Approximating Polygonal Objects by Deformable Smooth Surfaces

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Abstract. We propose a method to approximate a polygonal object by a deformable smooth surface, namely the t-skin defined by Edelsbrunner [5] for all $0 < t < 1$. We guarantee that they are homeomorphic and their Hausdorff distance is at most $\epsilon > 0$. This construction make it possible for fully automatic, smooth and robust deformation between two polygonal objects with different topologies. En route to our results, we also give an approximation of a polygonal object with a union of balls, which is a straightforward modification of our earlier work [4].

1 Introduction

Geometric deformation is a heavily studied topic in disciplines such as computer animation and physical simulation. Its challenges mainly are deformation between objects with different topologies, and maintaining a good quality mesh approximation of the deforming surface. Edelsbrunner defines a new paradigm for the surface representation to solve these problems, namely the *skin*. It provides a robust way of deforming one shape to another without any constraints on features such as topologies [2]. Moreover, the skin surfaces process nice properties such as curvature continuity which provide quality mesh approximation of the surface with guarantees such as triangle qualities [3]. However, most of the skin surface applications are still mainly on molecular modeling because of the intuition of its constitution by balls. The surface is not widely used in other fields because general geometric objects cannot be represented by the skin surfaces easily. This leaves a big gap between the nicely defined surfaces and its potential applications. We are trying to fill this gap in this paper.

1.1 Motivation and Related Works

Our main motivation for converting a polygonal object to a skin surface for deformation between objects. This is also a question asked by Amenta et. al in [1]. As noted earlier in some previous works [2, 5], deformation can be performed robustly and efficiently with the skin surface.

Moreover, our work here can also be viewed as a step toward converting an arbitrary smooth object into a provably accurate skin surface. In this regard, previous work has been done by Kruithof and Vegter [7]. For input the method requires a so-called r-admissible set of balls B which approximate the object well. Then, it expands all the weights of the balls by a carefully computed constant t, before taking the $\frac{1}{t}$-skin of the expanded balls to approximate the smooth object. There are two difficulties in such approach. First, such an r-admissible balls are not trivial to obtain. Furthermore, when the required factor t is closed to 1, the skin surface is almost the same as the union of balls, thus, does not give much improvement from the union of balls. On the other hand, our approach allows the freedom to choose any constant $0 < t < 1$ for defining the skin surface.

On top of the skin approximation, we also give an approximation of a polygonal object with a union of balls which has potential applications in computer graphics such as collision detection and deformation [6, 9, 10]. Ranjan and Fournier [9] proposed using a union of balls for object interpolation. Sharf and Shamir [10] also proposed using the same representation for shape matching. Those algorithms require a union of balls which accurately approximate the object as an input and to provide such a good set of balls at the beginning is still not trivial.

A comparison with our previous work. In [4], we proposed a method to construct of a set of weighted points whose alpha shape is the same as the input simplicial complex in $\mathbb{R}^d$, which we call the *subdividing alpha complex*, from which it is quite straightforward to obtain a set of balls which can be used to approximate the object. However, to construct the subdividing alpha complex, we need to make the assumption that the constrained triangulation of the input is given too.

In this paper the input is a piecewise linear complex which constitutes the boundary of the object. To avoid the assumption of constrained triangulation, we make use of the notion of *local gap size*(lgs) in the construction of the subdividing alpha complex.

1.2 Approach and Outline

First, we construct a set of balls whose alpha shape is the same as the boundary of the polygonal object, namely, the subdividing alpha complex. The radii of the balls constructed are at most $\epsilon > 0$.

Then, we fill the interior with balls according to the Voronoi complex. Specifically, we consider all the Voronoi vertices which are inside the object. Each Voronoi vertex determines an orthogonal ball. The set of all such orthogonal balls can be used to approximate the object. We will show that the union of such balls is homeomorphic to the object and furthermore, the Hausdorff distance between them is at most ϵ.

To obtain the skin approximation, we invert the weights of the balls that make up the subdividing alpha complex of the boundary. Those inverted balls, together with the balls in the interior of the object, generate a skin surface which is homeomorphic to the object and with the Hausdorff distance between them is at most ϵ.

Outline. This paper is organized as follow. In the next section we introduce some basic terminologies on piecewise linear complex(PLC) and alpha complex. In Section 3 we describe our method in constructing the subdividing alpha complex of a given PLC and the approximation of a polygonal object with a union of balls. Then we briefly review the definition of the skin surface in Section 4. The object approximation by the skin surface is described in Section 5. Finally, we end with some discussions in Section 6.

2 Notations and Basic Definitions

In this section we introduce a few basic definitions that we use throughout this paper: polygonal objects, piecewise linear complexes and alpha complexes.

Polygonal objects. A polygonal object $\mathcal{O} \subseteq \mathbb{R}^3$ is a compact 3-manifold whose boundary is a piecewise linear 2-manifold. Our algorithm takes as an input a *piecewise linear complex*(PLC) which constitutes the boundary of $\mathcal{O}$.

