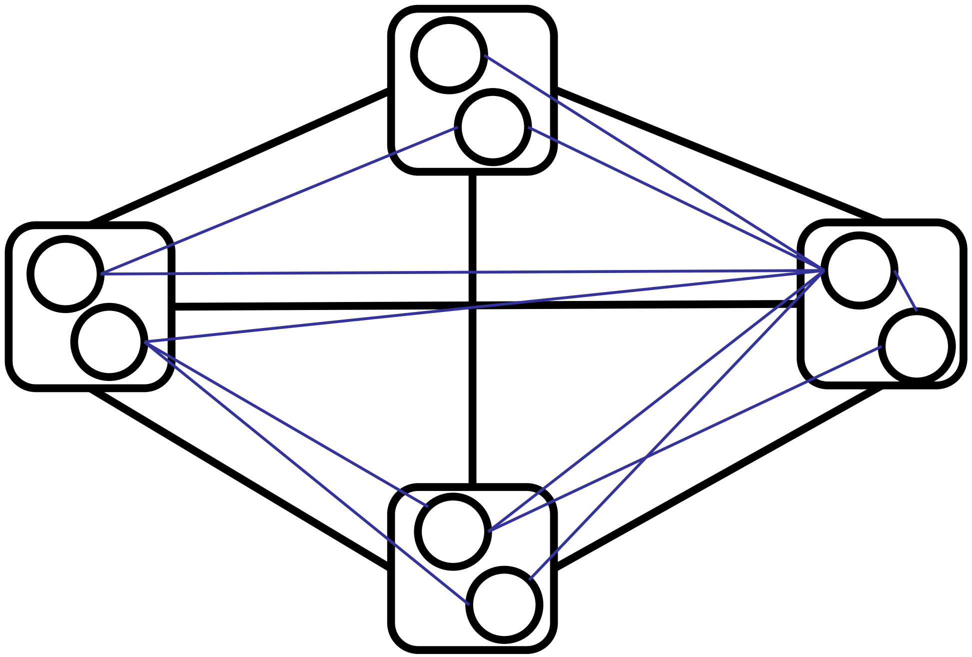
Boruvka's Algorithm

Repeat log n times:

1. Broadcast component id to all.
2. Broadcast MWOE to all.
3. Compute new component id.

Total running time:

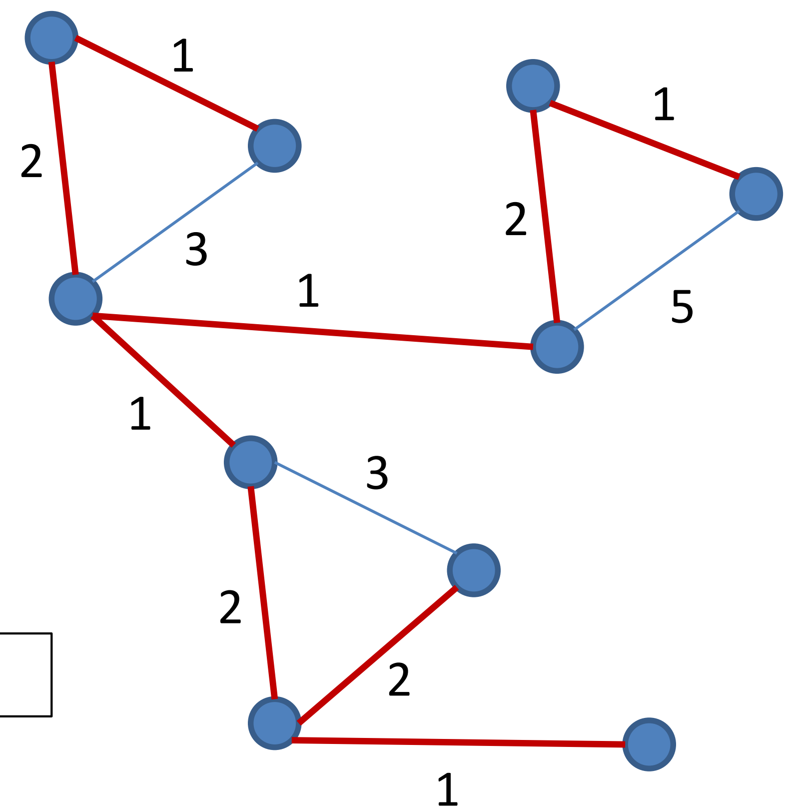
$$O\left(\frac{1}{B}\frac{n}{k}\log n\right)$$



Fully Distributed Model

CONGEST Model

Assume each node in the graph is its own machine.



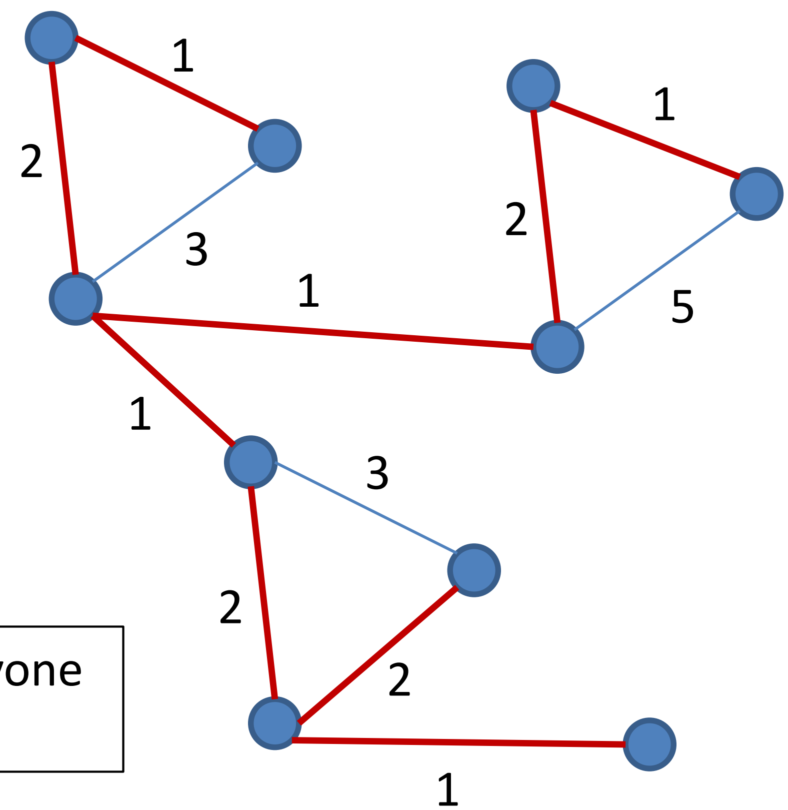
Almost like k=n?

Fully Distributed Model

CONGEST Model

Assume each node in the graph is its own machine.

Each edge in the graph is a real communication edge.



Cannot send message to everyone like in k-machine model.

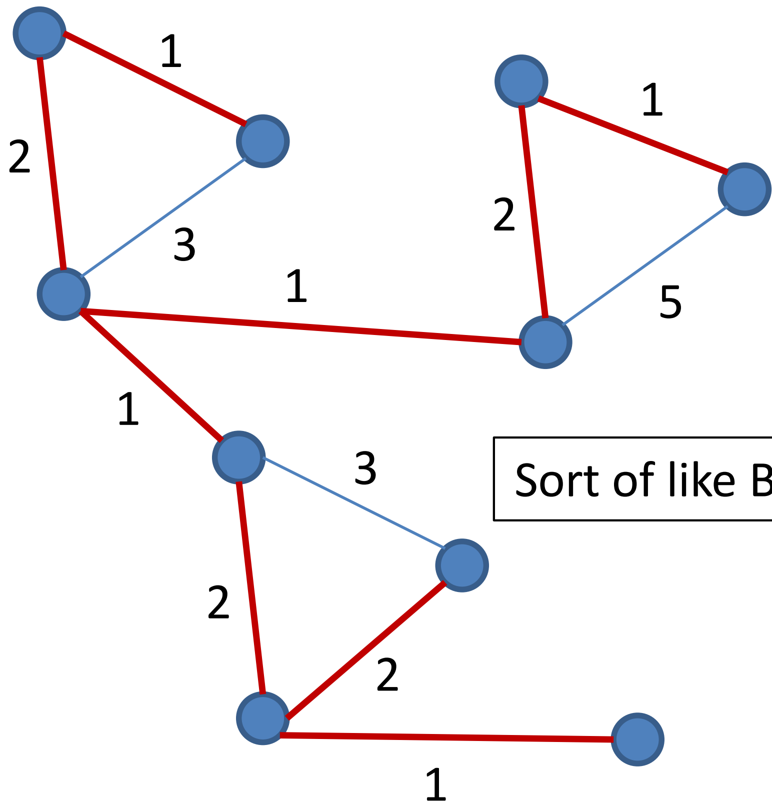
Fully Distributed Model

CONGEST Model

Assume each node in the graph is its own machine.

Each edge in the graph is a real communication edge.

Each edge carries 1 message per round.



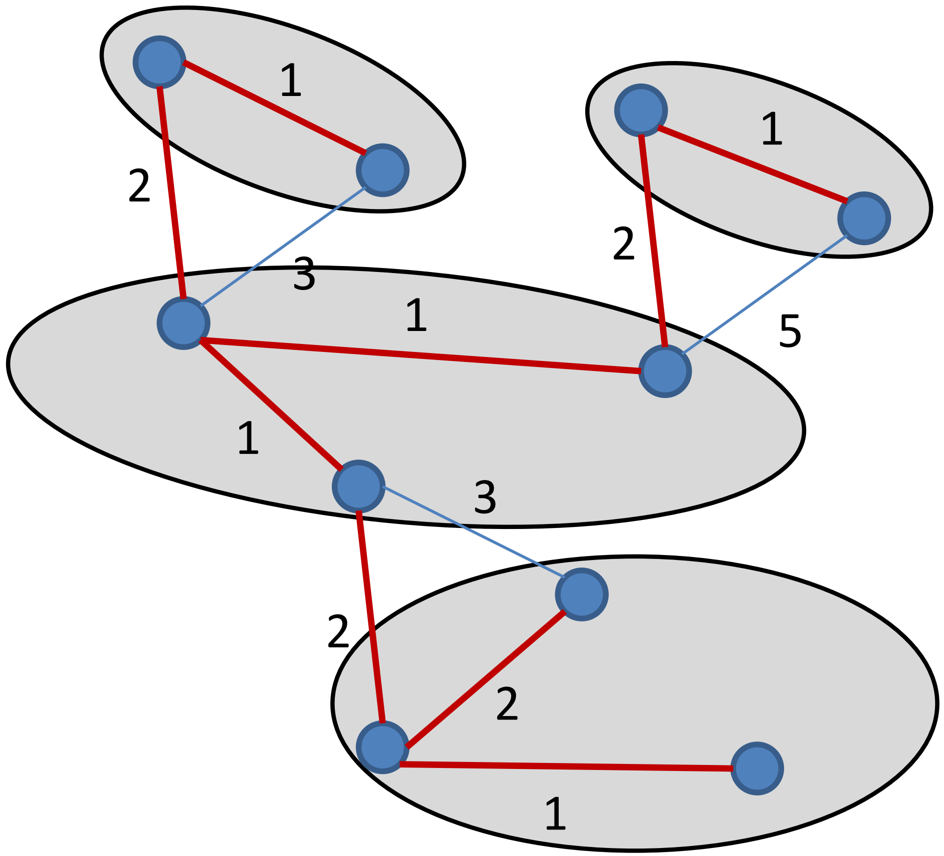
Sort of like B=log n

Fully Distributed Model

Boruvka's Algorithm

Key challenge:

Find minimum weight outgoing edge for a component.



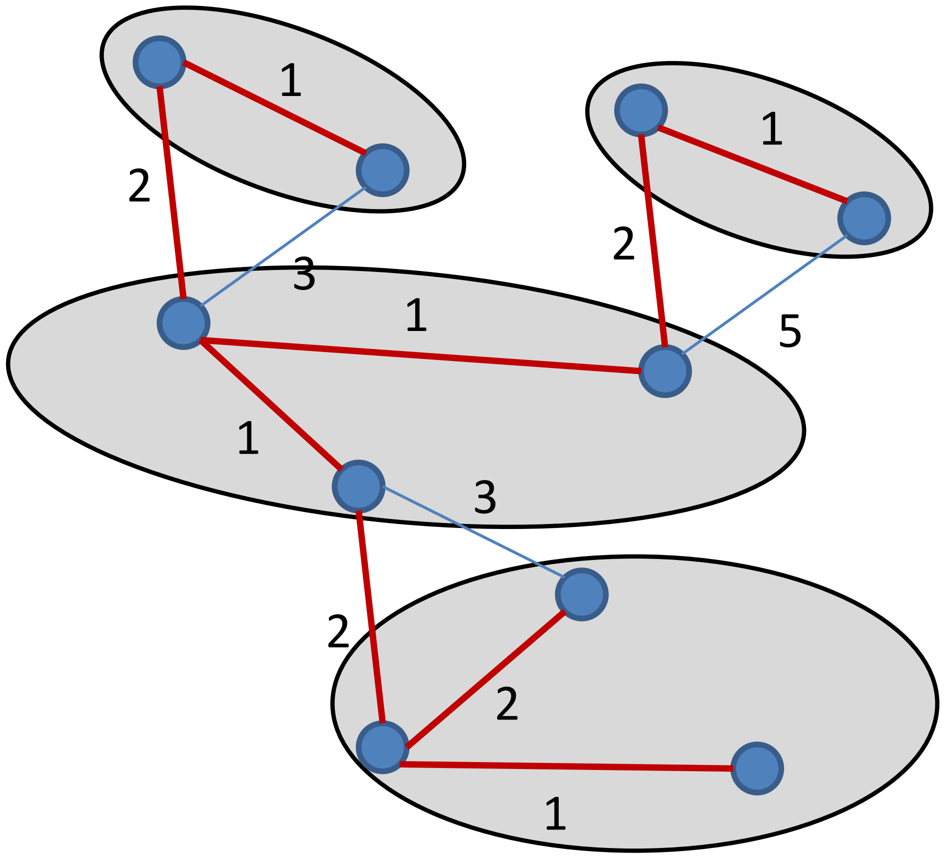
Fully Distributed Model

Boruvka's Algorithm

Step 1:

Each node sends a message to all its neighbors with its component id.

O(1) rounds

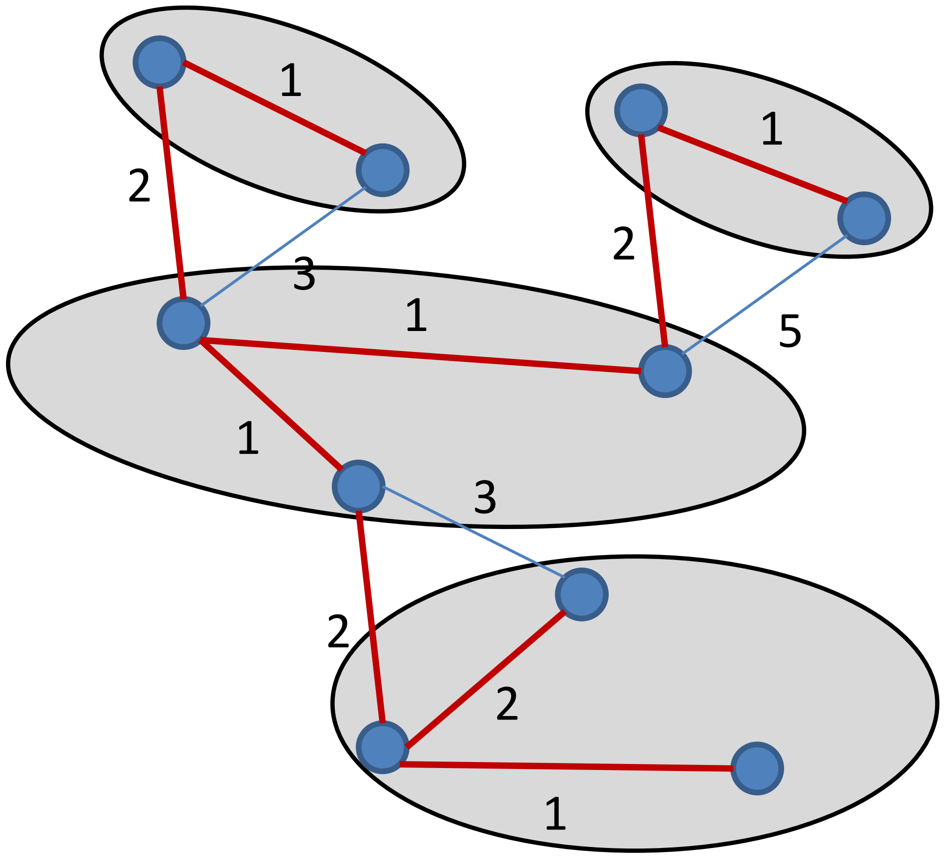


Fully Distributed Model

Boruvka's Algorithm

Step 2:

Each node computes its minimum weight outgoing edge to a different component.



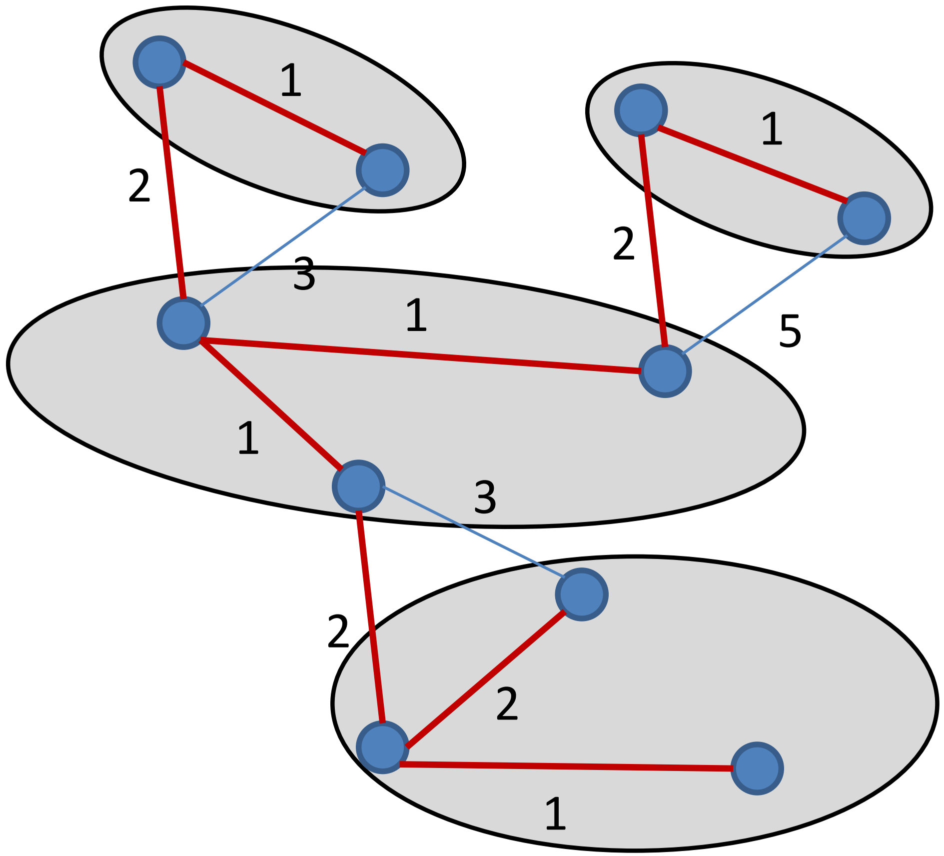
0 rounds

Fully Distributed Model

Boruvka's Algorithm

Step 3:

Send MWOE on the MST tree edges in your component.

