Weeks 9 and 10: Random Walks
— Random walk on a graph.
— Convergence to the stationary distribution (and coupling).
— Martingales and the Martingale stopping theorem
— Time reversible Markov Chains

Fundamental Theorem of Markov Chains

If X is a finite, irreducible, aperiodic Markov Chain, then

1) There exists a unique stationary distribution π

Strategy to find π: P^t

2) $\pi_i = \lim_{t\to\infty} P^t_{j,i}$, for all j

Pr(go from $j \to i$ in t steps)

$\pi_j = \lim_{t\to\infty} \frac{\#\text{ visits to } j}{t}$

3) $\pi_i = \frac{1}{h_{i,i}}$

E[# steps start at i, return to i for 1st time]

or

$\pi P = 1 \cdot \pi$

eigenvector, eigenvalue

unique when all eigenvalues $\lambda_2, \dots, \lambda_K < 1$

→ irreducible ⇒ strongly connected

→ aperiodic ⇒ not bipartite, not periodic
"self-loops" solve this problem

Stationary distribution

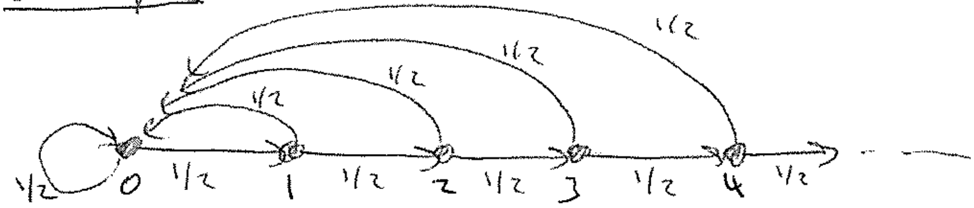
$\pi P = \pi$

Counterexample: $X_0 = 0$

$$X_{t+1} = \begin{cases} X_t + 1 & \text{wp } 2/3 \\ X_t - 1 & \text{wp } 1/3 \end{cases}$$

$\Rightarrow \pi_j = 0 \ \forall j$??

Example



$$\pi(j) = \sum_{i=1}^{\infty} \pi(i)\, P_{ij}$$

$$\pi(0) = \sum_{i=1}^{\infty} \pi(i)\, 1/2 = \frac{\sum \pi(i)}{2} = 1/2$$

$$\pi(j) = \pi(j-1) \cdot 1/2$$

$$= 1/2^j$$

Conclusion:
1/2 time in state 0
1/4 time in state 1
1/8 time in state 2

Problem? Not finite MC
However π is unique stationary distribution

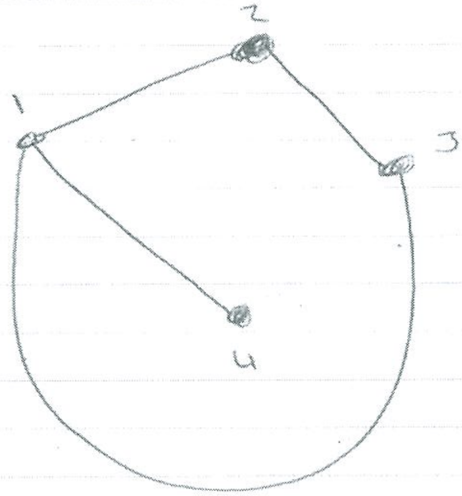
Goal: find the stationary distribution of a random walk on an undirected graph.

Random Walk on Undirected Graph

Let $G = (V, E)$ be:
- finite
- undirected
- connected
- not-bipartite ← add self-loops

why is "not bipartite" = "not periodic"

Ex



$\text{period}_j = \gcd(\text{all cycles through } j)$

$= \gcd(2, \text{odd})$

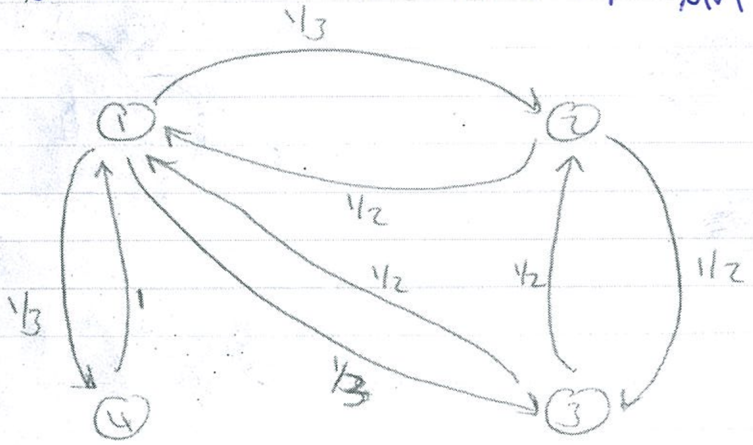
$= 1$

Questions:

How long to return home?