Piecewise linear complexes. In $\mathbb{R}^3$, a piecewise linear complex is a set $\mathcal{P}$ of vertices, line segments and polygons with the following conditions: i) all elements on the boundary of an element in $\mathcal{P}$ also belong to $\mathcal{P}$, and, ii) if any two elements intersect, their intersection is a lower dimensional element in $\mathcal{P}$. The underlying space of $\mathcal{P}$ is denoted by $|\mathcal{P}| = \bigcup_{\sigma \in \mathcal{P}} \sigma$.

The set of all vertices, edges and polygons are referred to as 0-, 1- and 2-skeletons, respectively. The *local gap size* is a function $lgs : |\mathcal{P}| \mapsto \mathbb{R}$ where $lgs(x)$ is the radius of the smallest ball centered on x that intersects an element of $\mathcal{P}$ that does not contain x.

Alpha complexes. We describe a *weighted point* $b \in \mathbb{R}^3 \times \mathbb{R}$ by its *location* $z_b \in \mathbb{R}^3$ and its *weight* $w_b \in \mathbb{R}$, written also as (z_b, w_b). A weighted point b can also be viewed a *ball* with center z_b and radius $\sqrt{w_b}$, that is, the set of points whose distance to z_b is less than or equal to $\sqrt{w_b}$. If w_b is negative then b is an imaginary ball, which is, an empty set. In this paper, we will use the terms *ball* and *weighted point* interchangeably.

The *weighted distance* of a point $p \in \mathbb{R}^3$ to a ball b is defined as

$$\pi_b(p) = \|pz_b\|^2 - w_b.$$

Two balls b_1 and b_2 are *orthogonal* to each other if $\|z_{b_1} z_{b_2}\|^2 = w_{b_1} + w_{b_2}$.

Given a finite set of balls B, each ball $b \in B$ defines a *Voronoi cell* ν_b which consists of the points in $\mathbb{R}^3$ with weighted distance to b less than or equal to any other ball in B. For $X \subseteq B$, the *Voronoi cell* of X is

$$\nu_X = \bigcap_{b \in X} \nu_b.$$

If ν_X consists of only one point then it is called a *Voronoi vertex*. The collection of all Voronoi cells is called the *Voronoi complex* of B,

$$V_B = \{\nu_X \mid X \subseteq B \text{ and } \nu_X \neq \emptyset\}.$$

For a set of balls X, we abuse the notation z_X to denote the set of the ballcenters of X. The *Delaunay complex* of B is the collection of simplices,

$$D_B = \{\text{conv}(z_X)) \mid \nu_X \in V_B\}.$$

The *alpha complex* of B is a subset of the Delaunay complex D_B which is defined as follow,

$$\mathcal{K}_B = \{\text{conv}(z_X)) \mid \bigcup X \cap \nu_X \neq \emptyset\}.$$

The *alpha shape* of B is the underlying space of $\mathcal{K}_B$, namely, $|\mathcal{K}_B|$. Note that if $\text{conv}(z_X) \in \mathcal{K}_B$ then $\bigcap X \neq \emptyset$.

3 Subdividing Alpha Complex

Given a PLC $\mathcal{P}$ and a set of balls B, we say $\mathcal{K}_B$ subdivides $\mathcal{P}$ if $|\mathcal{K}_B| = |\mathcal{P}|$. In this section, we show how to construct B such that $\mathcal{K}_B$ subdivides $\mathcal{P}$. For this we need the following Lemma 1 which is a straightforward generalization of Theorem 1 in [4]. The proof is quite tedious and can be found in the Appendix.

Lemma 1. *Let $\mathcal{P}$ be a PLC. If B is a set of balls that satisfies the following two conditions:*

C1. *For $X \subseteq B$, if $\bigcap X \neq \emptyset$ then $\text{conv}(z_X) \subseteq \sigma$ for some $\sigma \in \mathcal{P}$, and,*
C2. *For each $\sigma \in \mathcal{P}$, define $B(\sigma) = \{b \in B \mid b \cap \sigma \neq \emptyset\}$.*
Then we have: $z_{B(\sigma)} \subseteq \sigma \subseteq \bigcup B(\sigma)$,

then $\mathcal{K}_B$ subdivides $\mathcal{P}$.

We call $\mathcal{K}_B$ a *subdividing alpha complex*, or in short SAC, of $\mathcal{P}$. Furthermore, if all the weights in B are less than a real value ϵ, then $\mathcal{K}_B$ is called an ϵ-SAC of $\mathcal{P}$.

To construct such a set of balls, we first construct the ϵ-SAC of the 0-skeleton of $\mathcal{P}$, followed by the 1-skeleton and then the 2-skeleton of $\mathcal{P}$. The construction of the ϵ-SAC of the 0-skeleton of $\mathcal{P}$ is trivial. For each vertex v in $\mathcal{P}$, we add a ball with center v and radius $r = \min(\gamma \cdot lgs(v), \sqrt{\epsilon})$ where γ is a real number between 0 and 0.5. As defined in Lemma 1, $B(v)$ is the singleton set consists of this ball.