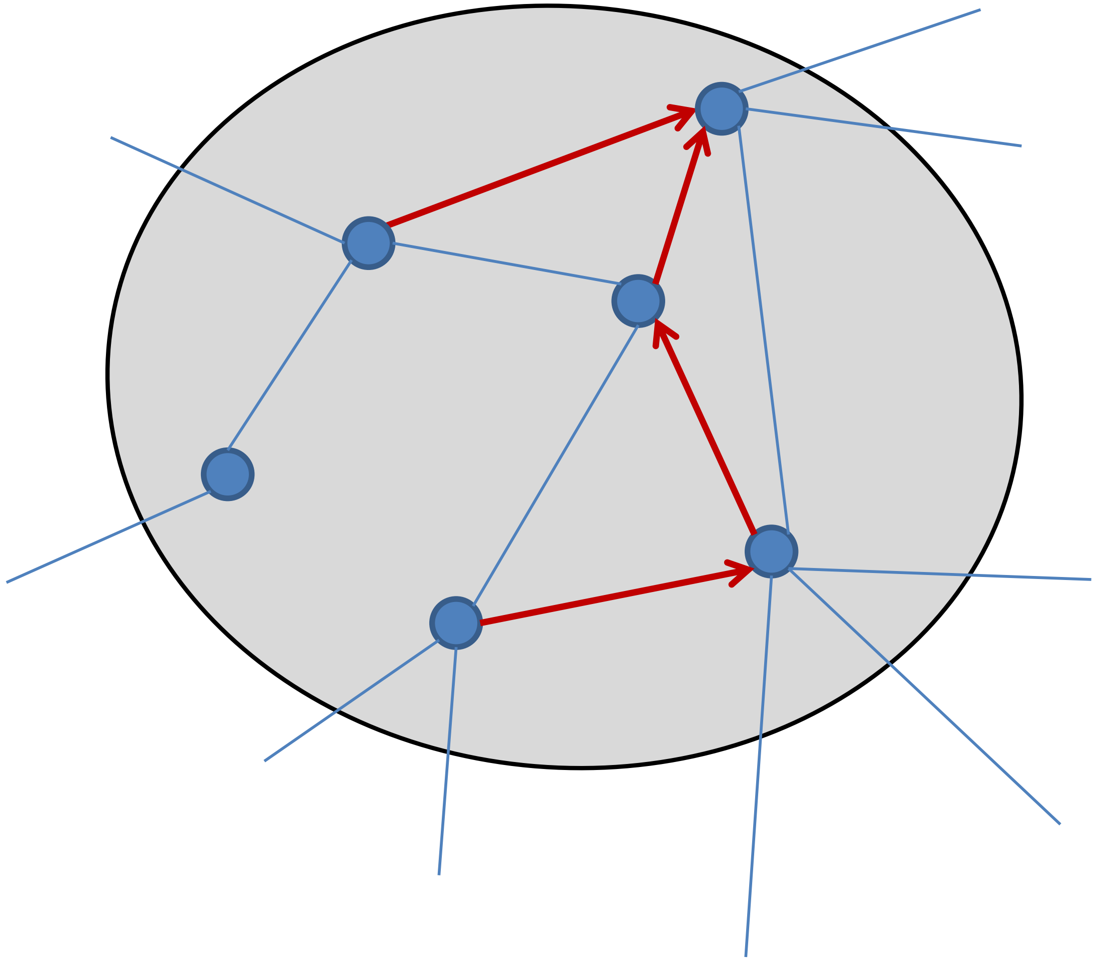


Fully Distributed Model

Boruvka's Algorithm

Detail:

Maintain MST fragment in component as a rooted tree.

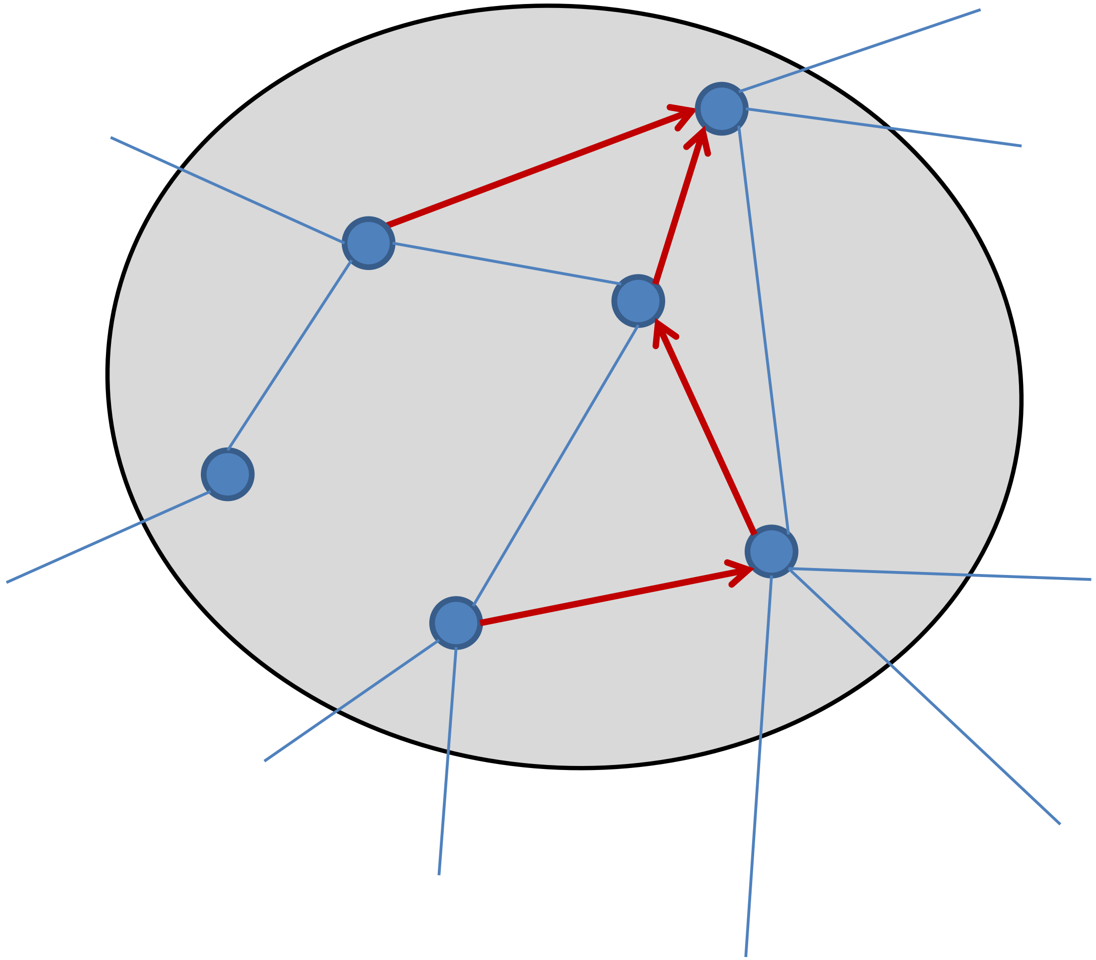


Fully Distributed Model

Boruvka's Algorithm

Detail:

Each node broadcasts MWOE up the tree to the root.

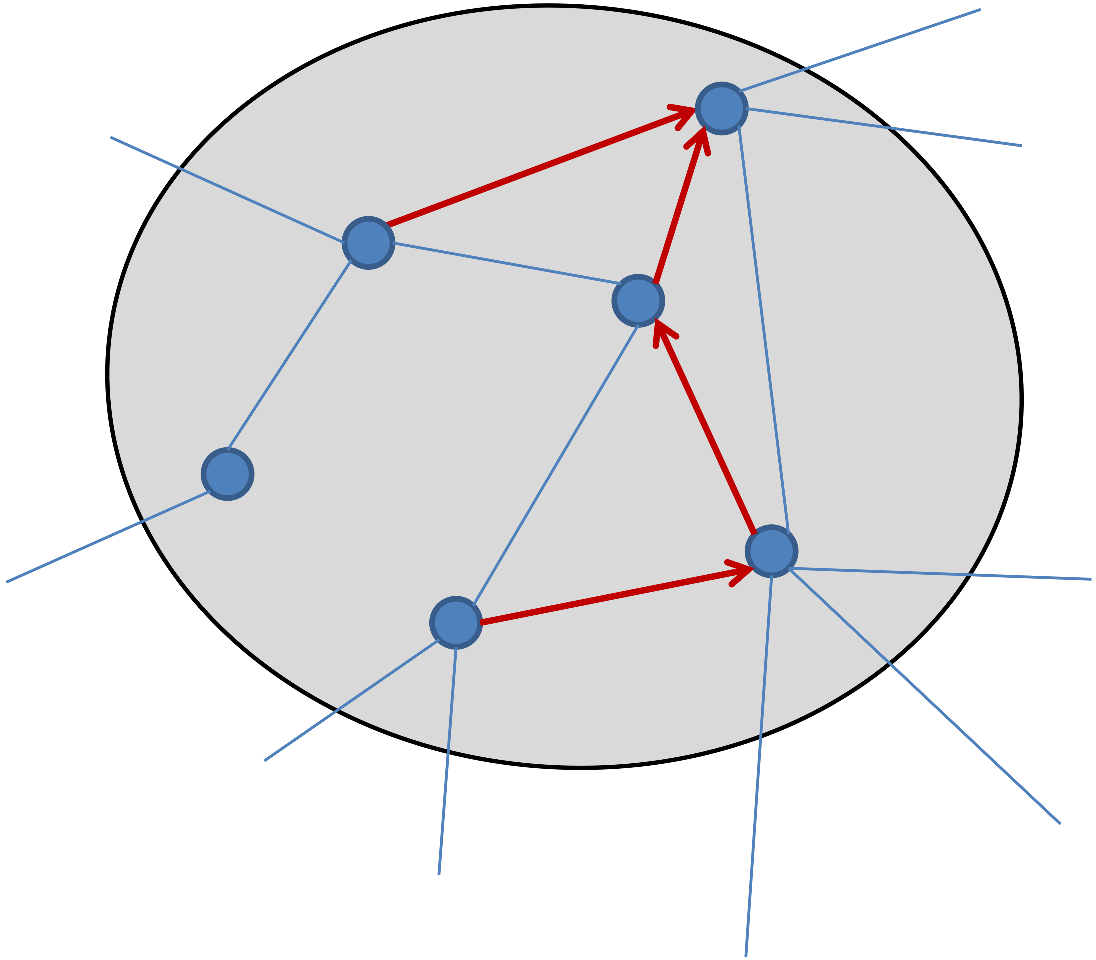


Fully Distributed Model

Boruvka's Algorithm

Detail:

Root chooses smallest weight MWOE.

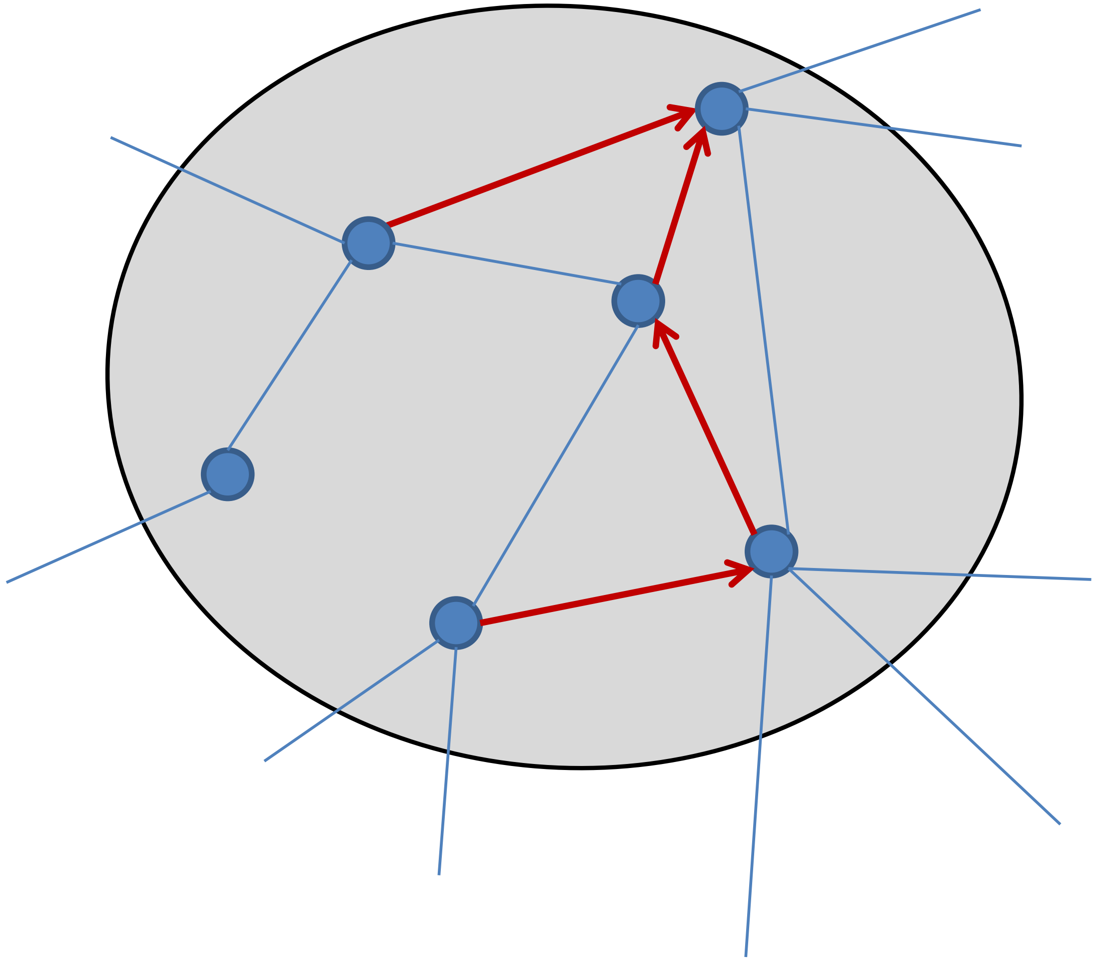


Fully Distributed Model

Boruvka's Algorithm

Detail:

Root broadcasts MWOE to everyone.

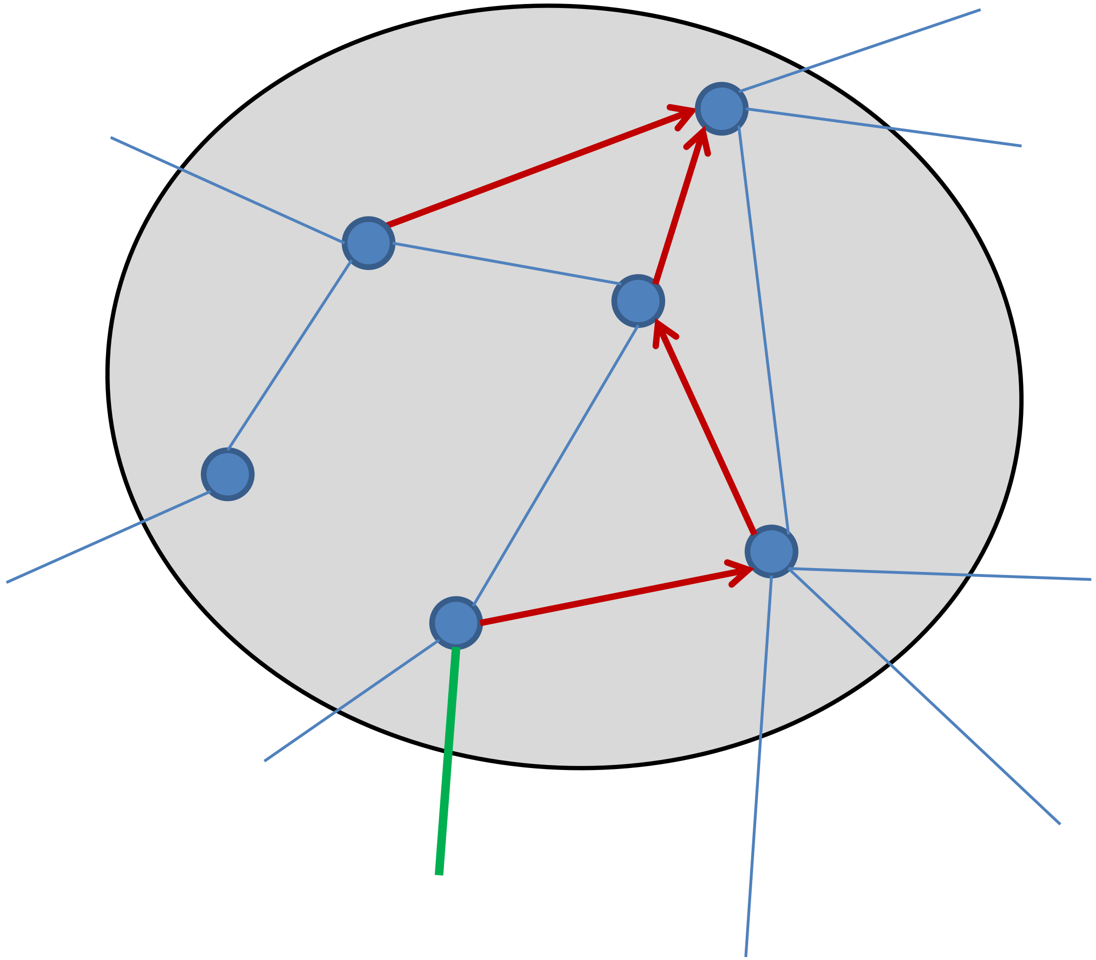


Fully Distributed Model

Boruvka's Algorithm

Detail:

Merge edge
is found.