How long to "cover" graph?

Unweighted: choose $u \in N(v)$ w.p. $\frac{1}{N(v)}$



$$\begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{P_{ij} = \frac{1}{d(i)}}$$

Random walk on G: $d(v)$ = degree v

$$Pr(X_{t+1} = w \mid X_t = v) = \begin{cases} 0 & \text{if } (v,w) \notin E \\ 1/d(v) & \text{if } (v,w) \in E \end{cases}$$

→ finite (by assumption)
→ aperiodic (not bipartite)
→ irreducible (connected)

↳ π exists
X converges to π
$\pi_v = 1/h_{v,v}$

π is distribution:

$$\boxed{\text{Note: } \sum_{j=1}^{n} \pi_j = \sum_{j=1}^{n} \frac{d(j)}{2E} = \frac{2E}{2E}}$$

<u>Claim</u> $\pi_v = \dfrac{d(v)}{2|E|}$

<u>Pf</u> π exists and is <u>unique</u>

<u>Verify</u> $\pi P = \pi$ $\quad [\ \pi\ ]\,[\ P\ ]$

↳ π is s.d.

$$\pi_v = \sum_{j=1}^{n} \pi_j P_{jv}$$

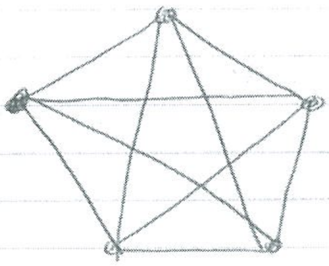
$$= \sum_{j \in N(v)} \pi_j P_{jv} = \sum_{j \in N(v)} \frac{d(j)}{2E} \cdot \frac{1}{d(j)}$$

$$= \sum_{j \in N(v)} \frac{1}{2E} = \frac{d(v)}{2E} \quad \checkmark$$

<u>Corollary</u> $E[\#\text{ steps to return to } v] = \dfrac{2E}{d(v)}$

$$\frac{d(v)}{2E} = \pi_v = \frac{1}{h_{v,v}}$$

<u>EX</u> Random walk on K_n:



$$h_{0,0} = \frac{2\binom{n}{2}}{n-1} = O(n)$$

<u>EX</u> Random walk on binary tree:

$$h_{0,0} = \frac{O(n)}{2} = O(n)$$

Define $h_{v,u} = E[\text{time go } v \to u]$

$E[\text{walk from } v \text{ to } u]$

Lemma If $(u,v) \in E$, then $h_{v,u} < 2|E|$

Proof $h_{u,u} = \sum_{w \in N(u)} Pr(u \to w) \cdot \left[E[w \to u \mid X = w] + 1 \right]$

law conditional expectation

$$= \frac{1}{d(u)} \sum_{w \in N(u)} (1 + h_{w,u})$$

$$= \frac{2E}{d(u)}$$ by Corollary, $\pi = \frac{d(u)}{2E}$

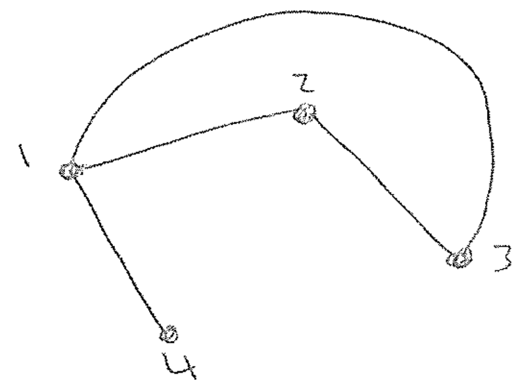
if regular graph

$$\Rightarrow 2E = \sum_{w \in N(u)} (1 + h_{w,u}) \geq d(u) h_{w,u}$$

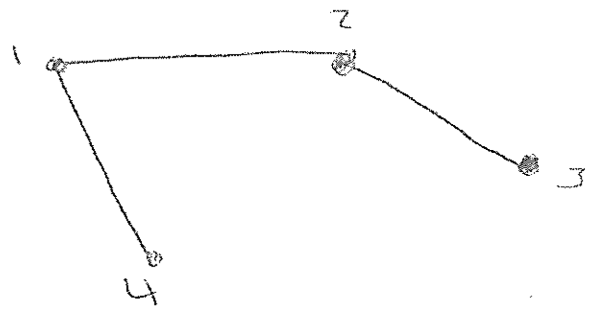
$$\Rightarrow h_{w,u} < 2E \text{ for all } w \in N(u)$$