To describe the construction of the ϵ-SAC of the 1- and 2-skeleton of $\mathcal{P}$, we need the notations of *restricted Voronoi complex*. The restricted Voronoi complex of a set of balls X on $\sigma \in \mathcal{P}$, denoted by $V_X(\sigma)$, is the complex which consists of $\nu_X \cap \sigma$, for all $\nu_X \in V_X$. A Voronoi vertex u in $V_X(\sigma)$ is called a *positive* vertex if $\pi_b(u) > 0$, for all $b \in X$. Note that such a vertex is outside every ball in X.

To determine whether a vertex is positive, it suffices to compute $\pi_{b'}(u)$ where u is the Voronoi vertex in the Voronoi cell of b'.

We construct the ϵ-SAC of the 1-skeleton of $\mathcal{P}$ according to Algorithm 1. The basic idea is to add a ball to a positive vertex in an edge until the edge is covered by the balls. In order to avoid unwanted elements other than the edge itself, we set the radius of every ball to be less than both $\sqrt{\epsilon}$ and γ times the *lgs* of the ballcenter, where γ is a real constant between 0 and 0.5.

Algorithm 1 Construction of the balls for the 1-skeleton

for all edge $\sigma \in \mathcal{P}$ **do**
    Let v_1, v_2 be the two vertices of σ.
    $X := B(v_1) \cup B(v_2)$
    while there exists a positive vertex u in $V_X(\sigma)$ **do**
        $r := \min(\gamma \cdot lgs(u), \sqrt{\epsilon})$
        $X := X \cup \{(u, r^2)\}$
    end while
    $B(\sigma) := X$
end for

The construction of the ϵ-SAC of the 2-skeleton of $\mathcal{P}$ is similar. For completeness, we present it as Algorithm 2.

Algorithm 2 Construction of the balls for the 2-skeleton

for all polygon $\sigma \in \mathcal{P}$ **do**
    Let $\tau_1, \ldots, \tau_m$ be the edges of σ.
    $X := B(\tau_1) \cup \cdots \cup B(\tau_m)$
    while there exists a positive vertex u in $V_X(\sigma)$ **do**
        $r := \min(\gamma \cdot lgs(u), \sqrt{\epsilon})$
        $X := X \cup \{(u, r^2)\}$
    end while
    $B(\sigma) := X$
end for

We claim that the alpha shape of the set $\bigcup_{\sigma \in \mathcal{P}} B(\sigma)$ produced is the ϵ-SAC of $\mathcal{P}$. That is, both Conditions C1 and C2 are satisfied as well as our algorithm terminates. Since every ball with center p has radius less than $0.5 \times lgs(p)$, it should be obvious that Condition C1 is satisfied. Condition C2 follows from Proposition 1 below. Lemma 2 establishes the termination of our algorithm.

Proposition 1. *Let X be a set of balls. Suppose $z_X \subseteq \sigma$. Then $\sigma \subseteq \bigcup X$ if and only if there is no positive vertex in* $\mathrm{V}_X(\sigma)$*.*

Proof. The "only if" part is immediate. We will show the "if" part. Suppose there is no positive Voronoi vertex in $V_X(\sigma)$. We claim that $\nu_b(\sigma) \subseteq b$ for all

$b \in X$. This claim follows from the fact that $\nu_b(\sigma)$ is the convex hull of its Voronoi vertices and bounded. Thus, by our assumption that all the Voronoi vertices are not positive, it is immediate that $\nu_b(\sigma) \subseteq b$ for any $b \in X$. Since σ is partitioned into $\nu_b(\sigma)$ for all $b \in X$, it follows that $\sigma \subseteq \bigcup X$.

To establish the termination of the algorithm, we need the following fact.

Fact 1. *Let $\sigma \in \mathcal{P}$. Suppose $\Gamma \subset \sigma$ is a closed region such that it does not intersect the boundary of σ. Then there exists a constant $c > 0$ such that for every point $p \in \Gamma$, $\mathrm{lgs}(p) > c$.*

Proof. We observe that *lgs* is a continuous function on σ. Thus, $\lim_{p_i \to p} lgs(p_i) = lgs(\lim_{p_i \to p}) = 0$ if and only if p is in the boundary of σ where $\{p_i\}$ is a convergent sequence of points in σ. The fact follows immediately.

Lemma 2. *Both algorithms 1 and 2 terminate.*

Proof. We first prove that Algorithm 1 terminates. It suffices to show that the `while`-loop does not iterate infinitely many times. The proof is by contradiction and it follows from the fact that each element σ in $\mathcal{P}$ is compact.