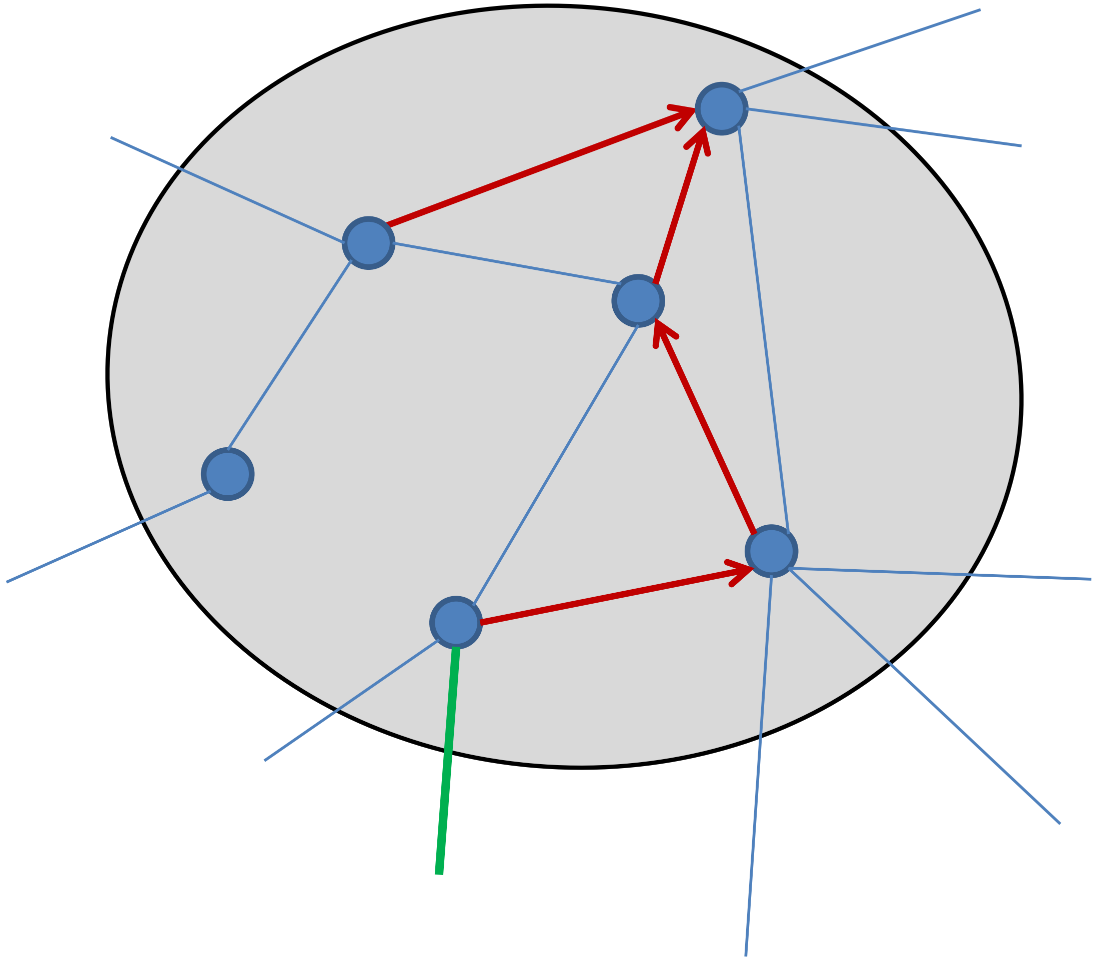


Fully Distributed Model

Boruvka's Algorithm

Detail:

Flood minimum component id through all merged components.

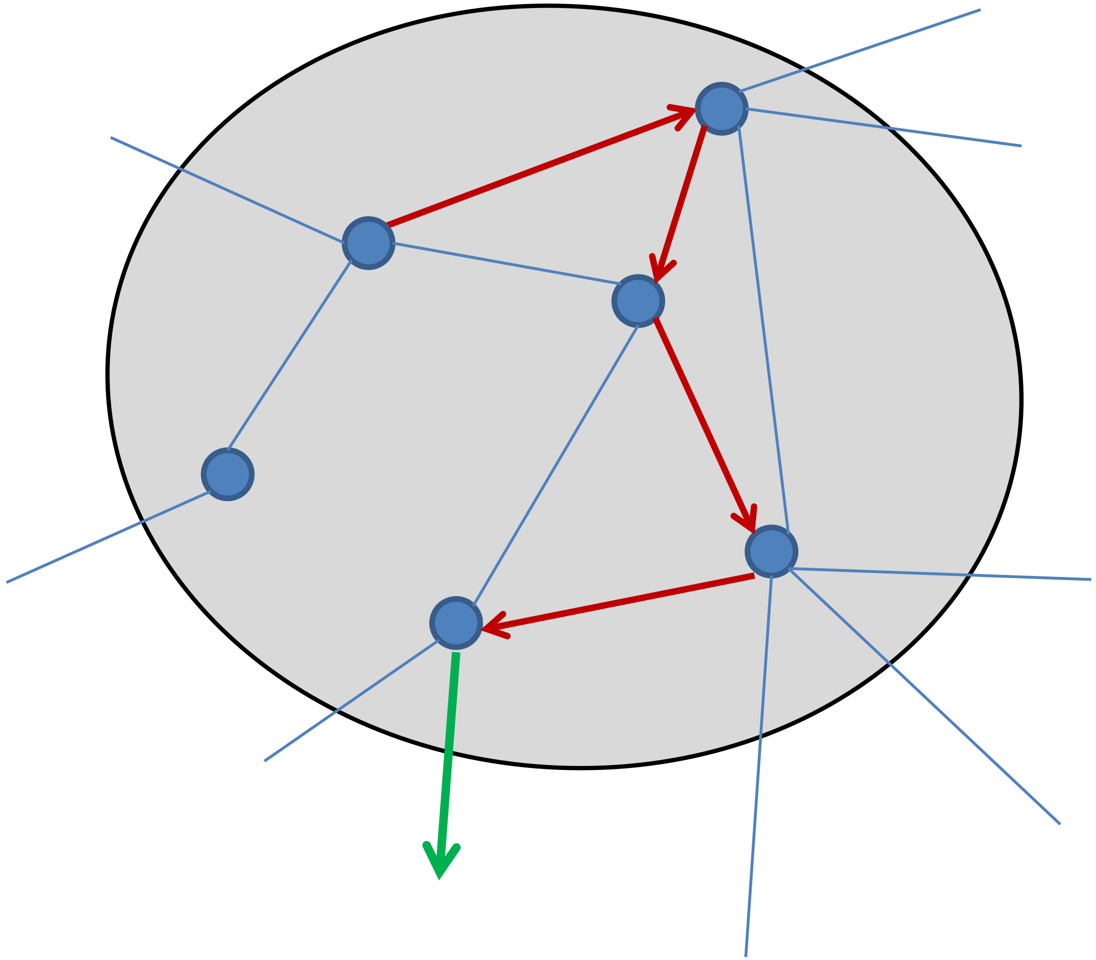


Fully Distributed Model

Boruvka's Algorithm

Detail:

If root is in another component, reorient tree.



Fully Distributed Model

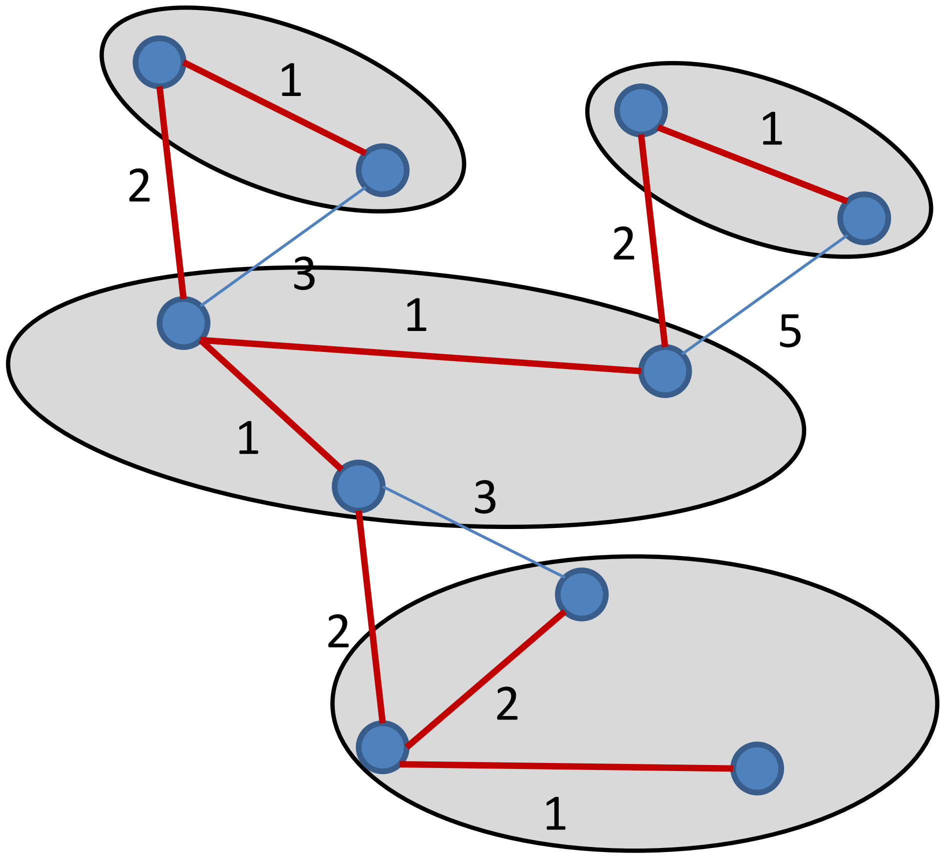
Boruvka's Algorithm

Step 3:

Send MWOE on the MST tree edges in your component.

And merge.

Any problem?

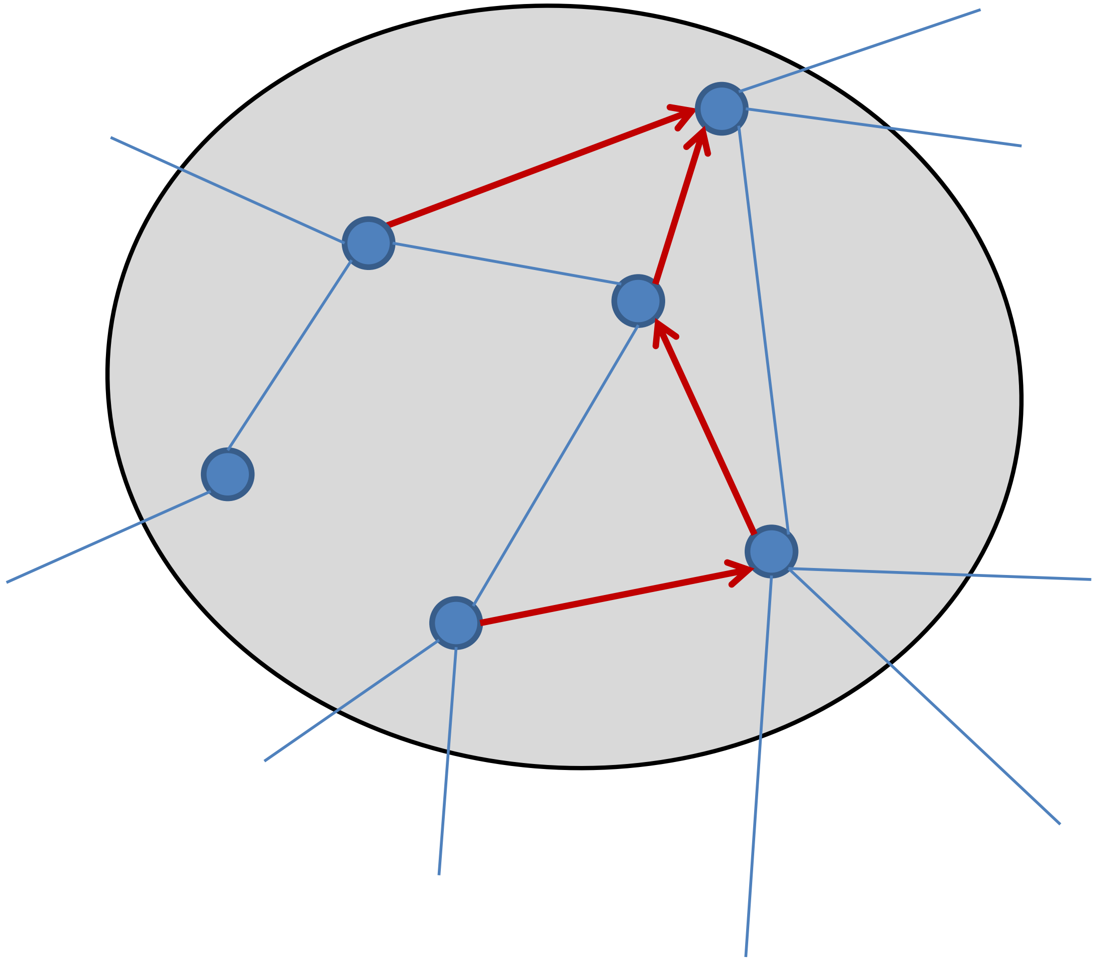


Fully Distributed Model

Boruvka's Algorithm

Detail:

Each node broadcasts MWOE up the tree to the root.



Time: $\Omega(n)$

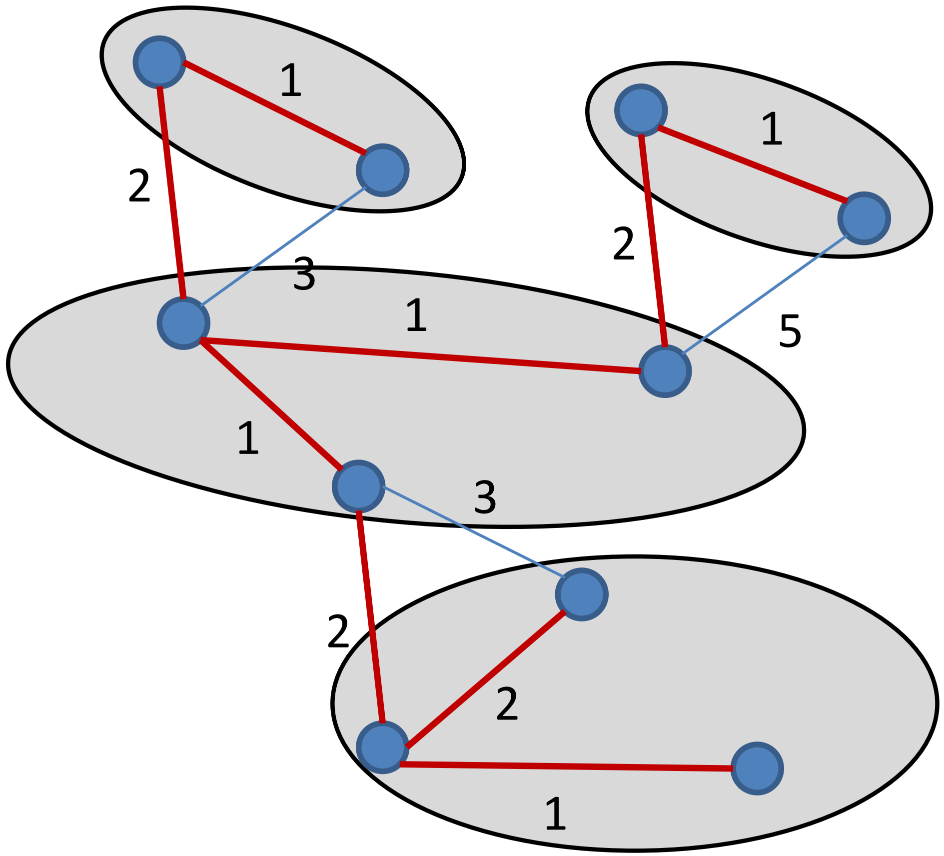
Fully Distributed Model

Boruvka's Algorithm

Step 3 (revised):

If component size is $< n^{1/2}$, then send MWOE on the MST tree edges in your component, and merge.

Time: $O(n^{1/2})$



Fully Distributed Model

Boruvka's Algorithm

Repeat until all components are size $>n^{1/2}$:

1. Find MWOE for each node.
2. Collect MWOE for each component at the root of the component, using the MST fragment edges.
3. Merge components.

Time: $O(n^{1/2} \log n)$

Fully Distributed Model

Boruvka’s Algorithm

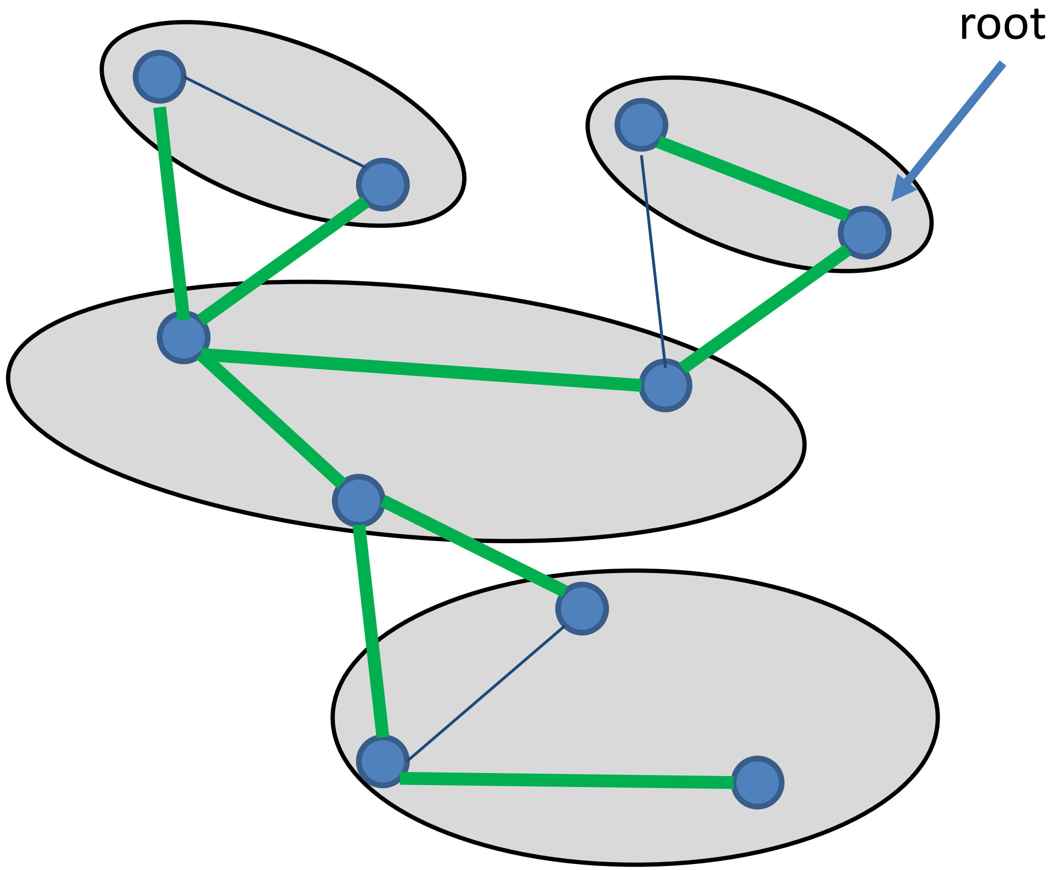
Idea 2: Use a BFS tree.