$h_{w,u} < \frac{2E}{d(u)}$ if regular

Assume random walk starts at node v.
How long until it visits every node? — "Cover time"



1) Choose any spanning tree $T = (V, E')$



edges $= V - 1$

2) Do DFS on tree

(1, 4), (4, 1), (1, 2), (2, 3), (3, 2), (2, 1)

Length of DFS $\leq 2|E'| \leq 2V$

3) Let Y_j = # steps from (v_j, v_{j+1}) in DFS

$$\underbrace{1, 4}_{Y_1}, \underbrace{1}_{Y_2}, \underbrace{2}_{Y_3}, \underbrace{3}_{Y_4}, \underbrace{2}_{Y_5}, \underbrace{1}_{Y_6}$$

$$E[Y_j] = h_{v_j, v_{j+1}} \le 2E$$

$$E[\text{cover time}] = \sum_{j=1}^{2V} E[Y_j]$$
$$\le \sum_{j=1}^{2V} 2E$$
$$\le 4EV$$

length of each step in DFS

\# steps in DFS

Conclusion: $E[\text{cover time}] \le 4EV$

<u>Corollary</u> For all G, $E[\text{cover}] \le O(n^3)$

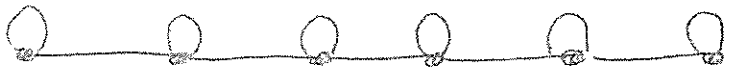
<u>Corollary</u>

$$E[2SAT] \le 4 \cdot (2n) \cdot n = O(n^2)$$

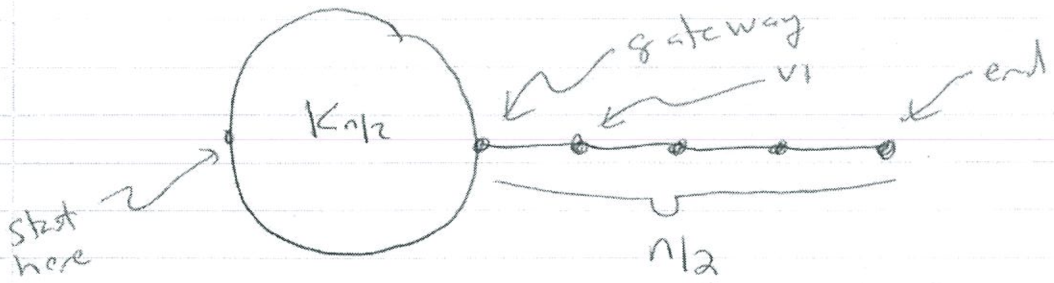
<u>Modify alg</u>

w.p. 1/3 do nothing

w.p. 2/3 update variable



Is this bound tight?



What is the cover time?

$$E[C] \le O(VE) \le O(n \cdot n^2) = O(n^3)$$

"Approximate" argument:

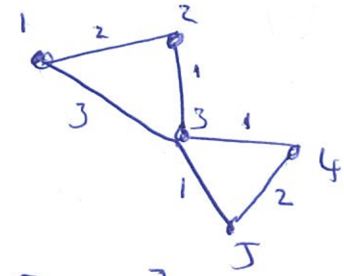
→ visit gateway about $1/n$ steps

→ visit (v_1 | gateway) = $1/n$
⇒ visit v_1 $1/n^2$ steps

→ visit ($v_{n/2}$ | v_1) = $1/n$ (or back to gateway)
⇒ visit $v_{n/2}$ $1/n^3$ steps

Make precise using Walds??

Let A = adjacency matrix
weighted, undirected
connected graph



$$\begin{bmatrix} 0 & 2 & 3 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} = A$$

D = degrees on diagonal

$$\begin{bmatrix} 5 & & & & \\ & 3 & & 0 & \\ & & 6 & & \\ & 0 & & 3 & \\ & & & & 3 \end{bmatrix} = D$$

$D^{-1}A$ = transition matrix $= \begin{bmatrix} 0 & 2/5 & 3/5 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 \\ 3/6 & 1/6 & 0 & 1/6 & 1/6 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1/3 & 2/3 & 0 \end{bmatrix}$

$$W = \frac{1}{2} I + \frac{1}{2} D^{-1} A = \begin{bmatrix} \frac{1}{2} & \frac{2}{10} & \frac{3}{10} & 0 & 0 \\ \frac{2}{6} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ \frac{3}{12} & \frac{1}{12} & \frac{1}{2} & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & 1/6 & 1/2 & 2/6 \\ 0 & 0 & 1/6 & 2/6 & 1/2 \end{bmatrix}$$

Claim: W has n eigenvectors/values:

$$\lambda_1, \lambda_2, \ldots, \lambda_n \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$
$$v_1, v_2, \ldots, v_n$$

All λ_j are real.

