Approximating Polyhedral Objects with Deformable Smooth Surfaces

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Abstract

We propose a method to approximate a polyhedral object with a deformable smooth surface, namely the t-skin defined by Edelsbrunner for all $0 < t < 1$. We guarantee that they are homeomorphic and their Hausdorff distance is at most $\varepsilon > 0$. This construction makes it possible for fully automatic, smooth and robust deformation between two polyhedral objects with different topologies. En route to our results, we also give an approximation of a polyhedral object with a union of balls.

1 Introduction

Geometric deformation is a heavily studied topic in disciplines such as computer animation and physical simulation. One of the main challenges is to perform deformation between objects with different topologies, while at the same time maintaining a good quality mesh approximation of the deforming surface.

Edelsbrunner defines a new paradigm for the surface representation to solve these problems, namely the *skin surface* [12] which is a smooth surface based on a finite set of balls. It provides a robust way of deforming one shape to another without any constraints on features such as topologies [4]. Moreover, the skin surfaces possess nice properties such as curvature continuity which provides quality mesh approximation of the surface [5, 6].

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However, most of the skin surface applications are still mainly on molecular modelling. The surface is not widely used in other fields because general geometric objects cannot be represented by the skin surfaces easily. This leaves a big gap between this nicely defined surface and its potential applications. We aim to fill this gap in this paper.

1.1 Motivation and Related Works

One of the main goals of the work by Amenta et. al in [1] is to convert a polyhedral object into a skin surface. We can view our work here as achieving this goal and the purpose of doing so is to perform deformation between polyhedral objects. As noted earlier in some previous works [12, 4], deformation can be performed robustly and efficiently if the object is represented by the skin surface [2, 8].

Moreover, our work here can also be viewed as a step towards converting an arbitrary smooth object into a provably accurate skin surface. In this regard, previous work has been done by Kruithof and Vegter [15]. For the input their method requires a set of r-admissible balls which approximate the object well. Then, it expands all the weights of the balls by a carefully computed constant t, before taking the $\frac{1}{t}$-skin of the expanded balls to approximate the smooth object.

However, we observe that there are at least two difficulties likely to occur in such approach. First, such a set of r-admissible balls is not trivial to obtain. Furthermore, when the computed factor t is closed to 1, the skin surface is almost the same as the union of balls, thus, does not give much improvement from the union of balls. On the other hand, our approach proposed here allows the freedom to choose any constant $0 < t < 1$ for defining the skin surface.

On top of the skin approximation, we also give an approximation of a polygonal object with a union of balls. Such approximation has potential applications in computer graphics such as collision detection and deformation [14, 18, 17]. Ranjan and Fournier [17] proposed using a union of balls for object interpolation. Sharf and Shamir [18] also proposed using the same representation for shape matching. Those algorithms require a union of balls which accurately approximate the object as an input and providing such a good set of balls at the beginning is still not trivial.

At last, as a by-product, our algorithm also gives the constrained Delaunay triangulation of a polyhedral object.

1.2 Approach

Given a polyhedral object, $\mathcal{O} \subset \mathbb{R}^3$, the first step is to construct a set of balls B whose alpha shape [11] is the same as the boundary of $\mathcal{O}$, namely, the *subdividing alpha complex*. All centers of the balls in B lie on the boundary of $\mathcal{O}$ and their radii are at most ε which is a positive number specified by the user. The set B covers the boundary and acts as a protecting layer similar to some previous work in Delaunay mesh generation and conforming Delaunay triangulation [9, 10].

In the second step, we fill the interior of $\mathcal{O}$ with another set of balls $B^{\perp}$. From the weighted Delaunay tetrahedralization of B, we extract all the tetrahedra in the interior of $\mathcal{O}$. Each ball in $B^{\perp}$ is an orthogonal ball of such tetrahedron. It is shown that the union of $B^{\perp}$, namely, the space that is occupied by balls in $B^{\perp}$, is homeomorphic to $\mathcal{O}$ and furthermore, the Hausdorff distance between them is at most ε.

To obtain the skin approximation, we construct a set of orthogonal balls B^* through some modifications of the balls B. The skin surface is proved to be homeomorphic to $\partial\mathcal{O}$ and their Hausdorff distance is at most ε as well.

1.3 Outline

This paper is organized as follows. We start by reviewing some basic concepts and results in Sections 2 and 3 that will be used throughout this paper, namely, weighted points, Delaunay complexes and alpha complexes. In Section 4 we introduce the concept of the subdividing alpha complex and propose the algorithms to compute it. We describe our method of ball approximation in Section 5 and the skin approximation in Section 6. Finally, we end with some concluding remarks in Section 7.

2 The Voronoi Complex of Weighted Points

In this section we will briefly review the basic definitions and notations of weighted points and their Voronoi complexes and Delaunay Complexes.