1. Find a BFS tree for the entire graph.
2. Collect MWOE for each component at the root of the BFS tree, using the BFS tree edges.
3. Merge components.

Fully Distributed Model

Boruvka's Algorithm

Idea 2: Use a BFS tree.



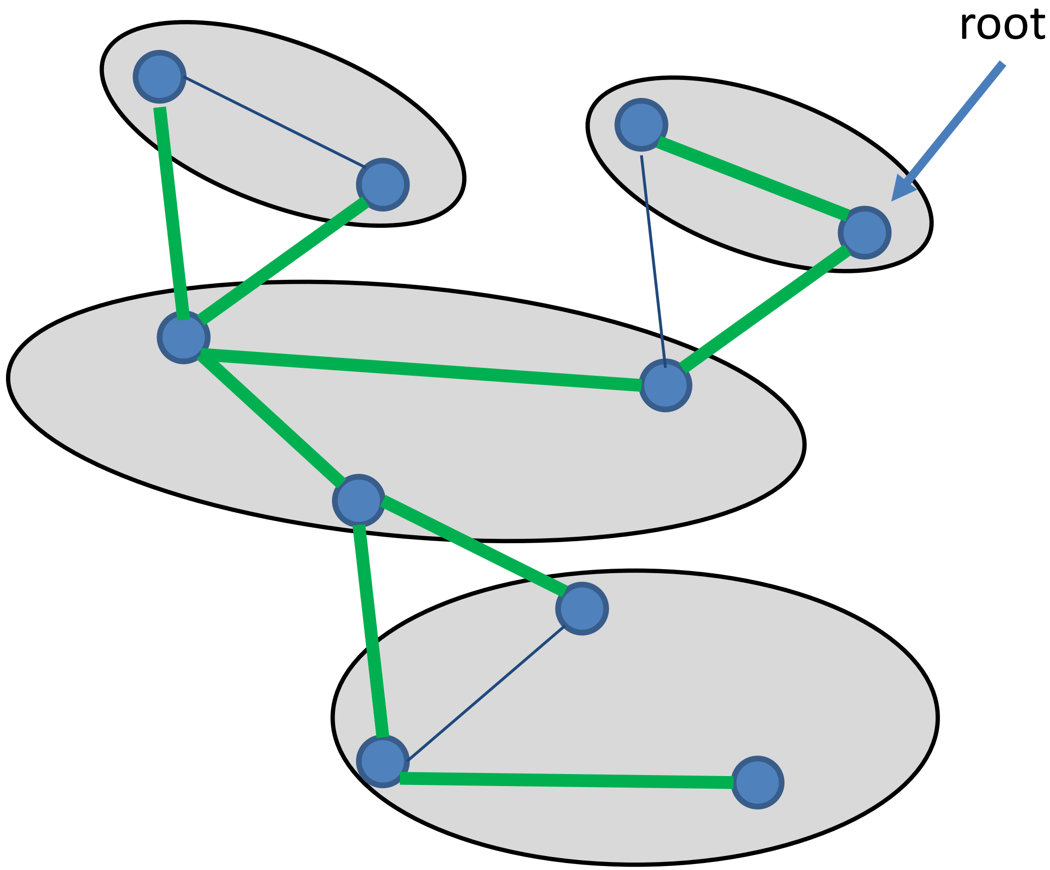
Fully Distributed Model

Boruvka's Algorithm

Idea 2: Use a BFS tree.

Easy to find.

Just have a root start broadcasting a message to all its neighbors.



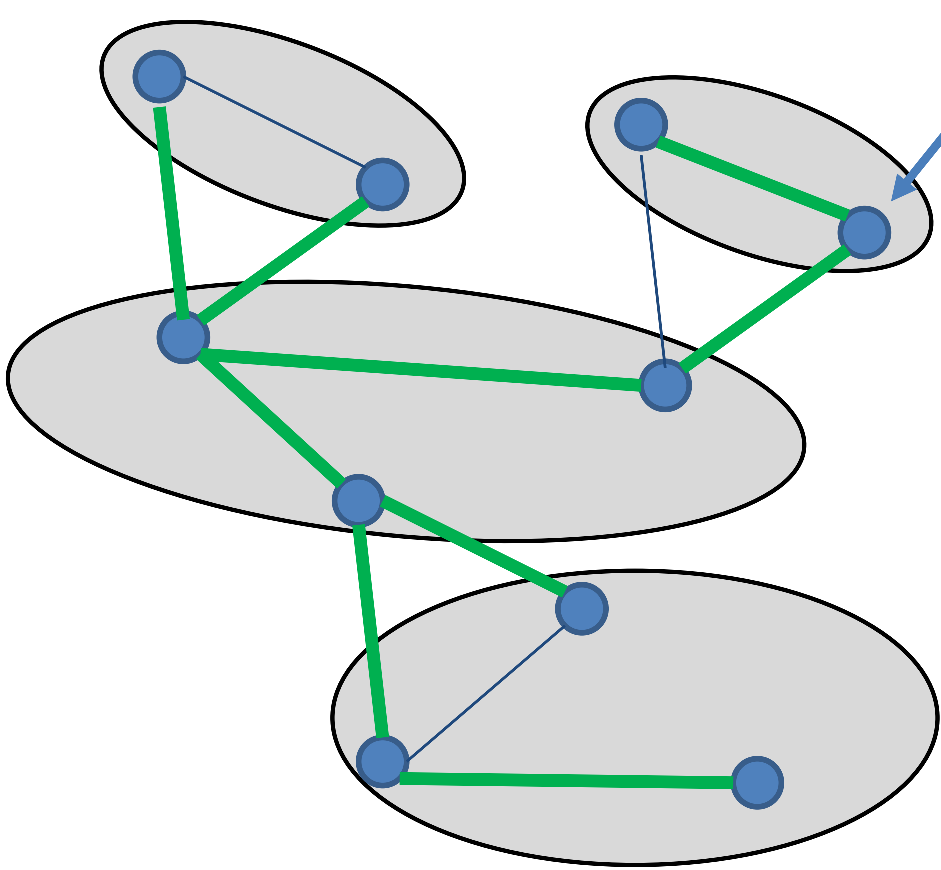
Fully Distributed Model

Boruvka's Algorithm

Idea 2: Use a BFS tree.

Easy to find.

When receive BFS message, then rebroadcast to your neighbors.



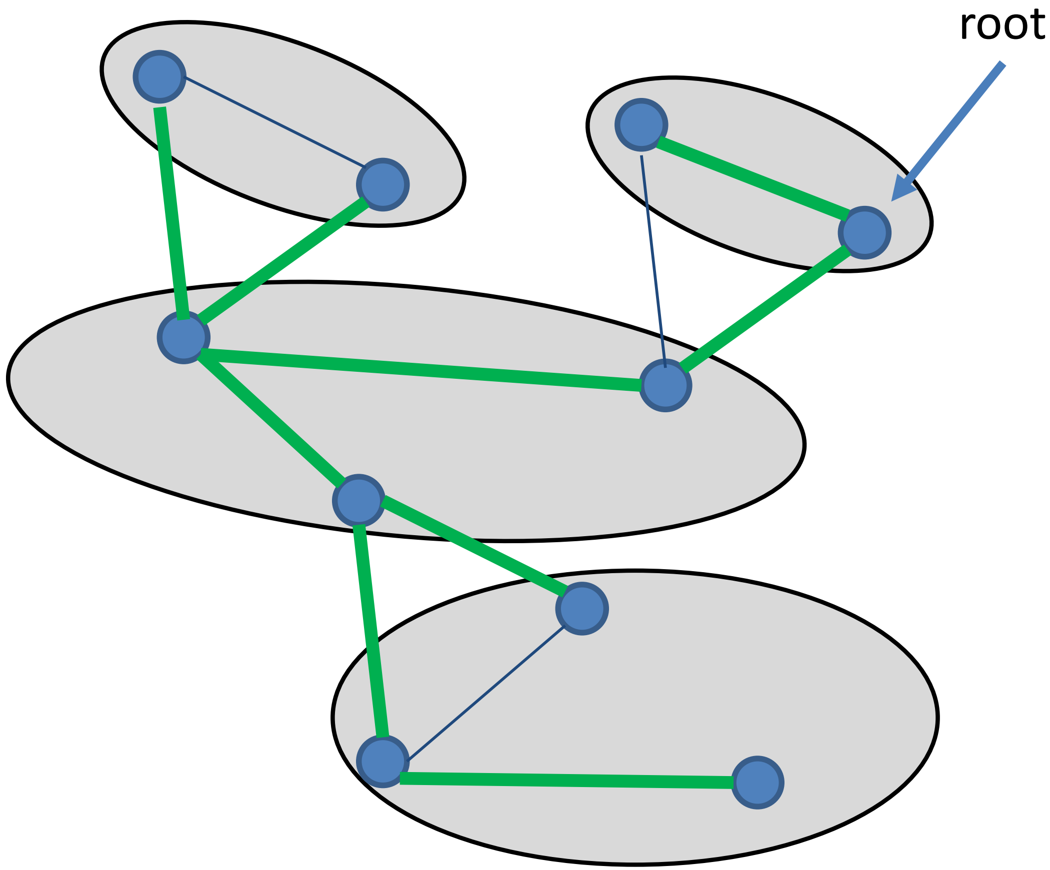
Fully Distributed Model

Boruvka's Algorithm

Idea 2: Use a BFS tree.

Easy to find.

Parent in BFS tree is first node that you received a message from.



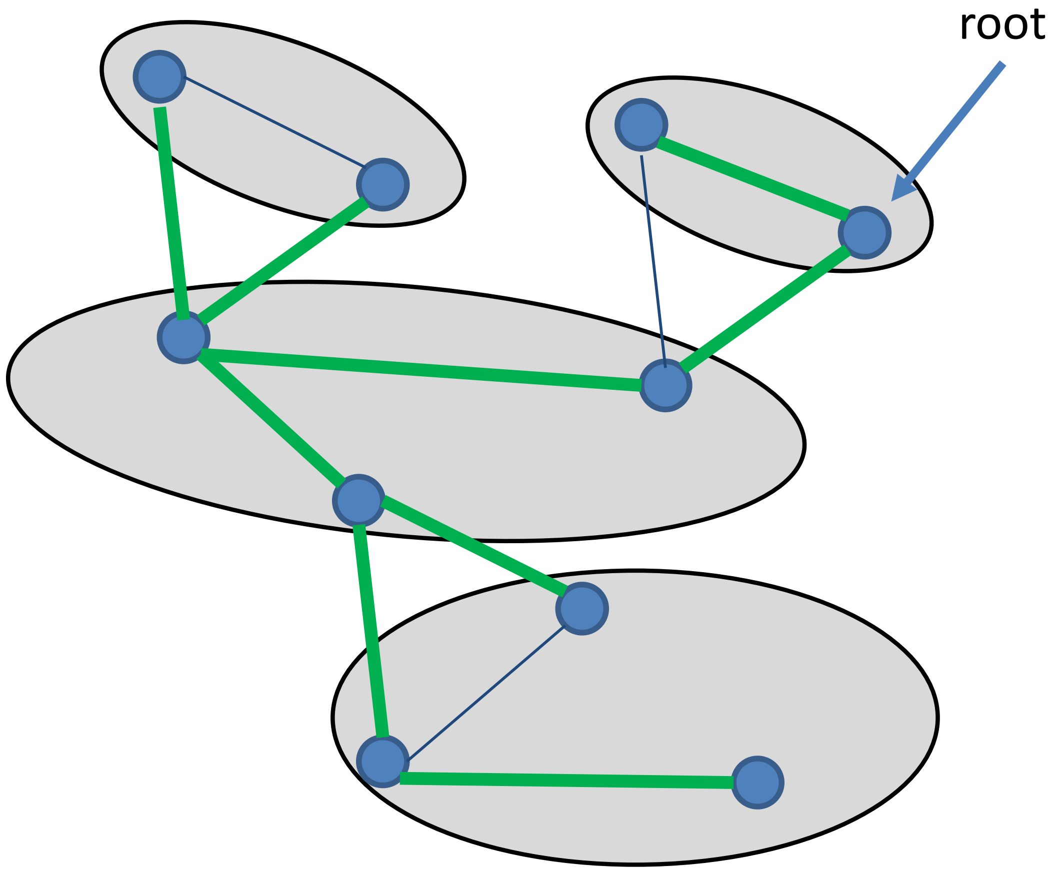
Fully Distributed Model

Boruvka's Algorithm

Idea 2: Use a BFS tree.

Max depth:
O(D)

D = diameter of
graph.

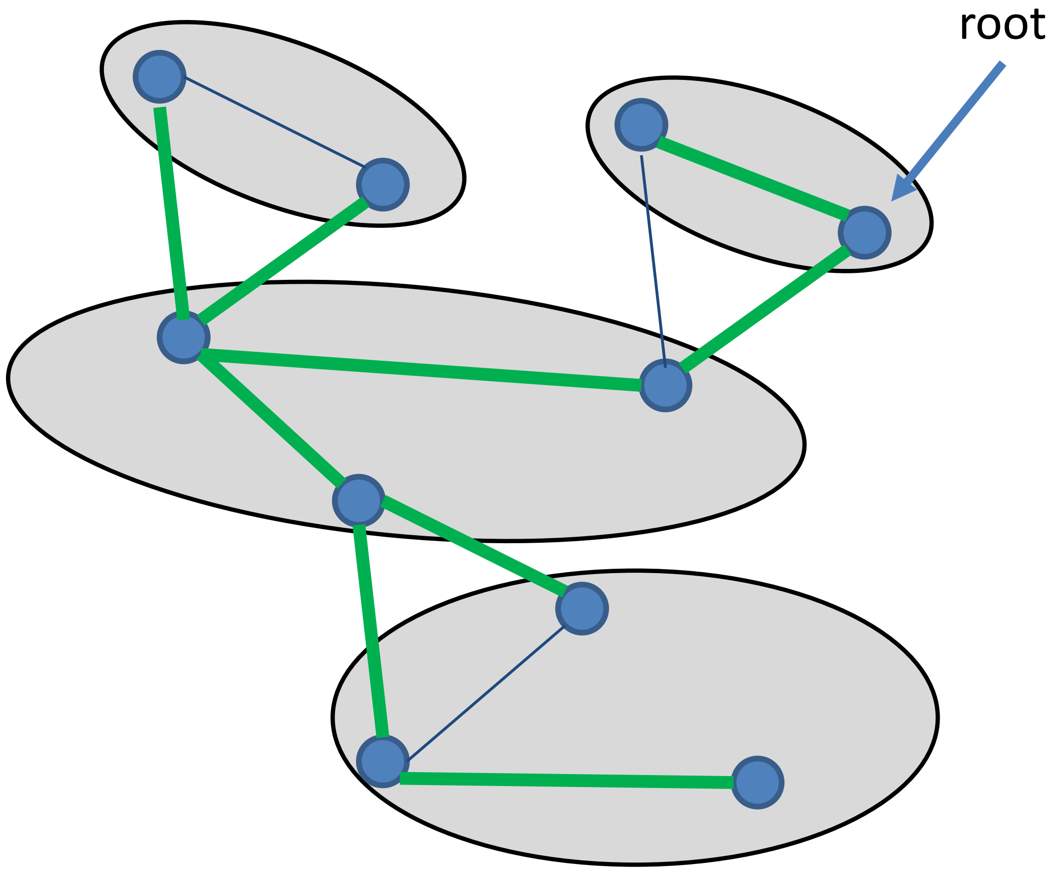


Fully Distributed Model

Boruvka's Algorithm

How to send MWOE up tree?

Wait until you
have received
all MWOE from
all your children.

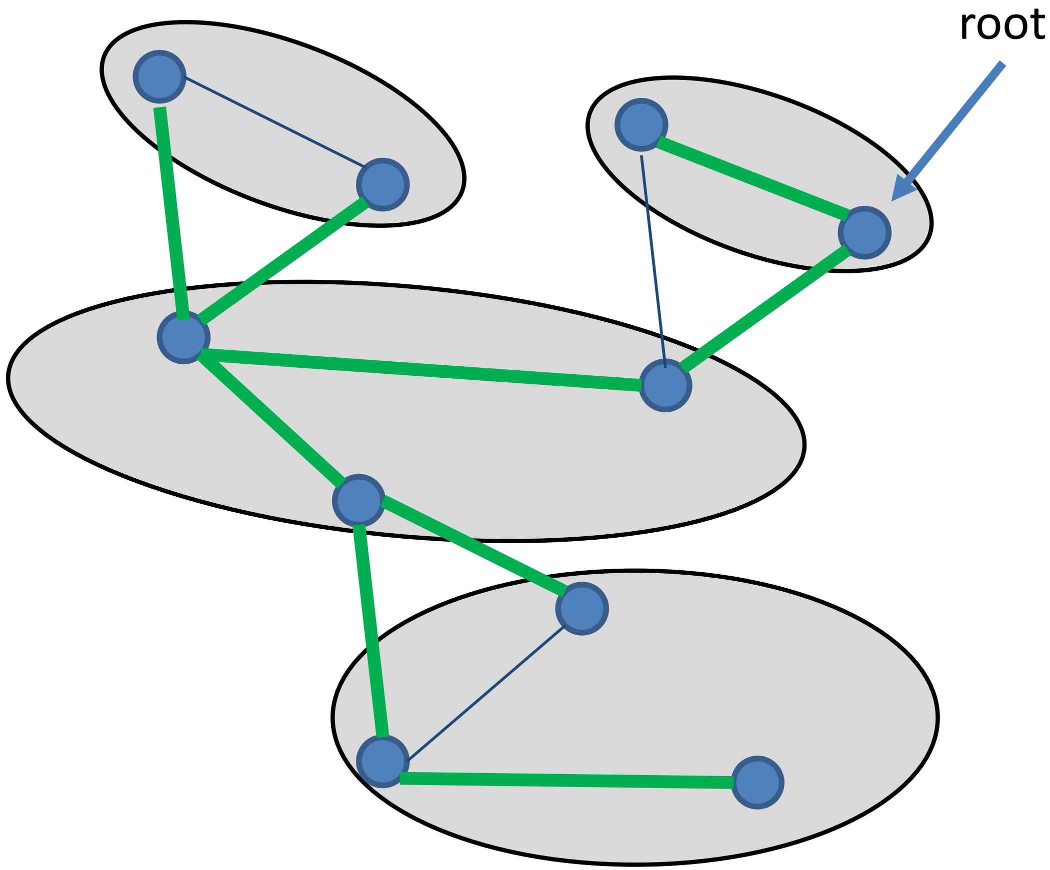


Fully Distributed Model

Boruvka's Algorithm

How to send MWOE up tree?

Compute one
min weight
edge for
each component.