All λ_j are $1 \geq \lambda_j \geq 0$

$\lambda_1 = 1, \quad v_1 = \pi$

$\lambda_2 < 1$

Proof

Look at:

$N = I - D^{-1/2} A D^{-1/2}$,

$W = I - \frac{1}{2} D^{1/2} N D^{-1/2}$

Claim: $\pi(j) = \dfrac{\overset{\text{degree}}{d(j)}}{\sum_{i=1}^{n} d(i)}$

Assume $v_0 = [0, 0, \ldots, 1, 0 \ldots, 0]$ index a

$v_0[a] = 1$

$v_0[x] = 0, \quad x \neq a$

Then:

There are many theorems like this that relate the speed of convergence to the second eigenvalue.

$$\left| \Pr(X_t = b \mid X_0 = a) - \pi(b) \right| \leq \sqrt{\frac{d(b)}{d(a)}} \lambda_2^t$$

$$\left| v_0 W^t(b) - \pi(b) \right| \leq \sqrt{\frac{d(b)}{d(a)}} \lambda_2^t$$

$$VD = \tfrac{1}{2} \| P_a^t - \pi \| \leq n^{3/2} \lambda_2^t \Rightarrow T(\varepsilon) = O\left(\frac{\log\left(\frac{1}{n}\right)}{\log(\lambda_2)} \right) = O\left(\frac{\log n}{\log\left(\frac{1}{\lambda_2}\right)} \right)$$
$$= O\left(\frac{\log n}{1 - \lambda_2} \right)$$

How long to converge to π?

Converge $\Leftrightarrow$ $|P^k - \pi|_1 \leq \varepsilon$

variation distance

Idea Coupling

Simulate using correlated random coins

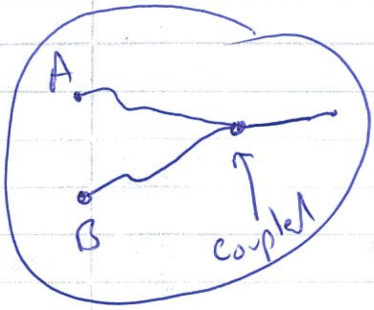
A $\leftarrow$ regular MC, starts in X_0

B $\leftarrow$ MC starts in π

Think random walks:

A is a random walk starting at X_0

B is a r.w starting in π



What happens if A meets B?

They couple!

A and B stay together forever.

V.D.

$$\|D_1 - D_2\| = \frac{1}{2} \sum_{x \in S} |D_1(x) - D_2(x)|$$

$$\tau(\varepsilon) = \max_{x \in S} \min_{t} \|P_x^t - \pi\| \leq \varepsilon$$

start in x
run t steps

mixing time

Coupling Lemma

If X, Y are coupled and $\Pr(X_T \neq Y_T \mid X_0 = x, Y_0 = y) \le \varepsilon$ for all x, y

then $\tau(\varepsilon) \le T$

Example: RW on line + self-loops

S.D. $\pi_j = 1/n$

A: start at v_1

B: start at v_j w.p. $1/n$

How long until A meets B?

Correlate walks:

B unchanged: wp $1/2$: A → left: $1/4$
A → right: $1/4$

B → left: $1/4$: wp $1/2$: A → still / unchanged
B → right: $1/4$

↓ Same distribution ↓ Same distribution

Correlated

Only A or B moves

If A walks from v_1 to v_n, then A hits B
↳ $O(n^2)$ time, in expectation

After A meets B: same movement

$\Rightarrow O(n^2 \log n)$ time, whp, A meets B

$\Rightarrow$ After $O(n^2 \log n)$ steps, A is polynomially close to uniform.

Alg for choosing random point on line:

1) Do a random walk for $O(n^2 \log n)$ time.

2) Return current location

Better example: Choose a <u>random</u> spanning tree in a graph.