Assume to the contrary that for some edge $\sigma = (v_1, v_2) \in \mathcal{P}$ the `while`-loop iterates infinitely many times. That is, it inserts infinitely many balls to $B(\sigma)$ whose centers are in the region $\sigma - (b_1 \cup b_2)$ where $b_i \in B(v_i)$ for $i = 1, 2$. By Fact 1, there exists a constant $c > 0$ such that all the radii of the balls are greater than c. By the compactness of σ, some centers of the balls converges. It means that there are balls inserted with centers inside another ball. Thus, it violates our construction that the balls are inserted with centers on positive vertices. Therefore, the `while`-loop iterates only finitely many times. The proof of the termination of Algorithm 2 is similar.

3.1 Approximating polygonal object with a union of balls

Let $\mathcal{O}$ be a polygonal object and $\mathcal{P}$ be its boundary. Let $\mathcal{K}_B$ be an ϵ^2-SAC of $\mathcal{P}$. Consider T, the set of all tetrahedra of D_B which is inside $\mathcal{O}$. Each tetrahedron in T determines an orthogonal ball and we call the set of all these balls $B^{\perp}$.

Note that every balls in $B^{\perp}$ has positive weight. Moreover, we also have $\mathcal{O} - \bigcup B \subseteq \bigcup B^{\perp} \subseteq \mathcal{O}$. We observe that $\bigcup B^{\perp}$ approximates the object well as stated below.

Theorem 1. $\bigcup B^{\perp}$ *is homeomorphic to $\mathcal{O}$ and the Hausdorff distance between them is at most ϵ.*

4 Skin Surface

The skin surface was first defined by Edelsbrunner [5] based on an algebraic structure of balls. In this section we briefly review both the algebra of balls and the definition of the skin surface. Readers interested in a detailed treatment of the algebra of balls may find the text by Pedoe [8] useful.

Algebra of balls. The algebra of balls is based on a bijection $\phi : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^4$ defined as

$$\phi(b) = (z_b, \|z_b\|^2 - w_b).$$

The space $\mathbb{R}^4$ together with the usual componentwise addition and scalar multiplication forms a vector space. The addition and scalar multiplication operations are defined on $\mathbb{R}^3 \times \mathbb{R}$ in such a way that ϕ is an isomorphism, that is,

$$\phi(b_1 + b_2) = \phi(b_1) + \phi(b_2),$$
$$\phi(\gamma \cdot b) = \gamma \cdot \phi(b),$$

where $b_1, b_2, b \in \mathbb{R}^3 \times \mathbb{R}$ and $\gamma \in \mathbb{R}$. One can easily verify that

$$b_1 + b_2 = (z_{b_1} + z_{b_2}, w_{b_1} + w_{b_2} + 2\langle z_{b_1}, z_{b_2}\rangle), \tag{1}$$
$$\gamma b = (\gamma z_b, \gamma w_b + (\gamma^2 - \gamma)\|z_b\|^2). \tag{2}$$

By the two operations above, the convex combination of a set of balls $B = \{b_1, \ldots, b_n\}$ is the set of balls $\text{conv}(B) = \{\sum_i \gamma_i b_i \mid \sum_i \gamma_i = 1 \text{ and } \gamma_i \geq 0 \text{ for all } i = 1, \ldots, n\}$. It is straightforward to verify that if a ball b is orthogonal to every ball $b_i \in \{b_1, \ldots, b_n\}$, then b is orthogonal to every ball $b' \in \text{conv}(b_1, \ldots, b_n)$.

Skin surfaces. Let b be a weighted point and $t \in \mathbb{R}$, we define $b^t = (z_b, tw_b)$. For a set of balls B, B^t is defined as $B^t = \{b^t \mid b \in B\}$.

For $0 \leq t \leq 1$, the skin body of a set of balls B is defined as

$$\text{bdy}^t(B) = \bigcup \text{conv}(B)^t,$$

that is, the set of points obtained by shrinking all balls in the convex combination of B. The skin surface is the boundary of the skin body of B, denoted by $\text{skin}^t(B)$. Note that $\bigcup B = \text{body}^1(B)$. We cite here an important relation between a union of balls $\bigcup B$ and the skin body that it generates.

Theorem 2. [5] *The union of balls $\bigcup B$ is homeomorphic to $bdy^t(B)$, for $0 < t < 1$.*

5 Approximating a Polygonal Object with the Skin Surface

So far, our method in approximating a polygonal object with a union of balls can be summarized as follow.

1. Construct a set of balls B such that $\mathcal{K}_B$ is an ϵ^2-SAC of the boundary of the object.
2. Compute the Voronoi complex of B and let $B^\perp$ be the set of all orthogonal balls with centers inside $\mathcal{O}$.
3. Output $B^\perp$ as the approximation of $\mathcal{O}$.

In this section we will show that the set of balls $B^{\perp} \cup B^{-1}$ will generate a skin body that approximates the object well too, as stated in Theorem 3 below.