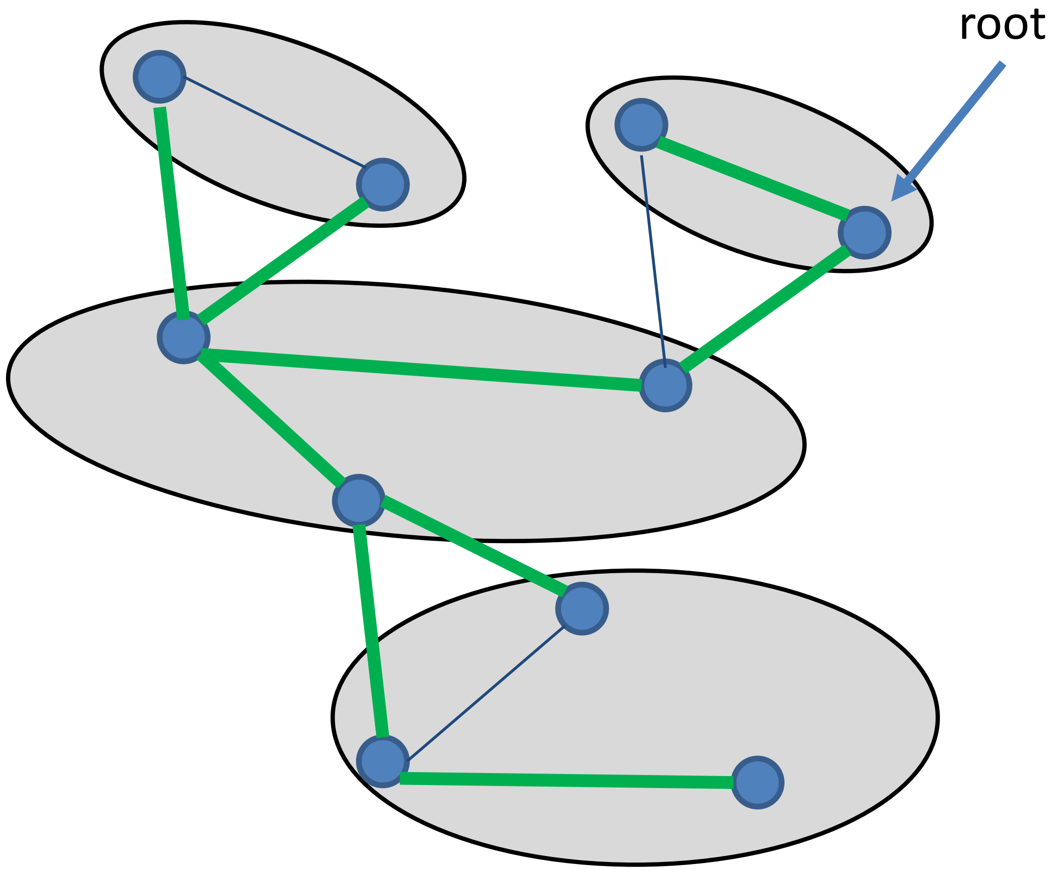


Fully Distributed Model

Boruvka's Algorithm

How to send MWOE up tree?

Send all
to your parent.



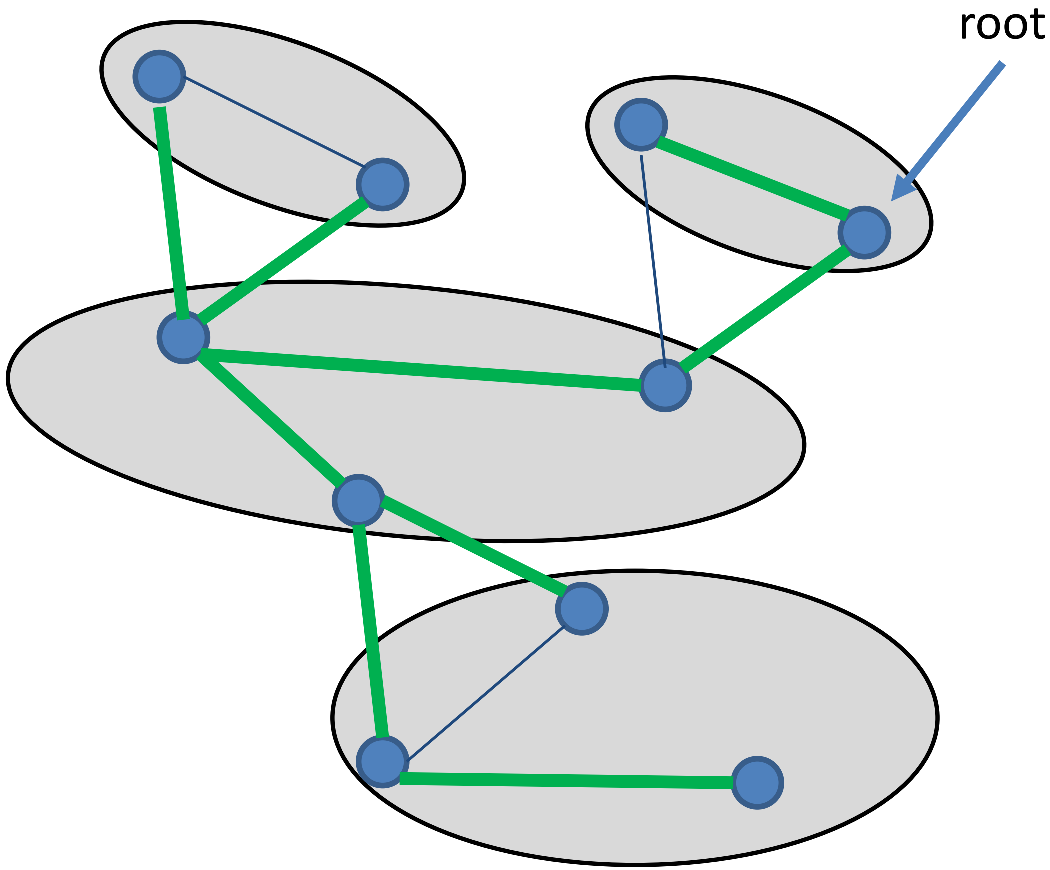
Fully Distributed Model

Boruvka's Algorithm

How to send MWOE up tree?

Send all
to your parent.

At most $n^{½}$ MWOE
to send to parent.



Fully Distributed Model

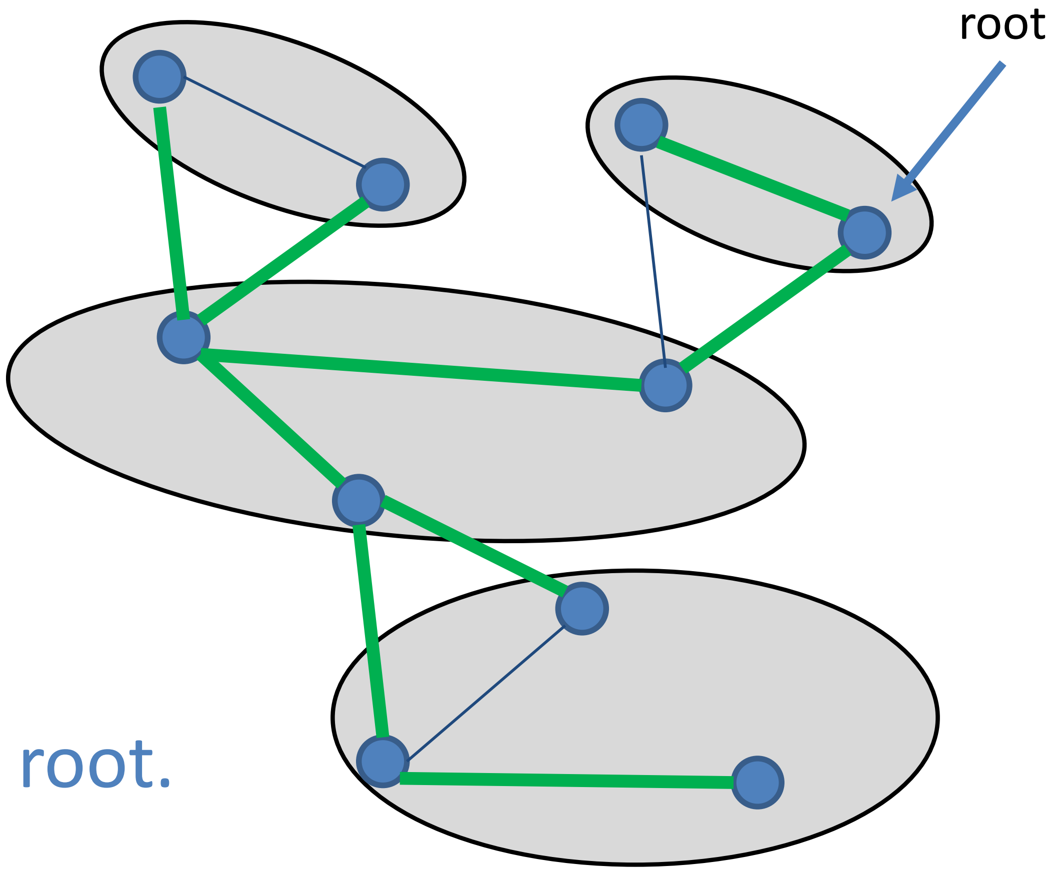
Boruvka's Algorithm

How to send MWOE up tree?

Send all
to your parent.

At most $n^{½}$ MWOE
to send to parent.

Takes at most $Dn^{½}$ time
for al messages to reach root.



Fully Distributed Model

Boruvka’s Algorithm

How to send MWOE up tree?

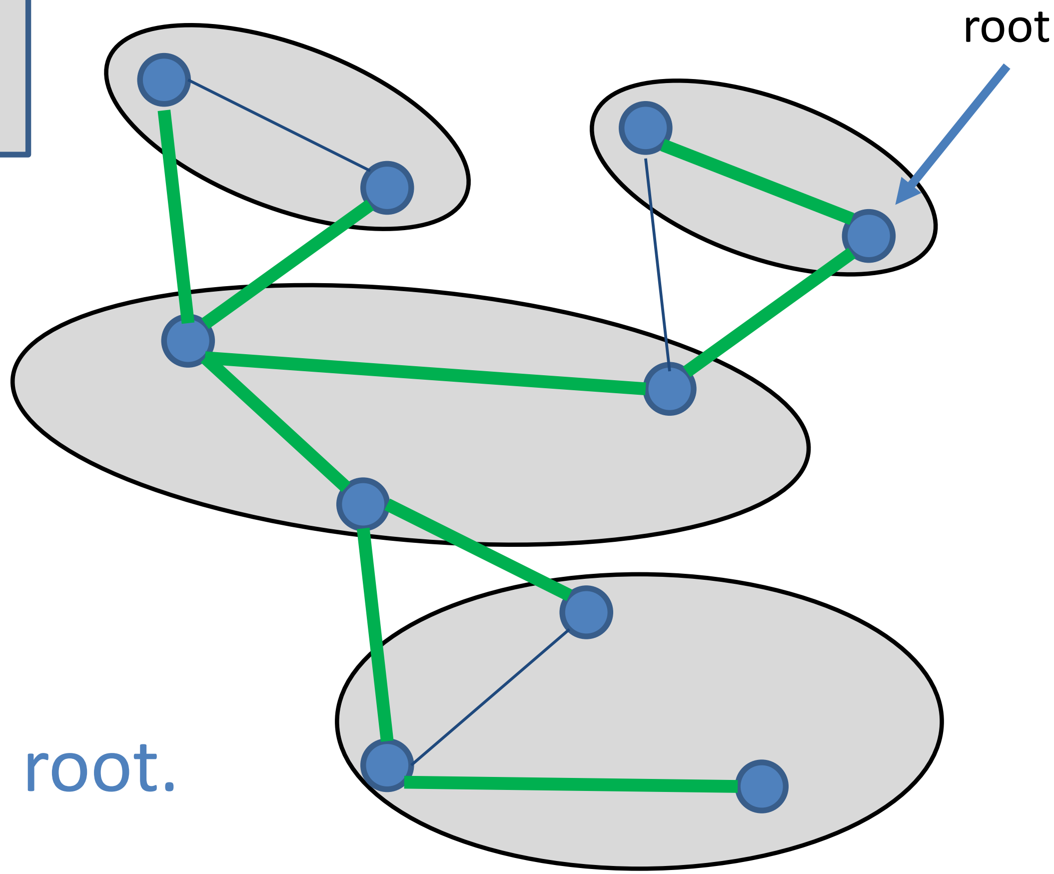
Key reason why we first had to build components of size $n^{½}$!

S

to your parent.

At most $n^{½}$ MWOE to send to parent.

Takes at most $Dn^{½}$ time for al messages to reach root.

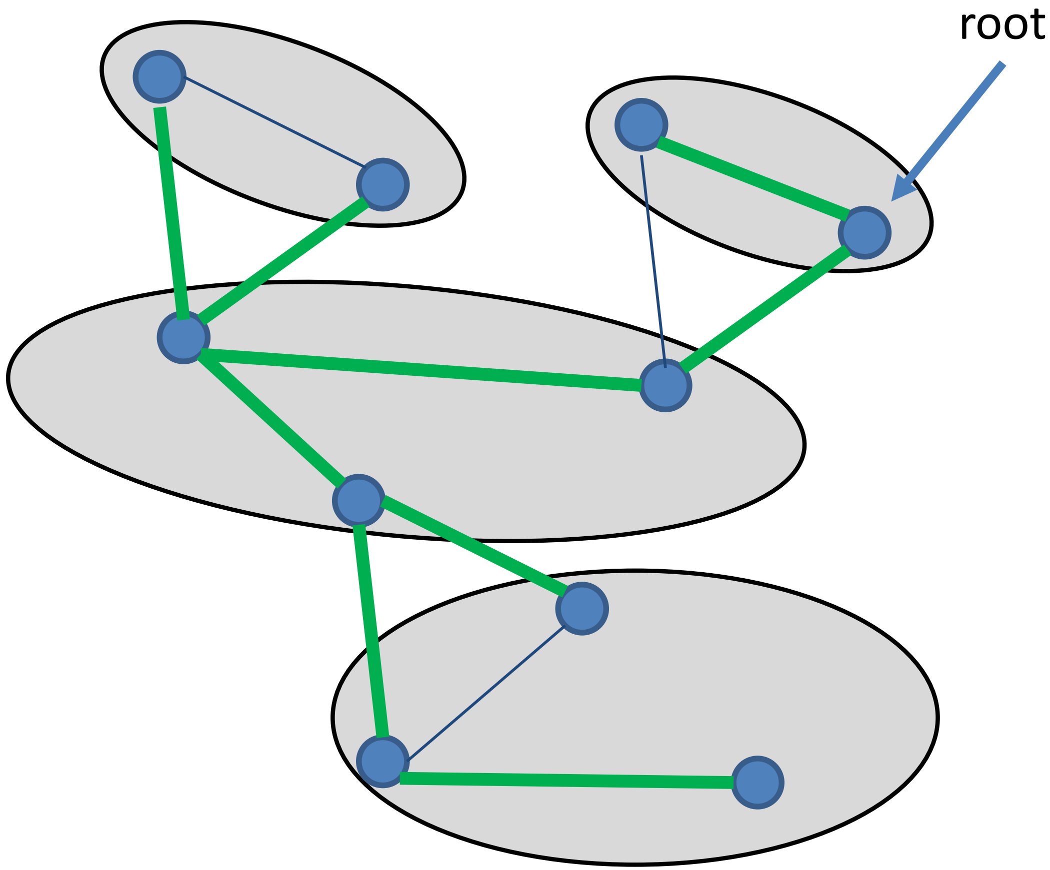


Fully Distributed Model

Boruvka's Algorithm

Improvement: first aggregate in $n^{½}$ sized base fragments.

Never more than $n^{½}$ MWOE to send to root total.



Key reason why we first had to build components of size $n^{½}$!

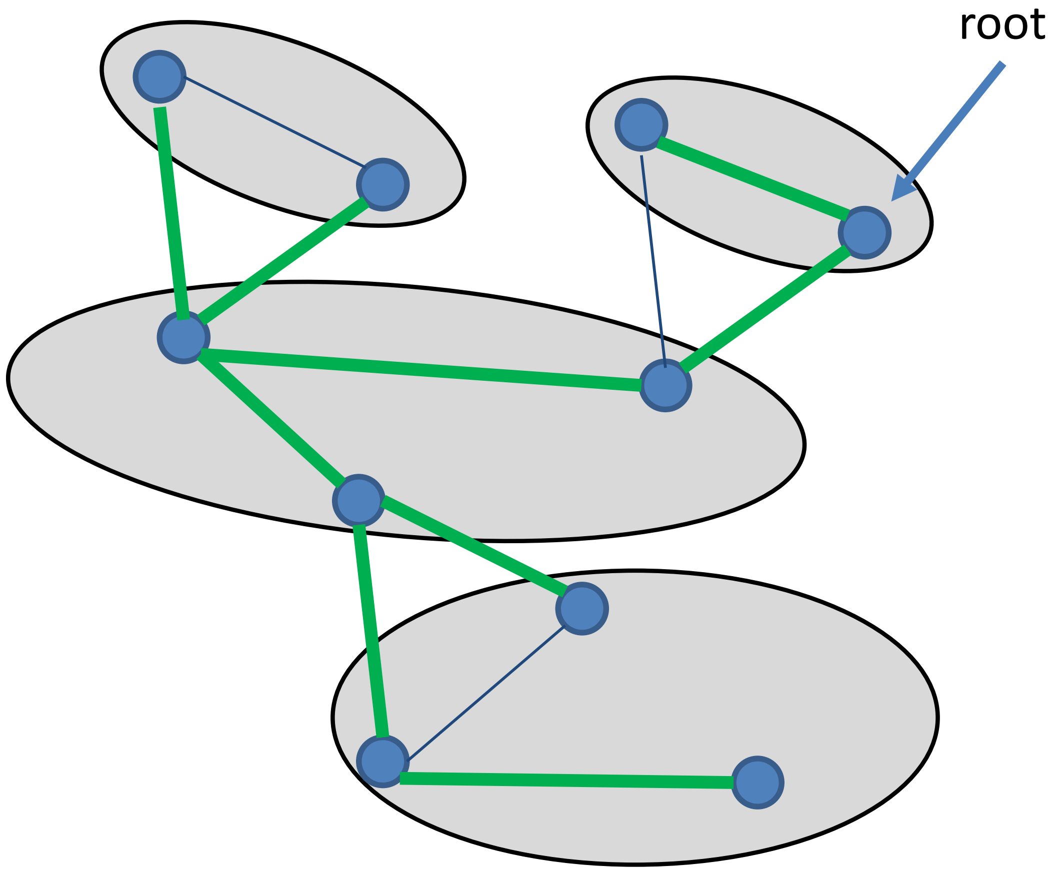
Fully Distributed Model

Boruvka's Algorithm

Improvement: pipeline.

Send on MWOE as soon as you receive it.

Never delayed by another MWOE more than *once*.