<u>Fact</u>

choose edge [illegible] first [illegible] node —

Example: Clique

On a clique, the mixing time is one!

Card Shuffling

Deck of n cards (How many states?)

Algorithm

Repeat
1) Choose random card
2) Place card on top.

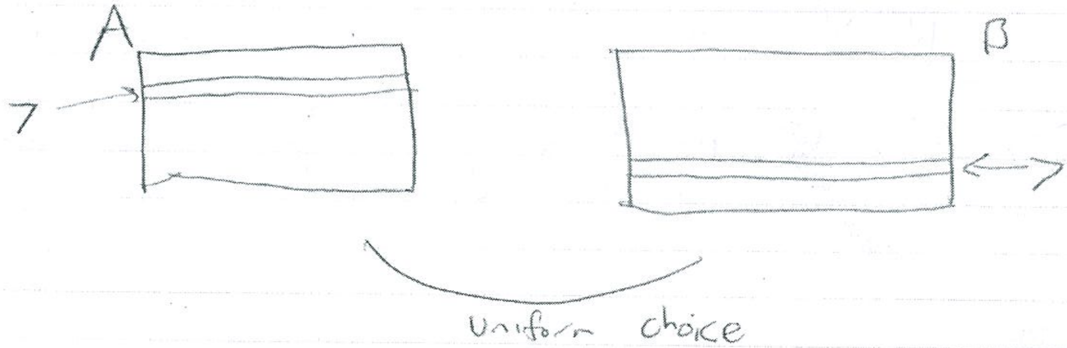
Markov Chain: states = permutations
transitions = put card on top

Stationary distribution π = Uniform over all permutations

How long until deck is shuffled?

Coupling: A is your deck
B is shuffled deck

Correlate: Choose same card in each deck



__Observe__

If we choose card X, then it is in the same position in both decks.

→ initially : 1

→ after each step : if chosen → 1

else → position $p+1$

When is $A == B$?

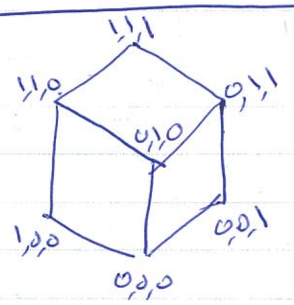
→ after each card has been chosen at least once

Coupon Collector!

After $O(n \log n)$ steps, $A = B$ whp.

⇒ Deck is shuffled.

Hypercube



A random walk on a hypercube is another good example of coupling.

$N = 2^n$ nodes: $X_1 X_2 X_3 \ldots, X_n$ $\quad X_j \in \{0,1\}$

R.W.: wp $1/2$ do nothing
wp $\frac{1}{2} \cdot 1/n$ flip neighbor n ⟺ choose coord j $\frac{1}{n}$
choose value v $1/2$
set $X_j = v$

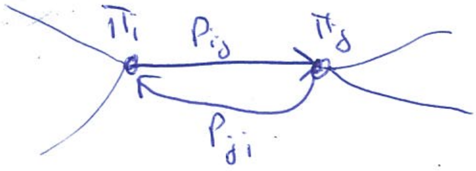
Couple: Both choose same coord, value.

$\Rightarrow O(n \log n) \geq T(\varepsilon)$

Time reversible Markov Chains are easier to analayze. Basically, any Time Reversible Markov Chain can be represented as a weighted undirected graph.

Time reversible:

For all i, j: $\pi_i P_{i,j} = \pi_j P_{ji}$

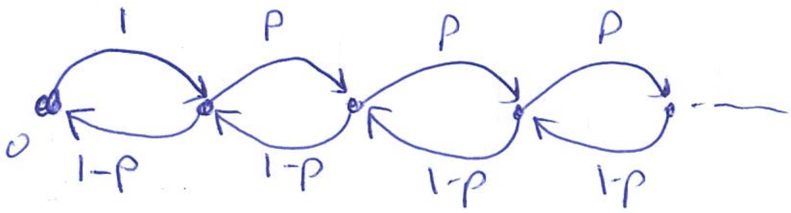


$\pi_i P_{ij} \approx$ frequency of $i \to j$

<u>Claim</u> If finite, irreducible:

if P is time reversible w.r.t. π, then π is unique stationary distribution

<u>Ex</u>



$$\boxed{p < \frac{1}{2}}$$

In this example, we want to find the stationary distribution for the random walk that goes right with probability p, left with probability 1-p, and "bounces" of zero, i.e., from zero it always goes left. By using the fact that it is time reversible, we get an easy calculation of the stationary distribution.