Theorem 3. *For all $0 \leq t \leq 1$, the skin body $bdy^t(B^{\perp} \cup B^{-1})$ is contained inside $\mathcal{O}$ and homeomorphic to it with Hausdorff distance between them is at most ϵ.*

Proof. All balls in B^{-1} have negative weights, thus, $\bigcup B^{\perp} \cup B^{-1} = \bigcup B^{\perp}$. By Theorem 1, $\bigcup B^{\perp} \subseteq \mathcal{O}$, thus, it follows that $\text{skin}^t(B^{\perp} \cup B^{-1}) \subseteq \bigcup B^{\perp} \cup B^{-1} = \bigcup B^{\perp} \subseteq \mathcal{O}$.

The homeomorphism follows from Theorem 2 that $\text{skin}^t(B^{\perp} \cup B^{-1})$ is homeomorphic to $\bigcup B^{\perp} \cup B^{-1} = \bigcup B^{\perp}$ which is homeomorphic to $\mathcal{O}$(Theorem 1).

The Hausdorff nearness from $\mathcal{O}$ to $\text{skin}^t(B^{\perp} \cup B^{-1})$ is more tedious. We present it in the next subsection.

5.1 Proof of the Hausdorff Nearness in Theorem 3

Note that for every point p in the object $\mathcal{O}$, there is a weighted point $b \in \text{conv}(B^{\perp} \cup B^{-1})$ such that $z_b = p$. In other words, $\mathcal{O} \subseteq \mathcal{Z}$ where $\mathcal{Z} = \{z_b \mid b \in \text{conv}(B^{\perp} \cup B^{-1})$. In view of this, it suffices to prove the following lemma.

Lemma 3. *For every ball $b \in \text{conv}(B^{\perp} \cup B^{-1})$ where $z_b \in \mathcal{O}$, if $w_b < 0$ then there exists a ball $b' \in \text{conv}(B^{\perp} \cup B^{-1})$ such that $w_{b'} > 0$ and $\|z_b z_{b'}\| \leq \epsilon$.*

We note that the object $\mathcal{O}$ can be partitioned into tetrahedra of Delaunay complex $D_{B^{\perp} \cup B}{}^{\star}$. We made a few simple observations concerning the tetrahedron of $D_{B^{\perp} \cup B}$ which is contained inside $\mathcal{O}$.

Fact 2. *Let $X = \{b_1, \ldots, b_4\}$ such that $\text{conv}(z_X)$ is a tetrahedron in $D_{B^{\perp} \cup B}$ and is contained inside $\mathcal{O}$. Then,*

1. *At least one of the balls in X is a ball in $B^{\perp}$.*
2. *If $b_i \in X \cap B^{\perp}$ and $b_j \in X \cap B$ then b_i and b_j are orthogonal to each other.*
3. *The simplex $\text{conv}(z_{B \cap X})$ is a simplex in $\mathcal{K}_B$, i.e. $\text{conv}(z_{B \cap X}) \subseteq |\mathcal{P}|$.*

Statements 1 and 2 are pretty straightforward. The intuition of Statement 3 is as follow. Let $X' = X \cap B$. It is clear when $\text{card}(X') = 1$. For $\text{card}(X) = 2$ or 3, assume to the contrary that $\text{conv}(z_{X'}) \notin \mathcal{K}_B$. Since $|\mathcal{K}_B| = |\mathcal{P}|$, the simplex $\text{conv}(z_{X'})$ is in the interior of $\mathcal{O}$. Then, there exist at least $5 - |X'|$ balls of $B^{\perp}$ which are orthogonal to every ball in $X'^{\star\star}$. These balls of $B^{\perp}$ make $\nu_X = \emptyset$, thus, yields a contradiction that $\text{conv}(z_X)$ is a Delaunay tetrahedron. Therefore, $\text{conv}(z_{X'}) \in \mathcal{K}_B$, where $X' = X \cap B$.

In view of Statement 3 in Fact 2, we categorize the tetrahedra of $D_{B^{\perp} \cup B}$ within $\mathcal{O}$ into four types according to $\text{card}(X \cap B)$. We illustrate it in Figure 1.

* Note that $D_{B^{\perp} \cup B}$ may not be the same as $D_{B^{\perp} \cup B^{-1}}$. The object $\mathcal{O}$ may not be partitioned into tetrahedra of $D_{B^{\perp} \cup B^{-1}}$.

** That is, if $\text{card}(X') = 2$, then $\dim(\text{conv}(z_{X'})) = 1$. So, $\text{conv}(z_{X'})$ is incident to at least three tetrahedra in D_B and each tetrahedron corresponds to one ball in $B^{\perp}$. Similarly, if $\text{card}(X') = 3$, then $\text{conv}(z_{X'})$ is incident to two tetrahedra in D_B and each tetrahedron correspond to one ball in $B^{\perp}$.