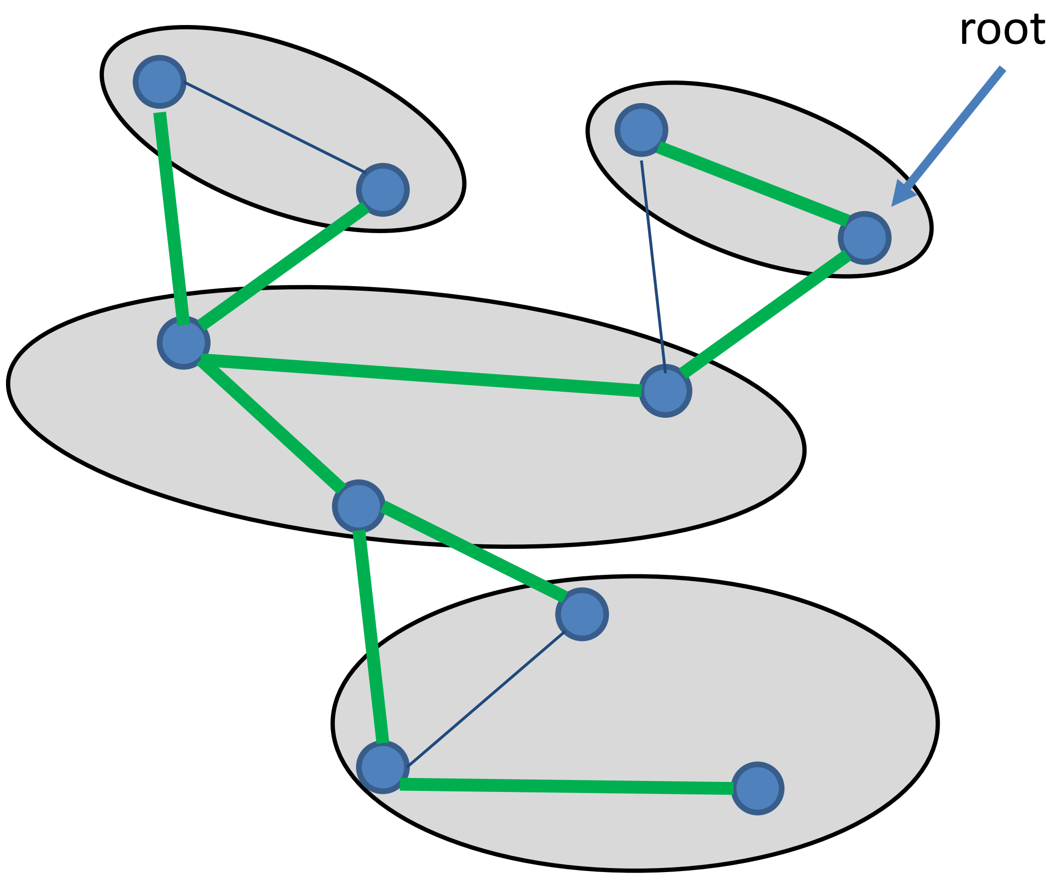


Fully Distributed Model

Boruvka's Algorithm

Conclusion.

$O(D + n^{½})$ time
to aggregate MWOE
and perform
merge.



Fully Distributed Model

Boruvka's Algorithm

Repeat:

1. Find MWOE for each node.
2. If component is $< n^{1/2}$ then aggregate MWOE in component. Otherwise aggregate on BFS tree.
3. Merge components.

Time: $O((D + n^{1/2})\log n)$

Fully Distributed Model

Minimum Spanning Tree

Can we do better than $n^{1/2}$?

Fully Distributed Model

Minimum Spanning Tree

Can we do better than $n^{1/2}$?

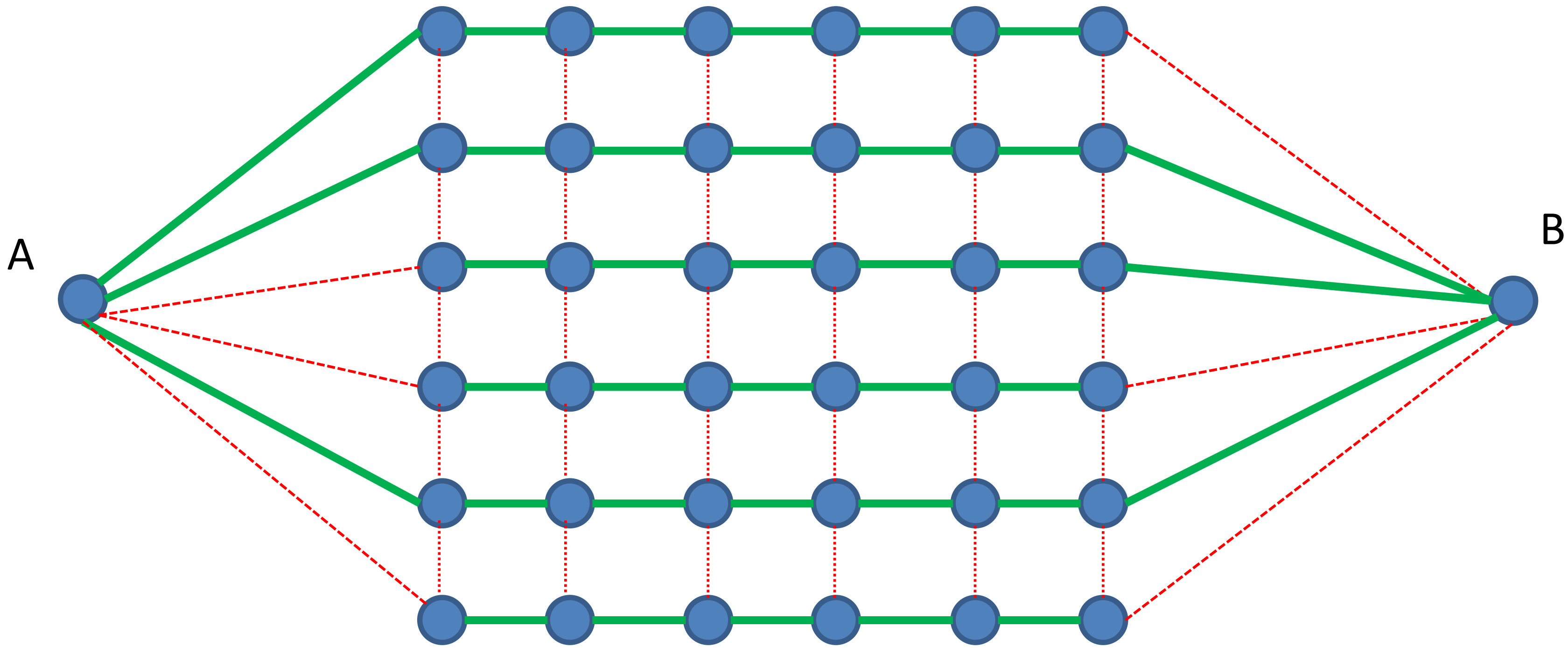
NO!

Lower Bound

Minimum Spanning Tree

$n^{1/2}$ by $n^{1/2}$ grid

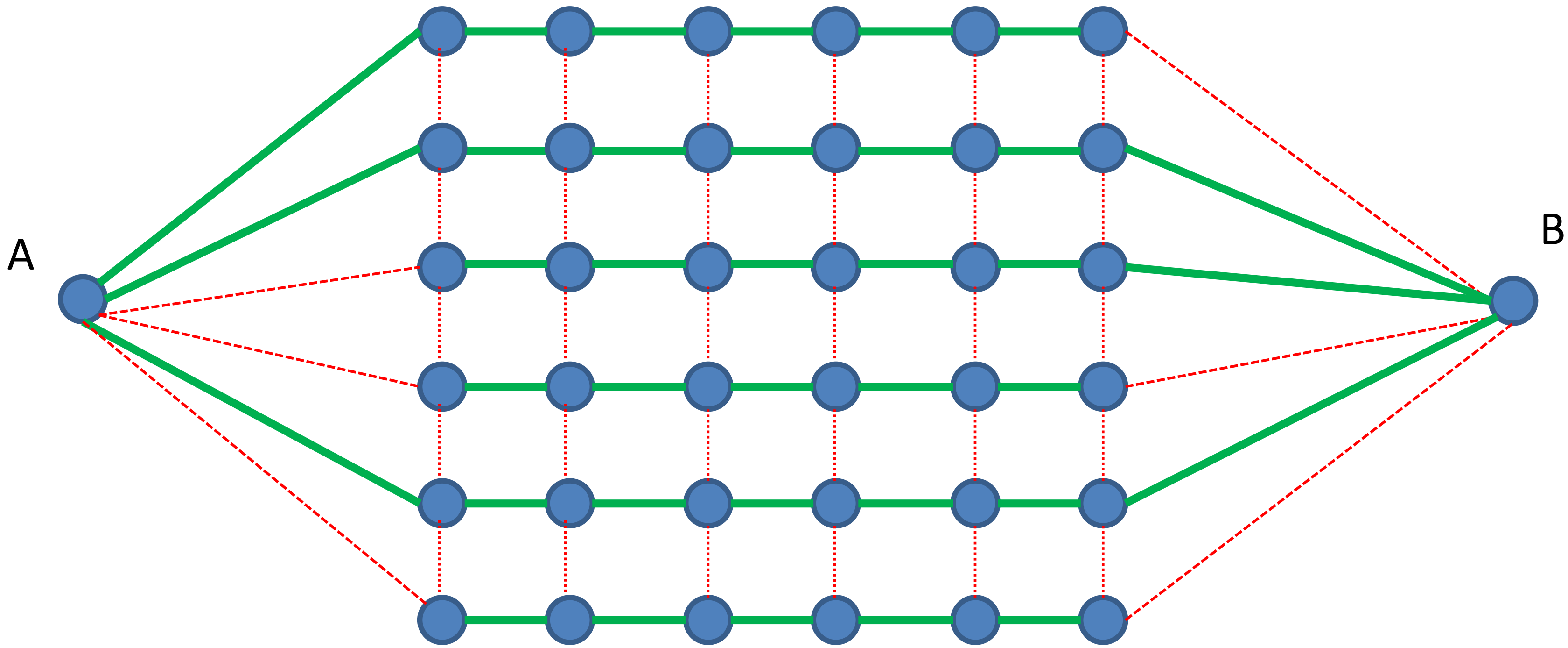
Two special nodes: A and B



Lower Bound

Minimum Spanning Tree

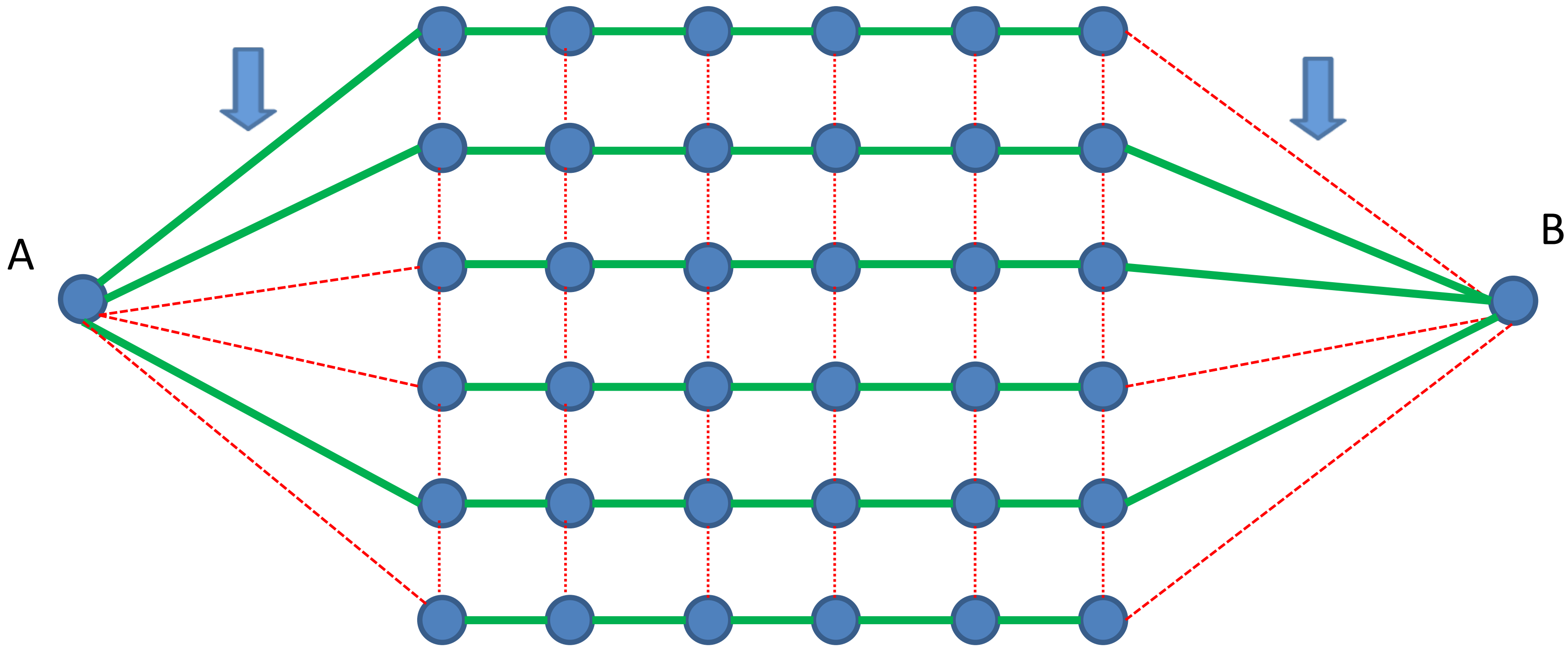
Thick green edges: light weight (should go in MST).
Dashed red edges: heavy weight (should NOT go in MST)



Lower Bound

Minimum Spanning Tree

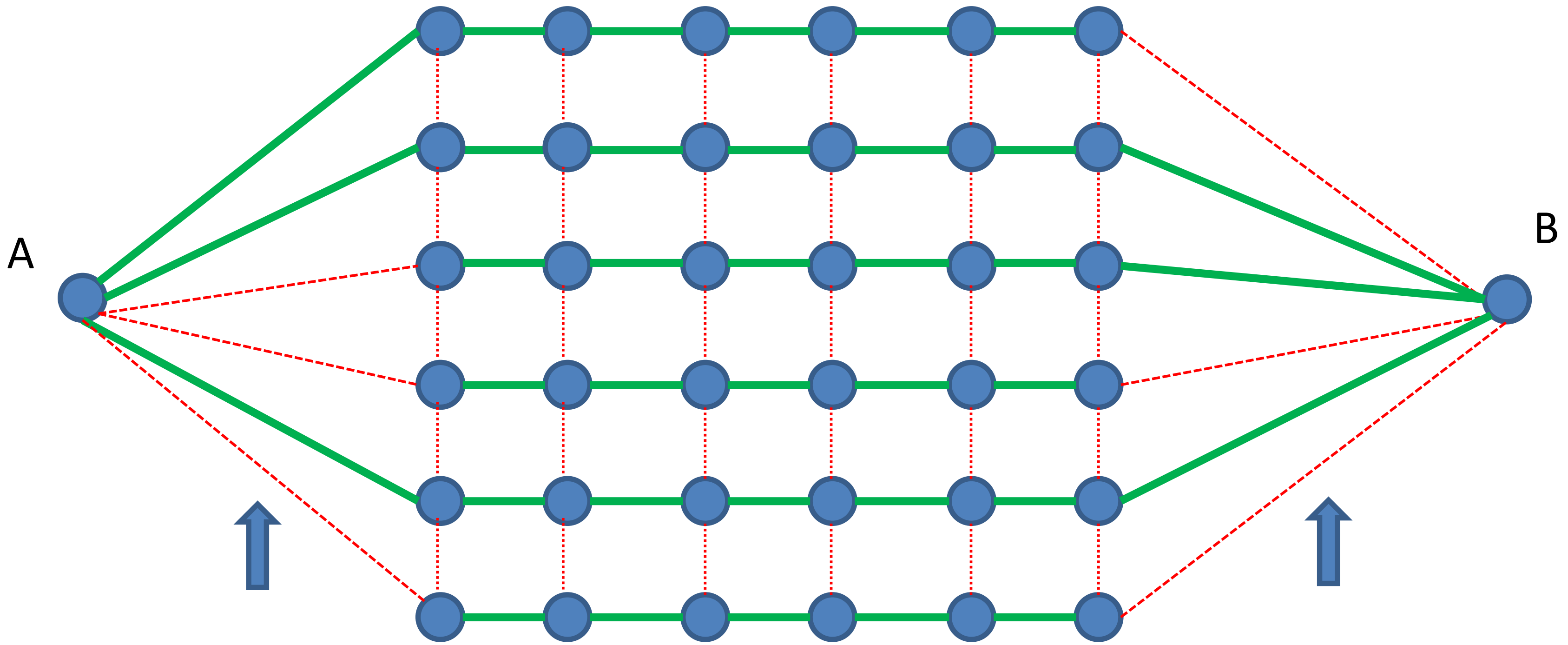
How do A and B decide which edges to include?
Must communicate with each other!



Lower Bound

Minimum Spanning Tree

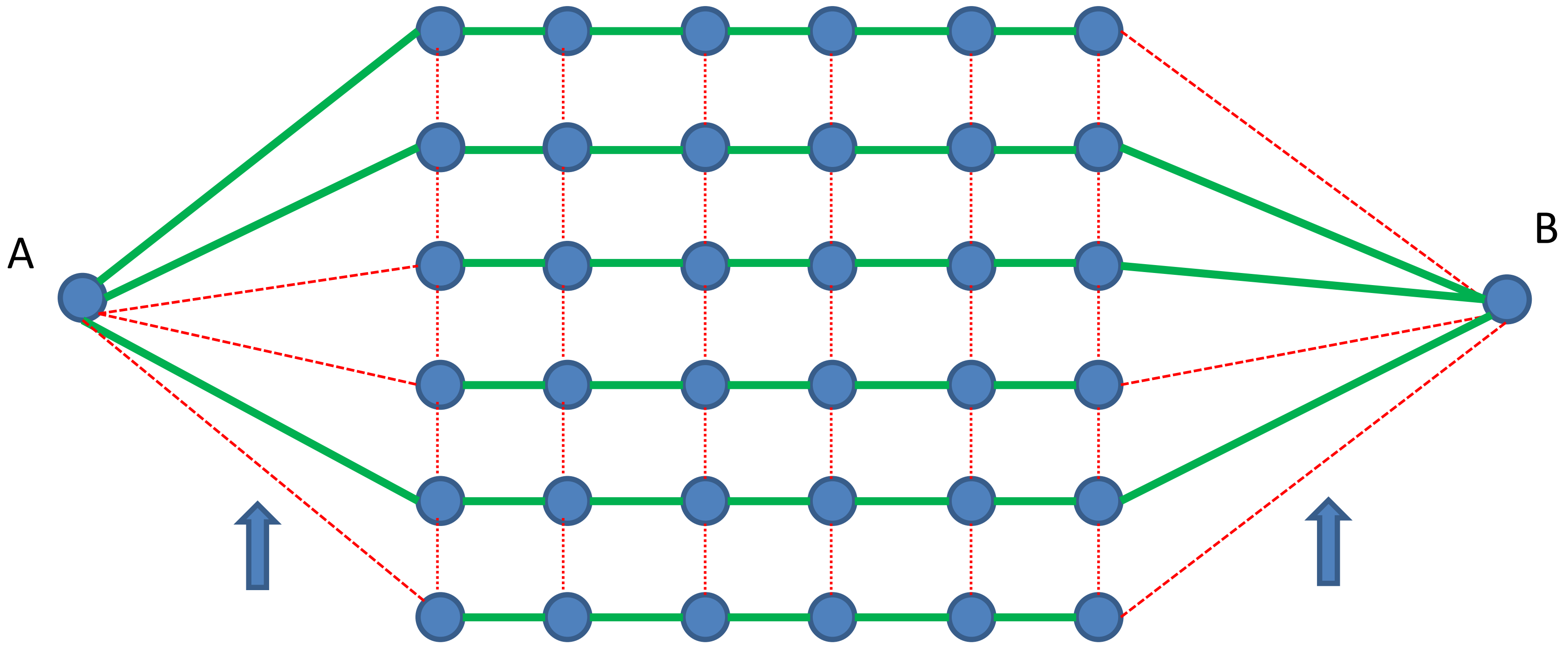
How do A and B decide which edges to include?
Must communicate with each other!