This chain represents a queue, and analyses like this show up in queuing theory.

Assume time reversible...

$$\left.\begin{aligned} \pi_0 \cdot 1 &= \pi_1 (1-p) \\ p\pi_j &= (1-p)\pi_{j+1} \end{aligned}\right\}$$

$$\pi_1 = \frac{\pi_0}{1-p}$$

$$\pi_2 = \left(\frac{p}{1-p}\right)\pi_1 = \left(\frac{p}{1-p}\right) \cdot \frac{1}{1-p}\pi_0$$

$$\pi_3 = \left(\frac{p}{1-p}\right)\pi_2 = \left(\frac{p}{1-p}\right)^2 \pi_1$$

$$= \left(\frac{p}{1-p}\right)^2 \frac{1}{1-p}\pi_0$$

$$\pi_j = \left(\frac{\rho}{1-\rho}\right)^{j-1} \cdot \frac{1}{1-\rho} \cdot \pi_0$$

$$\sum \pi_j = 1$$

$$\pi_0 \left(1 + \sum_{j=1}^{\infty} \frac{1}{1-\rho} \cdot \left(\frac{\rho}{1-\rho}\right)^{j-1}\right) = 1$$

$$\pi_0 \left(1 + \frac{1}{1-\rho}\left[\frac{1}{1-\frac{\rho}{1-\rho}}\right]\right) = 1 \qquad \boxed{\frac{\rho}{1-\rho} < 1}$$

$$\pi_0 \left(1 + \frac{1}{1-\rho}\left[\frac{1}{\frac{1-2\rho}{1-\rho}}\right]\right) = 1$$

$$\pi_0 \left(1 + \frac{1}{1-2\rho}\right) = 1$$

$$\pi_0 = \frac{1-2\rho}{2-2\rho}$$

$$\pi_j = \left(\frac{1-2\rho}{2(1-\rho)^2}\right) \left(\frac{\rho}{1-\rho}\right)^{j-1}$$

Metropolis - Hastings:

→ Graph G

→ target distribution: choose node u w.p. $\frac{w(u)}{W}$

$$W = \sum w(u)$$

E.g. uniform sampling

Idea: time reversible

Choose P_{ij} st $\pi_i P_{ij} = \pi_j P_{ji}$

$$\Rightarrow \frac{w(u)}{W} P_{uv} = \frac{w(v)}{W} P_{vu}$$

Let $A(u,v) = \min\left(1, \frac{w(v)}{w(u)} \cdot \frac{d(u)}{d(v)}\right)$

~~Let A(v,u) = min(1,~~

Define: $P_{uv} = \frac{1}{d(u)} \cdot A(u,v)$

Claim: $\pi_u = \frac{w(u)}{W}$

Proof: Show that balance equation holds

Martingale

A Martingale is a sequence where the expectation of the next item is equal to the previous item.

$Z_1, Z_2, \ldots$

$Z_{t+1} = f(Z_1, \ldots, Z_t)$

$E[|Z_n|] < \infty$

$E[Z_{n+1} \mid Z_0, \ldots, Z_n] = Z_n$

Ex.

$Z_0 = 100$

At time t bet ~~X~~ dollars:

$Z_{t+1} = \cancel{Z}_t + X$ wp $1/2$

$= Z_t - X$ wp $1/2$

$E[Z_{t+1} \mid Z_t] = Z_t$

Ex

Random Walk —

Nice: $E[X_t] = E[X_0]$

→ induction

$$E[Z_{i+1} \mid Z_0, \dots, Z_i] = Z_i$$

$$E[Z_{i+1}] = E\left[E[Z_{i+1} \mid Z_0, \dots, Z_i]\right] = E[Z_i]$$

Idea:

Bet 1 dollar.

Stop when $Z_t \geq 10$.

↳ at time T

Definition of a Stopping Time:

Stopping Time: T = when to stop

decide to stop at time t

based only on $Z_0, \dots, Z_t$

⇒ event $\{T = n\}$ depends only on

"first time condition XXX holds" ⇒ yes

"last time XXX occurs" ⇒ no

Question: $E[X_T] = X_0$??

Ex

"Stop when $X_t \geq 10$"

$$E[X_T] = X_0 = 0 \text{ ??}$$

No: $E[X_T] \geq 10$!!

Stopping Theorem

This theorem lets you show the expected value at a Stopping Time.

$Z_0, \ldots,$ is Martingale

T is stopping time.