1. Tetrahedron type I is a tetrahedron where $\mathrm{card}(X \cap B) = 1$.
   In Figure 1, $b_1 \in B$ and $b_2, b_3, b_4 \in B^{\perp}$.
2. Tetrahedron type II is a tetrahedron where $\mathrm{card}(X \cap B) = 2$.
   In Figure 1, $b_1, b_2 \in B$ and $b_3, b_4 \in B^{\perp}$.
3. Tetrahedron type III is a tetrahedron where $\mathrm{card}(X \cap B) = 3$.
   In Figure 1, $b_1, b_2, b_3 \in B$ and $b_4 \in B^{\perp}$.
4. Tetrahedron type IV is a tetrahedron where $\mathrm{card}(X \cap B) = 0$.
   In Figure 1, all $b_1, b_2, b_3, b_4 \in B^{\perp}$.

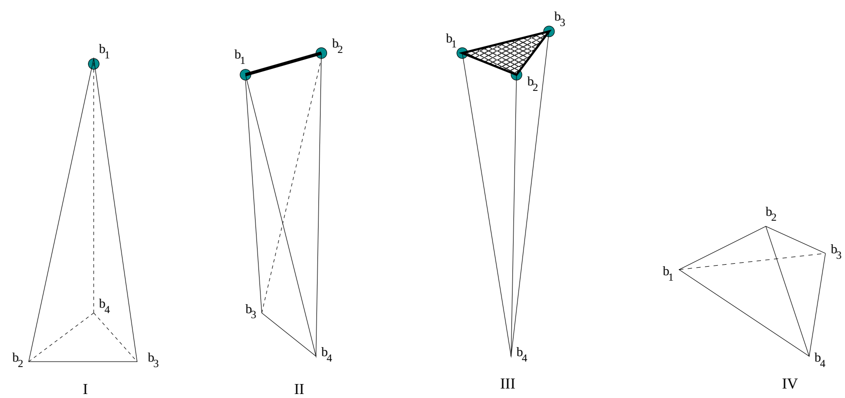

Fig. 1. The bold point in type I, the bold edge in type II and the the shaded triangle in the type III indicates that they are in $\mathcal{K}_B$, thus in the boundary of the object. None of the vertices in the type IV tetrahedron belongs to B.

In view of this, to prove Lemma 3 it is sufficient to prove the following.

Claim. Let $\mathrm{conv}(z_X) \in D_{B^{\perp} \cup B}$ and located inside $\mathcal{O}$. For every ball $b \in \mathrm{conv}(X)$, if $w_b < 0$ then there exists a ball $b' \in \mathrm{conv}(X)$ such that $w_{b'} > 0$ and $\|z_b z_{b'}\| \leq \epsilon$.

We divide the proof of the claim according to $|X \cap B|$, that is, the type of the tetrahedron that contains z_b. If $|X \cap B| = 4$ then all balls $b \in \mathrm{conv}(X)$ have weights $w_b > 0$.

The following Lemma 4 states that all points in $\mathrm{conv}(b_1^{-1}, b_2, b_3, b_4)$ (i.e. in tetrahedron type I) with negative weights are located within the ϵ-neighborhood of $z_{b_1^{-1}}$. This immediately implies the validity of claim for tetrahedron type I.

Lemma 4. *Let* $(p, w) \in \mathrm{conv}(b_1^{-1}, b_2, b_3, b_4)$*. If* $w \leq 0$ *then* $\|p z_{b_1^{-1}}\| \leq \epsilon$*.*

Proof. Let

$$\begin{aligned}(p, w) &= \gamma_1 b_1^{-1} + \gamma_2 b_2 + \gamma_3 b_3 + \gamma_4 b_4 \\ &= \gamma_1 b_1^{-1} + (1 - \gamma_1) b',\end{aligned}$$

where $b' = \frac{1}{1-\gamma_1}\sum_{i=2}^{4}\gamma_i b_i$ and $\sum \gamma_i = 1$ and $\gamma_i \geq 0$, for $i = 1, \ldots, 4$.

Since b_2, b_3, b_4 are all orthogonal to b_1, then b' is also orthogonal to b_1, i.e. $w_{b'} + w_{b_1} = \|z_{b_1} z_{b'}\|^2$.

We apply the formula of combination of weighted points:

$$w = (1-\gamma_1)w_{b'} + \gamma_1 w_{b_1^{-1}} + (\gamma_1^2 - \gamma_1)\|z_{b'} z_{b_1^{-1}}\|^2.$$