Lower Bound

Minimum Spanning Tree

Shortest path connecting A and B is $n^{½}$ so algorithm must take $n^{½}$ time.

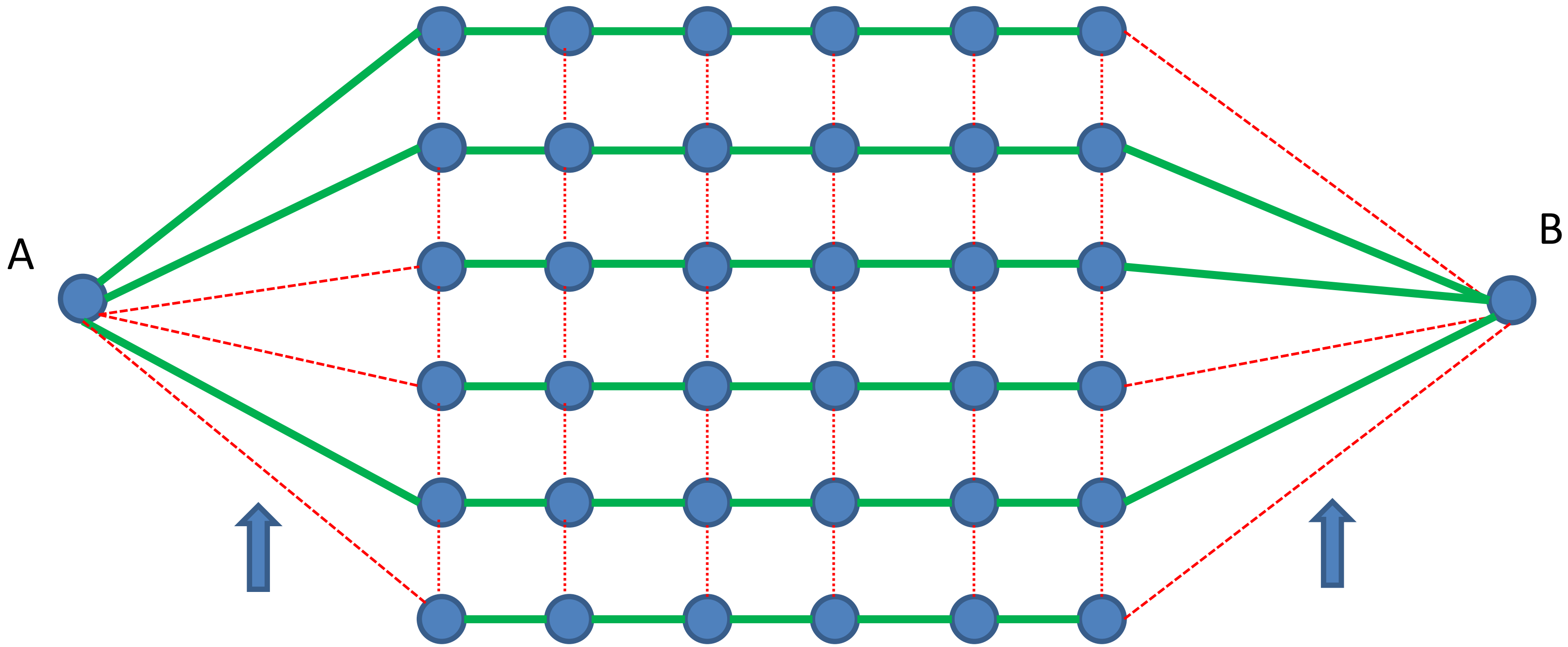


Any problem here?

Lower Bound

num Spanning Tree

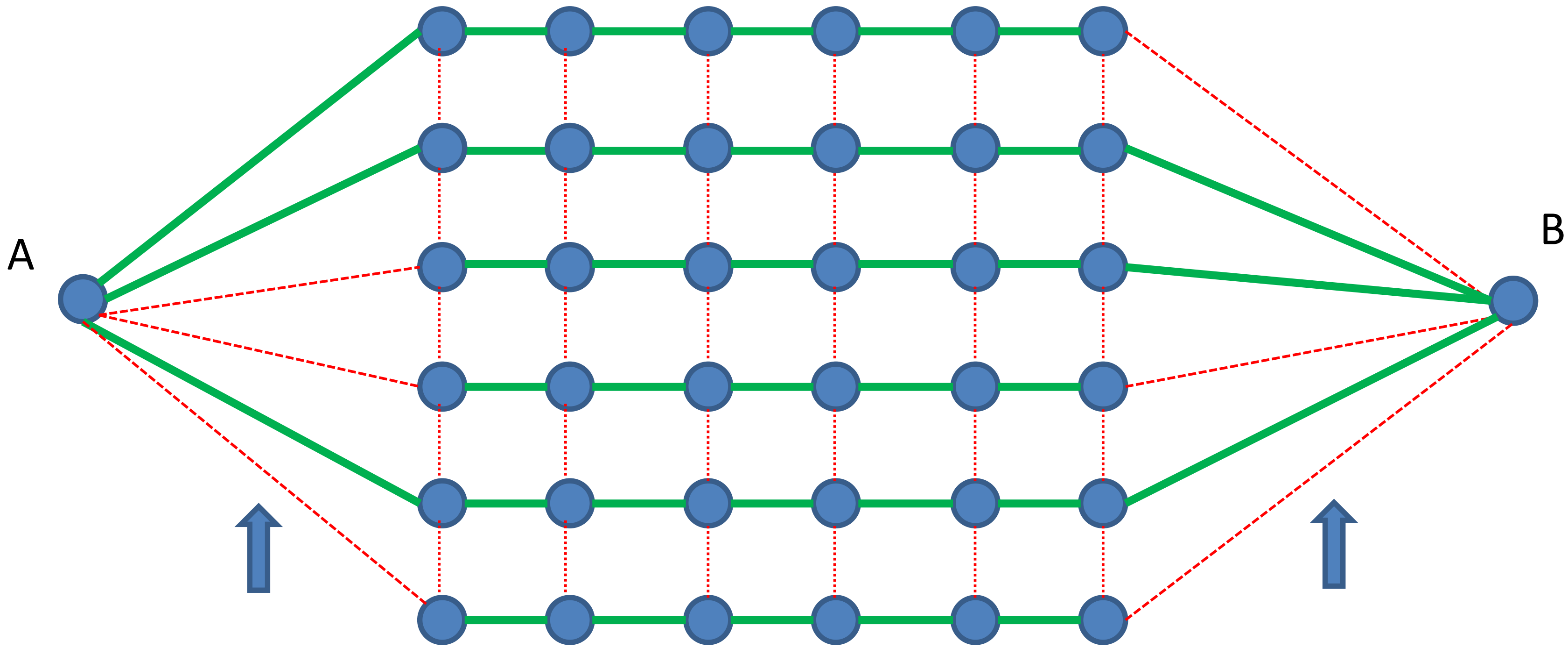
Shortest path connecting A and B is $n^{½}$ so algorithm must take $n^{½}$ time.



Lower Bound

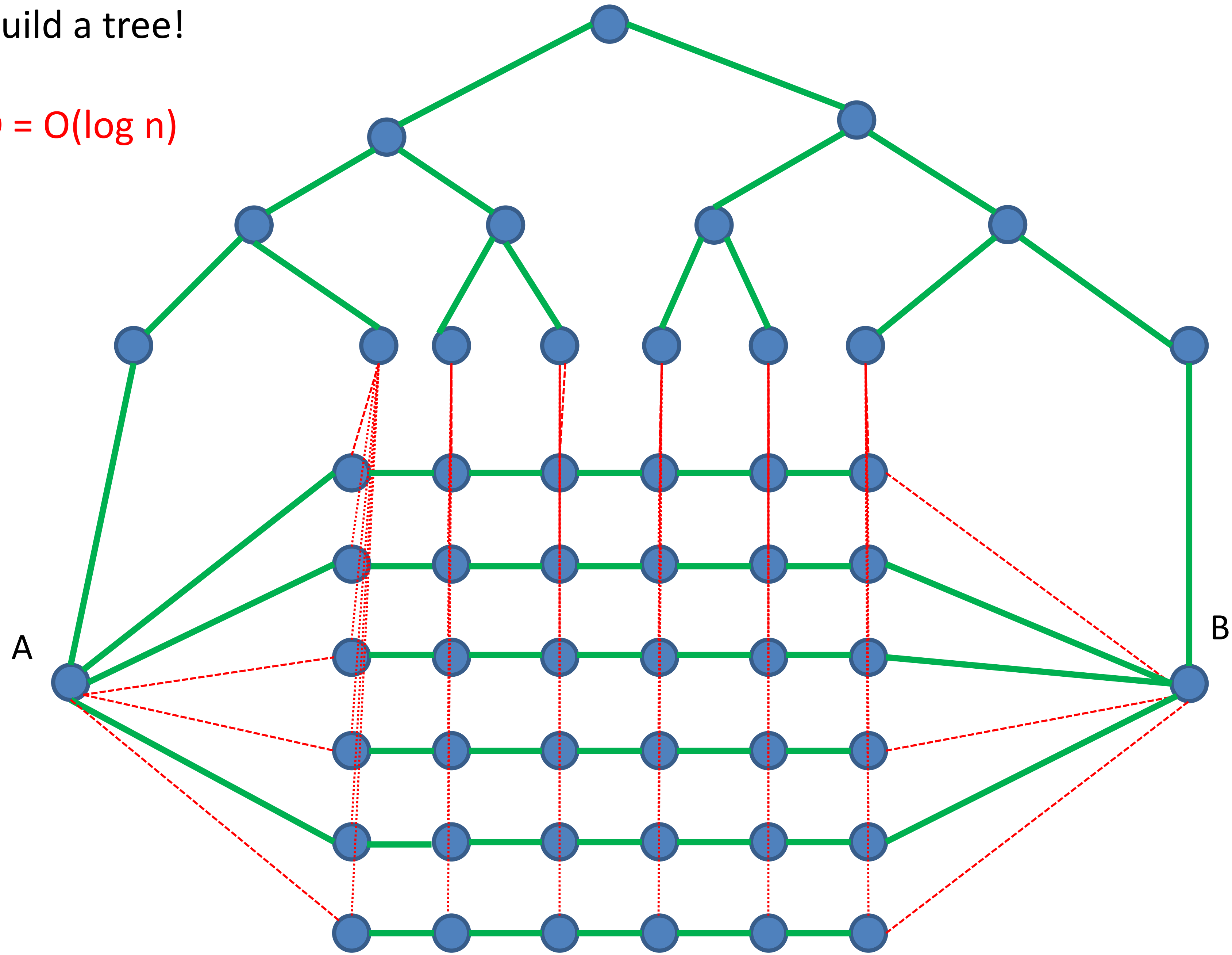
Minimum Spanning Tree

Diameter is $n^{1/2}$ so algorithm runs in O(D) time.

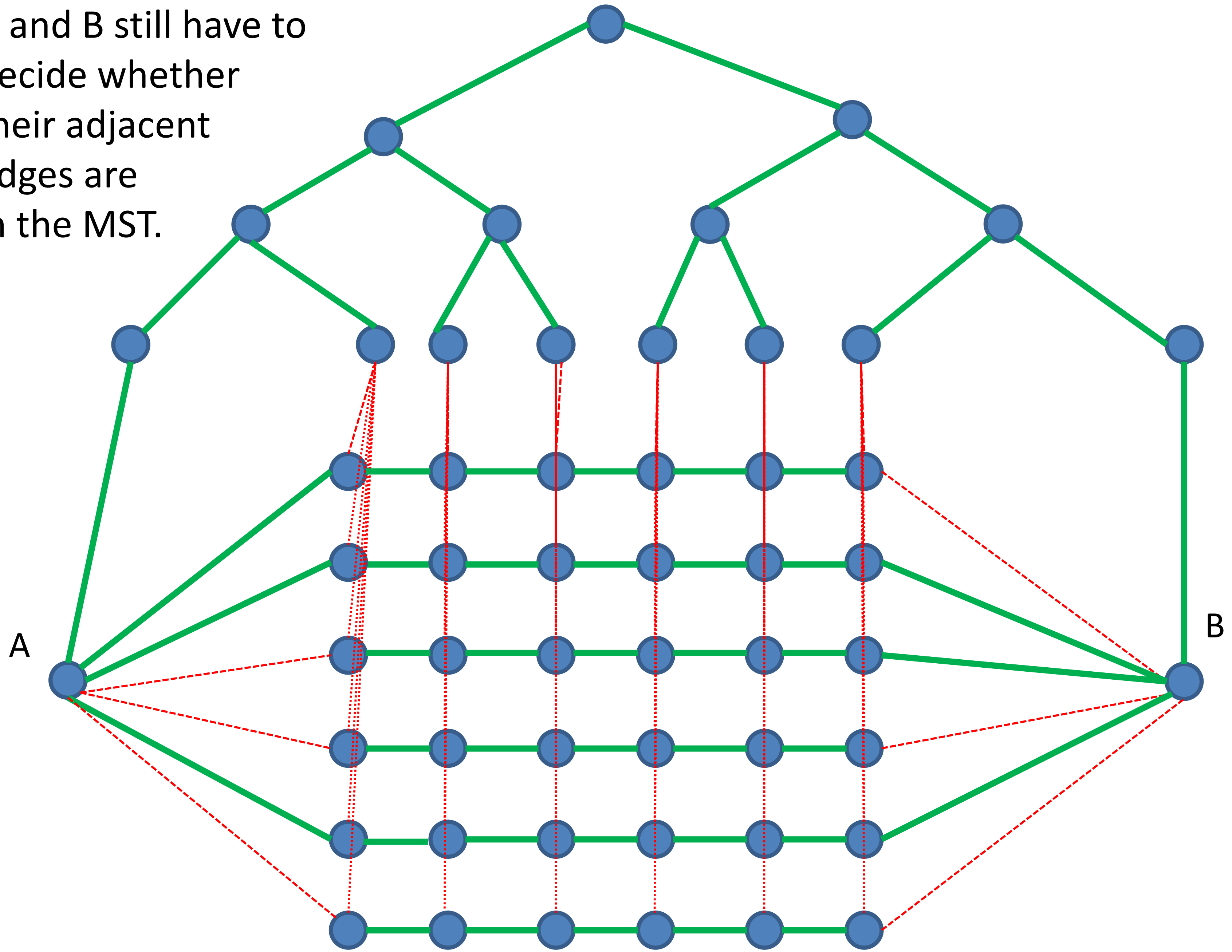


Build a tree!

$D = O(\log n)$

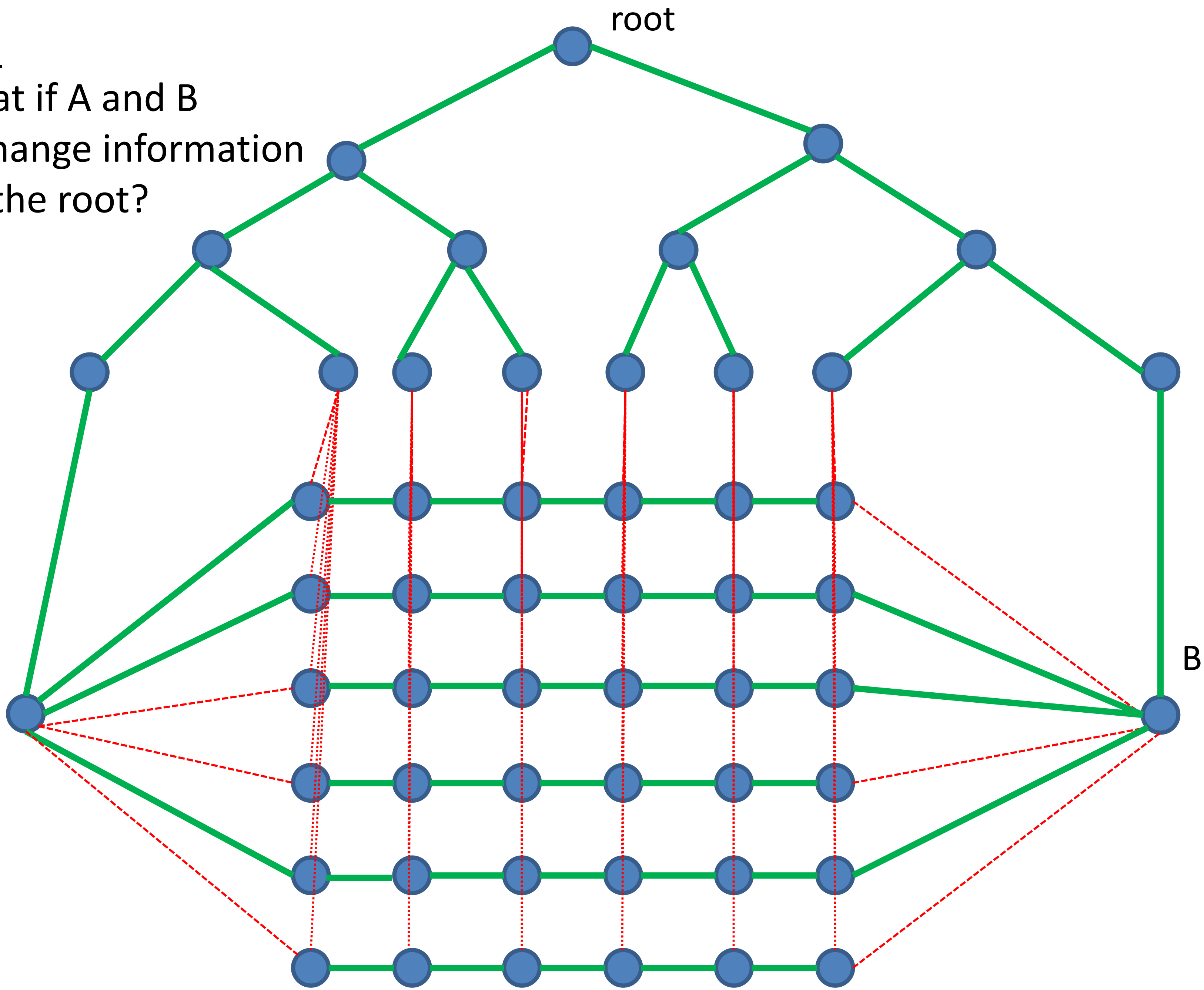


A and B still have to decide whether their adjacent edges are in the MST.