One of the following:

1. All $|Z_j| \leq C$ ← bounded Martingale
2. $T \leq C$ ← bounded stopping time
3. $E[T] < \infty$

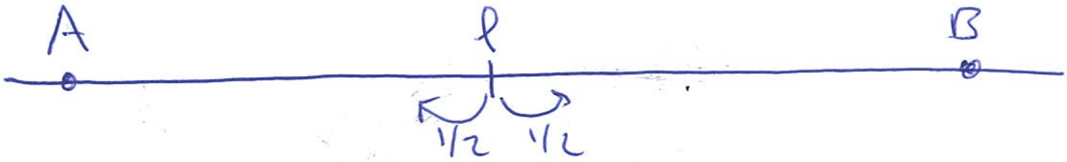
   and

   $E[|Z_{i+1} - Z_i| \mid Z_0, \ldots, Z_i] < C$

Then: $E[Z_T] = E[Z_0]$

Ex.

We can use the Stopping Time Theorem to analyze the (1/2, 1/2) random walk.



Random walk: 1/2, 1/2

Stop when $Z_t = A$ or $Z_t = B$

$$E[Z_T] = E[Z_0] = \ell$$

let $q = Pr(\text{stop at } B)$

$$E[Z_t] = q(B) + (1-q)A = \ell$$

$$qB + A - Aq = \ell$$

$$q = \frac{\ell - A}{B - A}$$

if $\ell = 1$, $A = 0$, $B = n \Rightarrow \frac{1}{n} = Pr(\text{stop at } B)$

This is the same result we found earlier via direction calculation: We are calculating the probability that a random walk reaches endpoint B before it reaches endpoint A, if it is walking on a line between points A and B.

$$Y_t = Z_t^2 - n$$

This is a neat trick for finding out in expectation how long it will take for the random walk to hit one of the two endpoints.

Show: Y_t is martingale

$$\Rightarrow E[Y_T] = E[Z_T^2 - T] = \ell^2 - 0$$

$$\Rightarrow E[T] = E[Z_t^2] - \ell^2$$

$$E[Z_t^2] = (1-q)A^2 + qB^2$$

$$= \left(\frac{B-\ell}{B-A}\right)A^2 + \left(\frac{\ell-A}{B-A}\right)B^2$$

if $A=0$: $\quad \frac{\ell}{n} \cdot n^2 = \ell n$

$B=n$:

$$\Rightarrow E[T] = \ell n - \ell^2 = \ell(n-\ell)$$

$$Y_t = Z_t^2 - t$$

Here we show that Yt really is a martingale.

Martingale :

$$E[Y_{t+1}] = E[Z_{t+1}^2] - (t+1)$$

$$= \frac{1}{2}[Z_t + 1]^2 + \frac{1}{2}[Z_t - 1]^2 - t - 1$$

$$= \frac{1}{2}(Z_t^2 + 2Z_t + 1) + \frac{1}{2}(Z_t^2 - 2Z_t + 1) - t - 1$$

$$= Z_t^2 + 1 - t - 1$$

$$= Z_t^2 - t$$

$$= Y_t$$

Assume $p = 1/3$

$(1-p) = 2/3$

$Y_t = 2^{Z_t}$

Y_t = martingale

Pr (win B) dollars?

What about a random walk that goes left with probability 1/3 and right with probability 2/3?

You can also analyze that using the Martingale stopping theorem.

Define the Martingale Yt as shown, prove that is a Martingale, and then apply the stopping theorem.

$$Y_t = 2^{Z_t}$$

First we show that it is a martingale.

Martingale:

$$E[Y_{t+1}] = E[2^{Z_{t+1}}]$$

$$= E\left[\frac{1}{3} 2^{Z_t+1} + \frac{2}{3} 2^{Z_t-1}\right]$$

$$= E\left[\frac{2}{3} 2^{Z_t} + \frac{1}{3} 2^{Z_t}\right]$$

$$= E[2^{Z_t}]$$

$$= Y_t$$

Then we analyze the stopping time to compute the probability of hitting B.

$$E[Z_T] = qB + (1-q)0$$

$$E[Y_T] = q2^B + (1-q)2^0 = 2^\ell$$

$$q2^B + 1 - q = 2^\ell$$

$$q(2^B - 1) = 2^\ell - 1$$

$$q = \frac{2^\ell - 1}{2^B - 1}$$

Chernoff Bound for Martingales

Azuma-Hoeffding Inequality

Assume (Z_j) is a Martingale,

$|Z_{t+1} - Z_t| < C_t$. Then:

$$\Pr\left[|Z_t - Z_0| \geq \lambda\right] \leq 2e^{-\frac{\lambda^2}{2\sum C_j^2}}$$