Since $w \leq 0$, we arrange the terms into

$$\gamma_1^2\|z_{b'} z_{b_1^{-1}}\|^2 - \gamma_1\|z_{b'} z_{b_1^{-1}}\|^2 - \gamma_1(w_{b'} + w_{b_1}) + w_{b'} \leq 0 \quad (3)$$

$$\gamma_1^2\|z_{b'} z_{b_1^{-1}}\|^2 - 2\gamma_1\|z_{b'} z_{b_1^{-1}}\|^2 \leq -w_{b'} \quad (4)$$

$$\gamma_1^2\|z_{b'} z_{b_1^{-1}}\|^2 - 2\gamma_1\|z_{b'} z_{b_1^{-1}}\|^2 + \|z_{b'} z_{b_1^{-1}}\|^2 \leq \|z_{b'} z_{b_1^{-1}}\|^2 - w_{b'} \quad (5)$$

$$(\gamma_1 - 1)^2\|z_{b'} z_{b_1^{-1}}\|^2 \leq w_{b_1} \quad (6)$$

$$(1-\gamma_1)^2\|z_{b'} z_{b_1^{-1}}\|^2 \leq \epsilon^2 \quad (7)$$

$$\|p z_{b_1^{-1}}\| \leq \epsilon \quad (8)$$

From Inequality 3 to Inequality 4 and Inequality 5 to Inequality 6, we apply $w_{b'} + w_{b_1} = \|z_{b_1} z_{b'}\|^2$. From Inequality 7 to Inequality 8, we apply $\|p z_{b_1^{-1}}\| = (1-\gamma_1)\|z_{b'} z_{b_1^{-1}}\|$.

The validity of the claim for tetrahedra type II and III is presented as Lemma 5 and 6 below. Lemma 5 states that all points in $\mathrm{conv}(b_1^{-1}, b_2^{-1}, b_3, b_4)$ (i.e. in tetrahedron type II) with negative weights are located within the ϵ-neighborhood of $\mathrm{conv}(z_{b_1^{-1}, b_2^{-1}})$. Similarly, Lemma 6 states that all points in $\mathrm{conv}(b_1^{-1}, b_2^{-1}, b_3^{-1}, b_4)$ (i.e. in tetrahedron type III) with negative weights are located within the ϵ-neighborhood of $\mathrm{conv}(z_{b_1^{-1}, b_2^{-1}, b_3^{-1}})$. Both proofs are just a slight twist of the proof of Lemma 4. For completeness, we present it below.

Lemma 5. *Let* $(p, w) = \mathrm{conv}(b_1^{-1}, b_2^{-1}, b_3, b_4)$*. If* $w \leq 0$ *then there exists* $b' \in \mathrm{conv}(b_1^{-1}, b_2^{-1})$ *such that* $\|p z_{b'}\| \leq \epsilon$*.*

Proof. Let

$$\begin{aligned}(p, w) &= \gamma_1 b_1^{-1} + \gamma_2 b_2^{-1} + \gamma_3 b_3 + \gamma_4 b_4 \\ &= \gamma b' + (1-\gamma) b'',\end{aligned}$$

where $\gamma = \gamma_1 + \gamma_2$ and $b' = \frac{1}{\gamma_1+\gamma_2}\sum_{i=1}^{2}\gamma_i b_i^{-1}$ and $b'' = \frac{1}{\gamma_3+\gamma_4}\sum_{i=3}^{4}\gamma_i b_i$.

Since b_1, b_2 are orthogonal to each of b_3, b_4, then b' is also orthogonal to b'', i.e. $w_{b'} + w_{b''} = \|z_{b'} z_{b''}\|^2$. Following the Inequalities 3 to 8 in the proof of Lemma 4, we can obtain $\|p z_{b'}\| \leq \epsilon$.

Lemma 6. *Let* $(p, w) = \mathrm{conv}(b_1^{-1}, b_2^{-1}, b_3^{-1}, b_4)$*. If* $w \leq 0$ *then there exists* $b' \in \mathrm{conv}(b_1^{-1}, b_2^{-1}, b_3^{-1})$ *such that* $\|p z_{b'}\| \leq \epsilon$*.*

Proof. Similar to the above, except that

$$\begin{aligned}(p, w) &= \gamma_1 b_1^{-1} + \gamma_2 b_2^{-1} + \gamma_3 b_3^{-1} + \gamma_4 b_4 \\ &= (1 - \gamma_4) b' + \gamma_4 b_4,\end{aligned}$$

where $b' = \frac{1}{\gamma_1+\gamma_2+\gamma_3} \sum_{i=1}^{3} \gamma_i b_i^{-1}$ and b' and b_4 are orthogonal. Similarly, following the Inequalities 3 to 8 in the proof of Lemma 4, we can obtain $\|pz_{b'}\| \leq \epsilon$.

6 Discussion

One future direction is to implement the same idea in approximating smooth objects with skin surfaces. Amenta et.al [1] showed that given a sufficiently dense sample points on a smooth surface, the set of polar balls obtained can be used to approximate the object well. There is analogy between such approach with our method here. We can view the ϵ-SAC constructed as the sample points and $B^{\perp}$ as the polar balls.

By appropriately assigning certain weights to the sample points and taking the polar balls, we hope to be able to approximate the smooth object by a skin surface. At this point, the usefulness of this idea is still under investigation.