__But:__

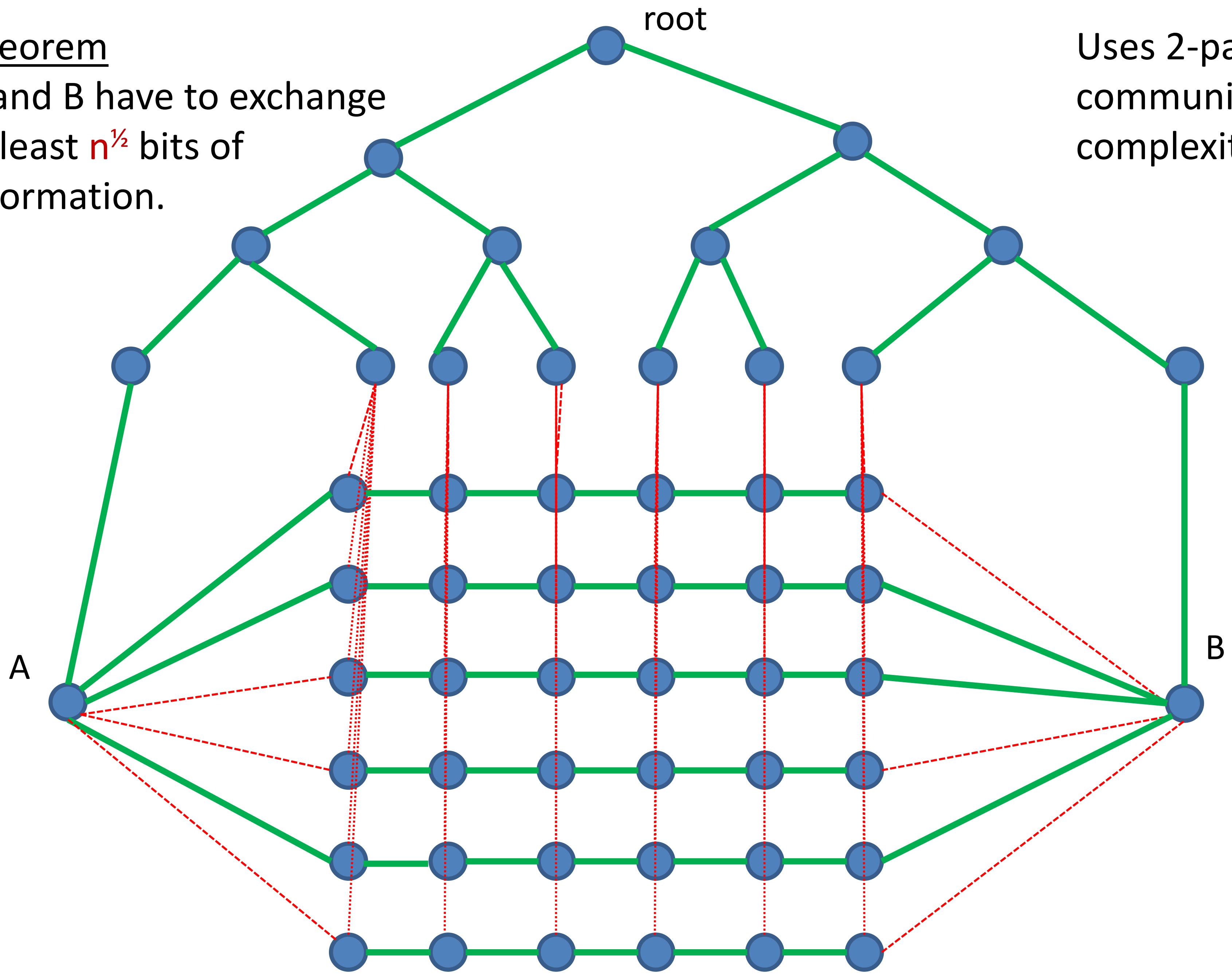
What if A and B exchange information via the root?



Theorem

A and B have to exchange at least $n^{½}$ bits of information.

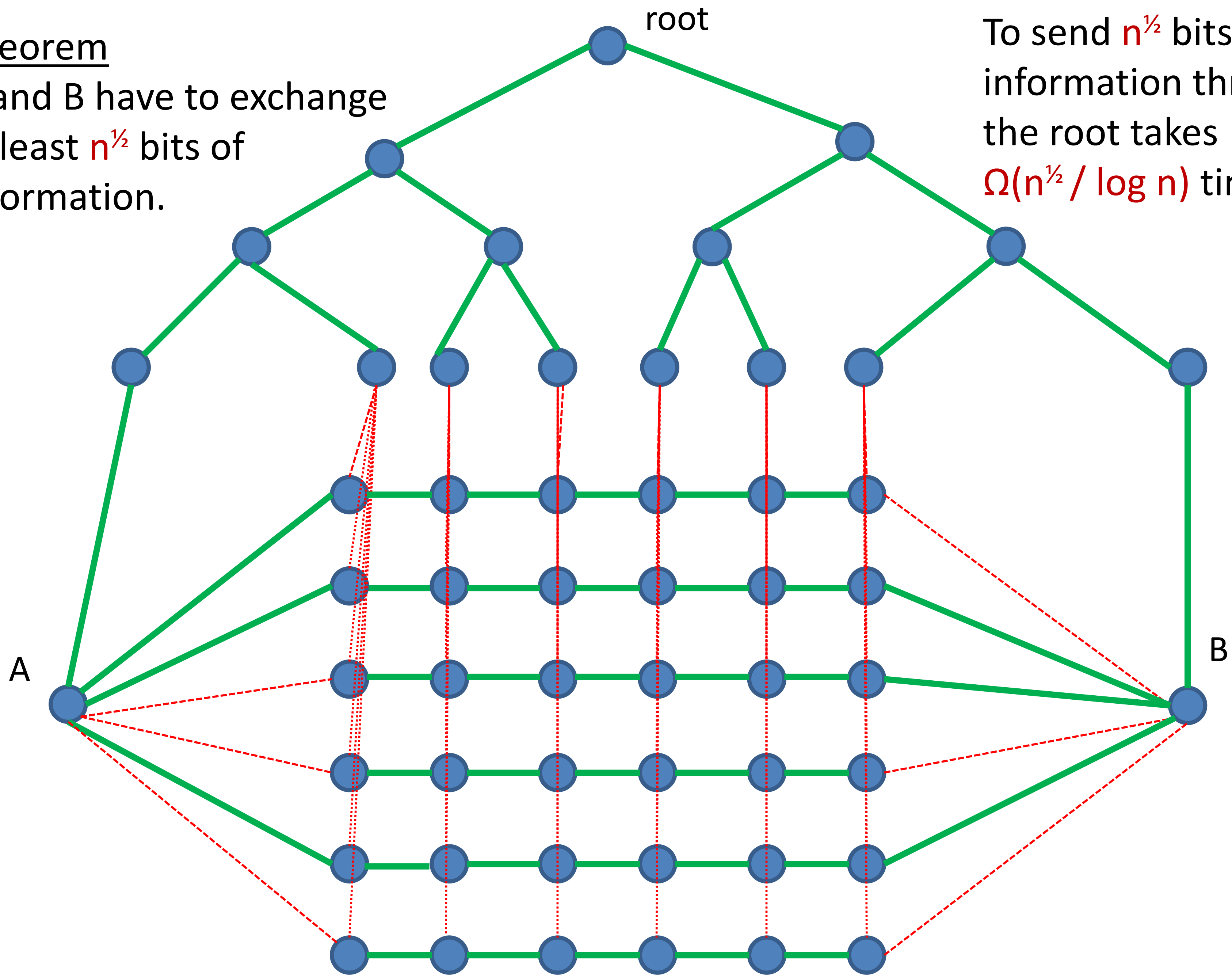
Uses 2-party communication complexity.



Theorem

A and B have to exchange at least $n^{1/2}$ bits of information.

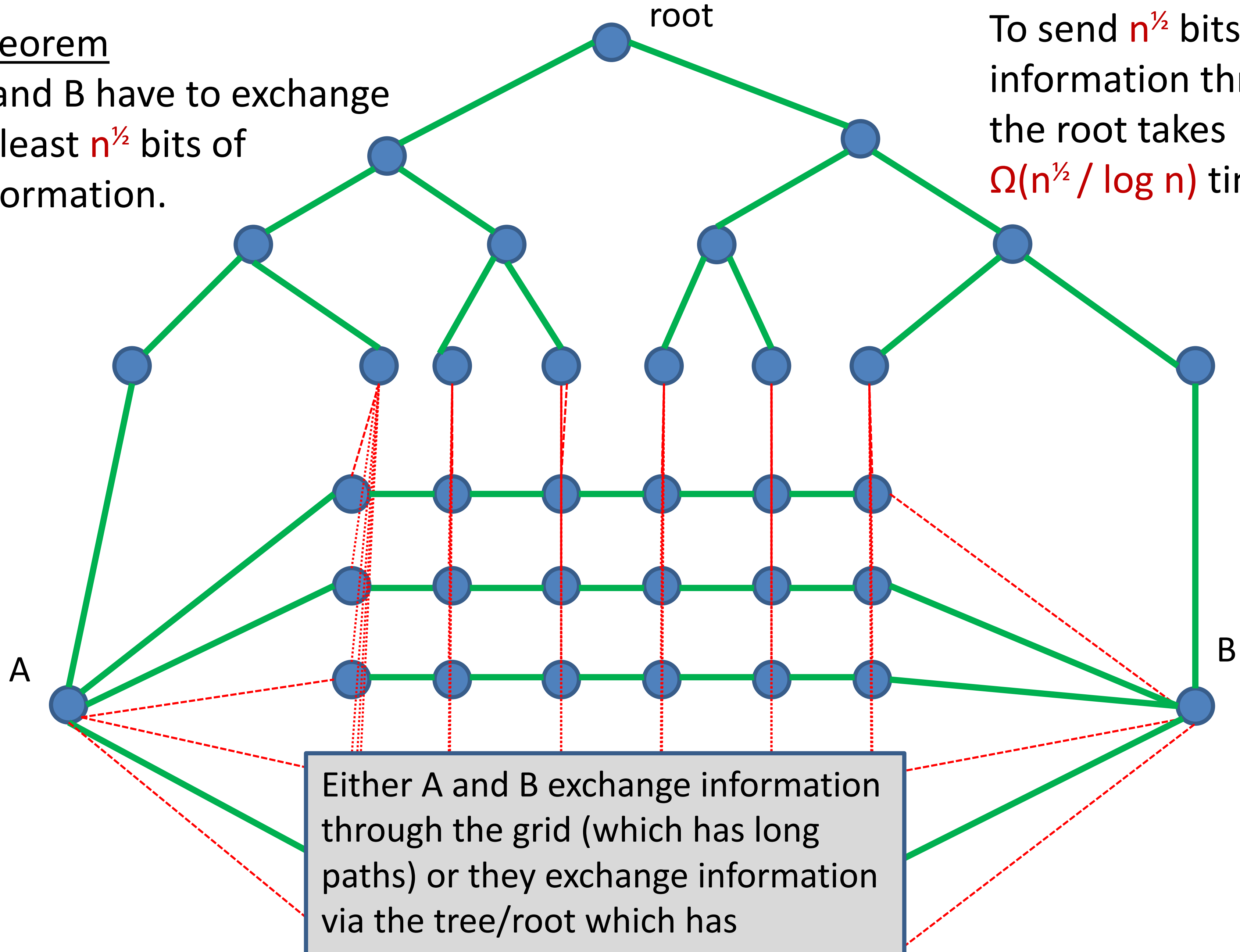
To send $n^{1/2}$ bits of information through the root takes $\Omega(n^{1/2} / \log n)$ time.



Theorem

A and B have to exchange at least $n^{½}$ bits of information.

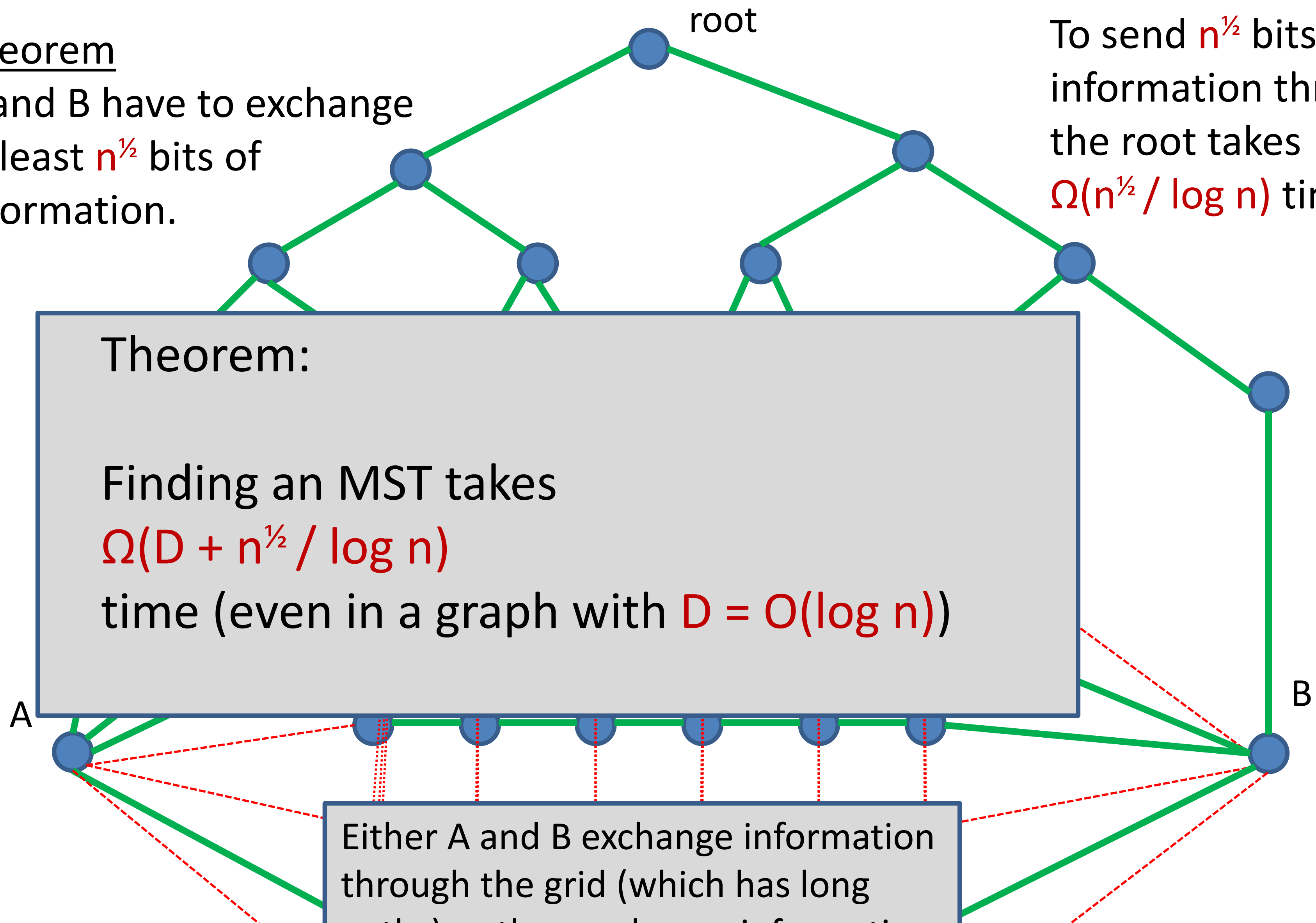
To send $n^{½}$ bits of information through the root takes $\Omega(n^{½} / \log n)$ time.



Either A and B exchange information through the grid (which has long paths) or they exchange information via the tree/root which has congestion.

Theorem
A and B have to exchange at least $n^{½}$ bits of information.

To send $n^{½}$ bits of information through the root takes $\Omega(n^{½} / \log n)$ time.



Theorem:

Finding an MST takes
$\Omega(D + n^{½} / \log n)$
time (even in a graph with $D = O(\log n)$)

Either A and B exchange information through the grid (which has long paths) or they exchange information via the tree/root which has congestion.

Summary

Today: k-Machine

k-Machine Model

- Cluster computing

Some simple examples

- Luby’s
- Bellman-Ford

Minimum Spanning Tree

- Basic algorithm
- Fully distributed algorithm
- Lower bound

Last Week: Map-Reduce

Map-Reduce Model

- Cluster computing

Some simple examples

- Word count
- Join

Algorithms

- Bellman-Ford
- PageRank

Design Some Algorithms

Design k-Machine algorithms:

Sorting

Finding a median

Prefix-Sum

Maximal matching

A little more:

What about PageRank?

PageRank (Last Week)

PageRank(G)

Choose a random node v (uniformly) from G

Repeat many times:

1. With probability ½: stay at node v.
2. With probability ½: choose a neighbor of v uniformly at random and go to that neighbor.

Assign to each node u the probability that you are at node u when the process terminates.

PageRank (Today)

PageRank(G) (Version 1)

Choose a random node v (uniformly) from G

Repeat many times:

1. With probability ε: restart at a new node chosen uniformly at random.
2. With probability (1- ε): choose a neighbor of v uniformly at random and go to that neighbor.

Assign to each node u the probability that you are at node u when the process terminates.

PageRank

Equivalent: (Version 2)

- Start a random walk at a random node v.
- At every step:
  1. With probability ε stop and return v.
  2. With probability $(1-\varepsilon)$ choose a neighbor uniformly at random and go there.

PageRank(v) = probability that process stops at v.

PageRank

1) Explain why the two versions are equivalent.

PageRank

Imagine running the process above $n \log n$ times.

If x random walks visit a node, then

$(\varepsilon x / n \log n)$

is a good estimate of the PageRank.

(Prove it? Essentially, just Chernoff Bounds.)

PageRank

1) Explain why the two versions are equivalent.

2) Give an algorithm for the k–machine model that runs the process n log n times in parallel and computes the PageRank. How long does it take?

Design Some Algorithms

Design k-Machine algorithms:

Sorting

Finding a median

Prefix-Sum

Maximal matching

A little more:

What about PageRank?

1) Explain why the two versions are equivalent.

2) Give an algorithm for the k–machine model that runs the process n log n times in parallel and computes the PageRank. How long does it take?