Ex: Betting

$Z_t = Z_{t-1} \pm 1$ w.p. $1/2$

$Z_0 = Z_0$, $C_t \leq 1$

$$\Pr\left[|Z_t - Z_0| \geq \sqrt{4n\lg n}\right] \leq 2e^{\frac{-4n\lg n}{2n}}$$

$$\leq \frac{1}{n}$$

Balls-in-Bins

n balls

n bins

How many empty bins?

$$\Pr(\text{bin } j \text{ empty}) = \left(1-\frac{1}{n}\right)^n \approx \frac{1}{e}$$

$$A = E[\#\text{ empty bins}] = \frac{n}{e}$$

Prove: w.h.p. $\frac{n}{e} \pm \Theta(\sqrt{n \log n})$ empty bins

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$$Z_0 = E[A] = \frac{n}{e}$$

throw ball 1

$$Z_1 = E[A \mid X_1]$$

$$Z_2 = E[A \mid X_1, X_2]$$

$$Z_3 = E[A \mid X_1, X_2, X_3]$$

$\vdots$

$$Z_n = E[A \mid X_1, X_2, \dots, X_n] = \#\text{ empty bins} = A$$

after all balls thrown

Claim: (Z_j) is a martingale!

$$E[Z_1 \mid Z_0] = Z_0$$

$$E[Z_2 \mid Z_1, Z_0] = Z_1$$

$$\Rightarrow E[Z_n] = E[Z_0] = \frac{n}{e}$$

$$E_{X_1,\dots,X_n}\left[E_A[A \mid X_1,\dots,X_n]\right] = E[A]$$

Doob Martingale

→ Sequence of R.V. $X_1, X_2, \ldots, X_n$.

→ Bounded function $f(X_1, \ldots, X_n) \to \mathbb{R}$

Define $Z_j = E[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_j]$

Ex: X_j = bin in which ball j lands

f = # of empty bins

Z_j = E[# empty bins after j balls thrown]

Theorem: Z_j is a Martingale

Need to show: $E[Z_t \mid \underbrace{Z_1, \ldots, Z_{t-1}}_{\text{depend on } X_1, \ldots, X_{t-1}}] = Z_{t-1}$

$$\ldots = E[Z_t \mid X_1, \ldots, X_{t-1}] = E[E[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_t] \mid X_1, \ldots, X_{t-1}]$$

definition of Z_t

$$= E[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_{t-1}]$$

$$= Z_{t-1}$$

Balls in Bins

Z_t is a Doob Martingale.

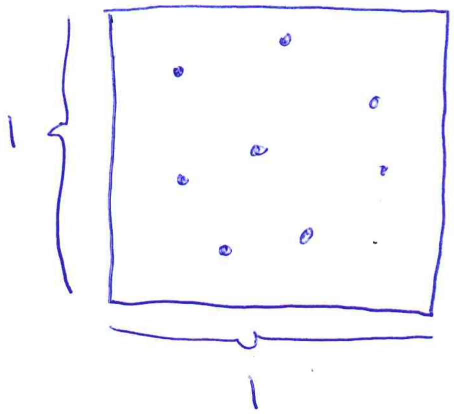
$c_t \leq 1$: one ball only changes the answer by at most 1

$$\Pr\left[\,|\underbrace{Z_n}_{\#\text{ empty bins}} - \underbrace{Z_0}_{\frac{n}{e}}| \geq \sqrt{4n\lg n}\,\right] \leq 2e^{-\frac{4n\lg n}{2n}}$$

$$\leq \frac{1}{n}$$

$\Rightarrow$ w.p. $\geq \left(1-\frac{1}{n}\right)$, # empty bins $= \frac{n}{e} \pm \sqrt{4n\lg n}$

④ Euclidean TSP



n random points on flat Euclidean plane

TSP: find shortest path to visit all points.

There exists a $(1+\varepsilon)$-approximation algorithm.

Prove: The length of the TSP $\to T$ route is ~~$\Theta(\sqrt{n})$~~ w.h.p.
$\Theta(\sqrt{n \log n})$

1) $E[T] = \Theta(\sqrt{n})$

Show: $E[T] \le 3\sqrt{n}$ ← easy, always true

$E[T] \ge \frac{1}{512}\sqrt{n}$ ← every node is connected to at least one neighbor!

2) Define Doob Martingale and show that wp $\ge (1 - \frac{1}{n})$, $T = \Theta(\sqrt{n})$.