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Appendix. Proof of Theorem 1

It is obvious that the following two properties are sufficient conditions for subdividing alpha complex:

P1. Every simplex in $\mathcal{K}_B$ is contained in an element in $\mathcal{P}$.
P2. Every element in $\mathcal{P}$ is a union of some simplices in $\mathcal{K}_B$.

We divide the theorem into two lemmas. Lemma 7 states that Condition C1 imply property P1, while Lemma 8 states that Condition C2 implies property P2.

Lemma 7. *If B satisfies Condition $C1$, then every simplex in $\mathcal{K}_B$ is contained in an element in $\mathcal{P}$, that is, property P1.*

Proof. It is immediate that every vertex in $\mathcal{K}_B$ is inside an element in $\mathcal{P}$. Let $\text{conv}(z_X)$ be a simplex in $\mathcal{K}_B$. By the remark in the definition of alpha complex, $\bigcap X \neq \emptyset$. Then, by Condition C1, there is an element $\sigma \in \mathcal{P}$ such that $\text{conv}(z_X) \subseteq \sigma$.

Lemma 8. *If B satisfies Condition C2, then every element in $\mathcal{P}$ is a union of some simplices in $\mathcal{K}_B$, that is, property P2.*

Proof. We divide the proof into two stages:

Stage 1. We show that for every $\sigma \in \mathcal{P}$, $\sigma \subseteq |\mathcal{K}_{B(\sigma)}|$, where $B(\sigma)$ is as defined in Lemma 1.
Stage 2. We show that $\mathcal{K}_{B(\sigma)} \subseteq \mathcal{K}_B$.

The first stage is further divided into three parts according to the dimension of σ.

1. $\dim \sigma = 0$.
   Then $B(\sigma)$ consists of a ball with center on σ. Thus, $\sigma = \mathcal{K}_{B(\sigma)}$.
2. $\dim \sigma = 1$.
   Since $z_{B(\sigma)} \subseteq \sigma$, σ is partitioned into the Delaunay edges of $D_{B(\sigma)}$[⋆⋆⋆]. Furthermore, because $\sigma \subseteq \bigcup B(\sigma)$, every Delaunay edges $\text{conv}(z_{b_1,b_2}) \in D_{B(\sigma)}$, where $b_1, b_2 \in B(\sigma)$, is covered by $b_1 \cup b_2$.
   The midpoint $p \in \text{conv}(z_{b_1,b_2})$, where $\pi_{b_1}(p) = \pi_{b_2}(p)$, is in the Voronoi cell ν_{b_1,b_2} with respect to the Voronoi complex $V_{B(\sigma)}$. Thus, $p \in (b_1 \cup b_2) \cap \nu_{b_1,b_2}$ and it implies $\text{conv}(z_{b_1,b_2}) \in \mathcal{K}_{B(\sigma)}$. Therefore, σ is partitioned into $\mathcal{K}_{B(\sigma)}$.
3. $\dim \sigma = 2$.
   The reasoning is similar to the above case. Since $z_{B(\sigma)} \subseteq \sigma$, σ is partitioned into the Delaunay triangles of $D_{B(\sigma)}$. Furthermore, because $\sigma \subseteq \bigcup B(\sigma)$, every Delaunay triangles $\text{conv}(z_{b_1,b_2,b_3}) \in D_{B(\sigma)}$, where $b_1, b_2, b_3 \in B(\sigma)$, is covered by $b_1 \cup b_2 \cup b_3$.
   The midpoint $p \in \text{conv}(z_{b_1,b_2,b_3})$, where $\pi_{b_1}(p) = \pi_{b_2}(p) = \pi_{b_3}(p)$, is in the Voronoi cell ν_{b_1,b_2,b_3} w.r.t. $V_{B(\sigma)}$. Thus, $p \in (b_1 \cup b_2 \cup b_3) \cap \nu_{b_1,b_2,b_3}$ and it implies $\text{conv}(z_{b_1,b_2,b_3}) \in \mathcal{K}_{B(\sigma)}$. Therefore, σ is partitioned into $\mathcal{K}_{B(\sigma)}$.

Now we show that $\mathcal{K}_{B(\sigma)} \subseteq \mathcal{K}_B$ for every $\sigma \in \mathcal{P}$. Note that every ball $b \in B - B(\sigma)$, $b \cap \sigma = \emptyset$. Thus, for every midpoint $p \in \text{conv}(z_X)$, where $X \subseteq B(\sigma)$ and $\pi_{b'}(p) = \pi_{b''}(p)$ for all $b', b'' \in X$, p is in ν_X w.r.t. $V_{B(\sigma)}$. Therefore, $p \in \bigcup X \cap \nu_X$ and it implies $\text{conv}(z_X) \in \mathcal{K}_B$.

⋆⋆⋆ Recall that if v_1, v_2 are vertices of σ, then $B(v_1), B(v_2) \subseteq B(\sigma)$. Similarly, if σ is a polygon and $e_1, \ldots, e_m$ are the edges then $B(e_1), \ldots, B(e_m) \subseteq B(\sigma)$.