2.1 The Weighted Points

We describe a *weighted point* $b \in \mathbb{R}^d \times \mathbb{R}$ by its location $z_b \in \mathbb{R}^d$ and its weight $w_b \in \mathbb{R}$. The weighted point b can also be written as (z_b, w_b). We assume that a point $p \in \mathbb{R}^d$ is a point of zero weight when the weight is not specified.

A weighted point b can be alternatively interpreted as *an open ball* with center z_b and radius $\sqrt{w_b}$, which is the set of points $\{p \in \mathbb{R}^d \mid \|p - z_b\|^2 < w_b\}$. However, if w_b is zero, we treat b as the set containing the point z_b only. If w_b is negative then we treat b as an empty set.

For a set of weighted points $X = \{b_1, \ldots, b_n\}$, we use the notation $\bigcup X$ to denote $b_1 \cup \cdots \cup b_n$ where each b_i is viewed as a ball. Similarly, we write $\bigcap X$ to denote $b_1 \cap \cdots b_n$. In this paper the terms *ball* and *weighted point* will be used interchangeably.

Affine Hulls of Balls. Given a set of balls $B = \{b_1, b_2, ..., b_n\}$, we define the *affine hull* * of B as

$$\mathrm{aff}(B) \quad = \quad \Big\{ \sum_{i=0}^{n} \lambda_i b_i \mid \sum_{i=0}^{n} \lambda_i = 1 \Big\}. \tag{1}$$

To complete the definition, we need the addition and scalar multiplication of weighted points. Define a bijective lifting map $\phi : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^4$ such that for a ball $b = (z_b, w_b)$, $\phi(b)$ has the first three coordinates same as z_b and the last coordinate of $\phi(b)$ is $\|z_b\|^2 - w_b$. The addition and scalar multiplication operations are defined on $\mathbb{R}^3 \times \mathbb{R}$ in such a way that ϕ is a vector space isomorphism, that is,

$$\begin{aligned} b_1 + b_2 &= \phi^{-1}(\phi(b_1) + \phi(b_2)), \\ \gamma \cdot b_1 &= \phi^{-1}(\gamma \cdot \phi(b_1)), \end{aligned}$$

where $b_1, b_2 \in \mathbb{R}^3 \times \mathbb{R}$ and $\gamma \in \mathbb{R}$.

Orthogonal Balls. The *weighted distance* between two weighted points b_1 and b_2 is defined as follows:

$$\pi_{b_1}(b_2) = \pi_{b_2}(b_1) = \|z_{b_1} - z_{b_2}\|^2 - w_{b_1} - w_{b_2}.$$

A point $p \in \mathbb{R}^d$ is inside the ball b if and only if $\pi_b(p) < 0$. Two weighted points, b_1 and b_2, are said to be *orthogonal* to each other if the weighted

*This affine hull definition is only for a set of balls. For affine hull of an unweighted point set, we still use the conventional definition.

distance between them is zero, denoted as $b_1 \perp b_2$. Note that if $b_1 \perp b_2$ and the weight of b_1 is positive, z_1 is out of b_2.

We write $B_1 \perp B_2$ if $b_1 \perp b_2$ for all $b_1 \in B_1$ and $b_2 \in B_2$. From the thesis of Cheng [3], it is proven that

$$B_1 \perp B_2 \Rightarrow \mathrm{aff}(B_1) \perp \mathrm{aff}(B_2). \tag{2}$$

2.2 The Voronoi Complexes and Delaunay Complexes

A Voronoi complex is a partition of the space $\mathbb{R}^d$ according to a finite set of balls. Let $B = \{b_1, \ldots, b_n\}$ be a set of n balls. The *Voronoi cell* of the ball b_i, with respect to B, is

$$\nu_{b_i} = \{p \in \mathbb{R}^d \mid \pi_{b_i}(p) \leq \pi_{b_j}(p) \text{ for all } j = 1, \ldots, n\}.$$

For a set of balls $X \subseteq B$, the *Voronoi cell* of X with respect to B is

$$\nu_X = \bigcap_{b \in X} \nu_b.$$

For every point $p \in \nu_X$, we have $\pi_{b_i}(p) = \pi_{b_j}(p)$ for all $b_i, b_j \in X$. The set ν_X is known to be convex. The *dimension* of ν_X is defined as the dimension of the minimal affine space that contains ν_X. If the dimension of ν_X is zero, then ν_X consists of only one point. We call this point a *Voronoi vertex.*

The *Voronoi complex* of B, V_B, is the collection of all the non-empty Voronoi cells:

$$V_B = \{\nu_X \mid X \subseteq B \text{ and } \nu_X \neq \emptyset\}.$$

Throughout this paper, we make an important but standard assumption regarding V_B:

General Position Assumption. *Let $B \subseteq \mathbb{R}^d \times \mathbb{R}$ be a finite number of set of balls and let $X \subseteq B$. Suppose $\nu_X \neq \emptyset$. Then $1 \leq \mathrm{card}(X) \leq d + 1$ and the dimension of ν_X is $d + 1 - \mathrm{card}(X)$.*

Such assumption can be achieved by small perturbation on either the weights or the positions of the balls in X. (See, for example, [13])

Associated Orthogonal Balls. We can associate a Voronoi cell ν_{b_i} with the set of balls

$$\hat{\nu}_{b_i} \quad = \quad \{(p, w) \mid p \in \nu_{b_i} \text{ and } w = \pi_{b_i}(p)\}.$$

This set of balls is called the *associated orthogonal balls* of ν_{b_i}. If b is an associated orthogonal ball of ν_{b_i} then b is orthogonal to b_i and for all $b_j \neq b_i$, $\pi_{b_j}(b) \geq 0^{\dagger}$.

Similarly, the associated orthogonal balls of ν_X, where $X \subseteq B$, can be defined as

$$\hat{\nu}_X = \{(p, w) \mid p \in \nu_X \text{ and } w = \pi_{b_i}(p) \text{ for some } b_i \in X\},$$

and if b is an associated orthogonal ball of ν_X then b is orthogonal to every ball in X and for all $b_j \notin X$, $\pi_{b_j}(b) \geq 0$.

Proposition 1 *Let $X, Y \subseteq B$ such that $\nu_X, \nu_Y \neq \emptyset$. Let $b_x \in \hat{\nu}_X$ and $b_y \in \hat{\nu}_Y$. Then, for every ball $b \in X$, $\pi_{b_y}(b) \geq \pi_{b_x}(b) = 0$.*

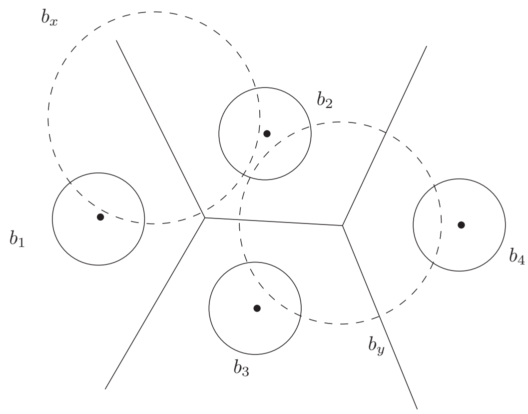

Figure 1: An illustration of Proposition 1. Let $X = \{b_1, b_2\}$ and $Y = \{b_2, b_3, b_4\}$. The ball $b_x \in \hat{\nu}_X$ and $b_y \in \hat{\nu}_Y$. We have $\pi_{b_y}(b_1), \pi_{b_y}(b_2) \geq 0$ and $\pi_{b_x}(b_1), \pi_{b_x}(b_2) = 0$. To be exact, $\pi_{b_y}(b_2)$ is actually 0.

Proof. We illustrate this proof in Figure 1. Let $b' \in Y$. By definition, $b_y = (z_{b_y}, \pi_{b'}(z_{b_y}))$. Since $z_{b_y} \in \nu_{b'}$, $\pi_b(z_{b_y}) \geq \pi_{b'}(z_{b_y})$. Thus, $\pi_b(b_y) \geq \pi_{b'}(b_y) = 0$.

In addition, $b_x = (z_{b_x}, \pi_b(b_x))$, thus, $\pi_b(b_x) = 0$. Therefore, $\pi_b(b_y) \geq \pi_b(b_x)$, or, equivalently, $\pi_{b_y}(b) \geq \pi_{b_x}(b)$. This proves our proposition. $\square$

For a set of balls X, we abuse the notation z_X to denote the set of the ball centers of X, that is, $z_X = \{z_b \mid b \in X\}$. The *Delaunay complex* of B is

†Let z_b be the center of b, $\pi_{b_j}(z_b) \geq \pi_{b_i}(z_b) \Rightarrow \pi_{b_j}(b) \geq \pi_{b_i}(b) \geq 0$.

the collection of simplices,

$$D_B = \{\mathrm{conv}(z_X) \mid \nu_X \in V_B\}.$$

We call a simplex in D_B a Delaunay simplex.

Assuming the general position assumption, if $\nu_X \neq \emptyset$ then the dimension of $\mathrm{conv}(z_X)$ is $\mathrm{card}(X) - 1$. So, if $\mathrm{card}(X) = d + 1$ and $\mathrm{conv}(z_X)$ is Delaunay then the associated orthogonal ball $\hat{\nu}_X$ consists of only one ball b, where b is orthogonal to every ball in X. The center of b is on the Voronoi vertex ν_X.

3 The Alpha Complexes

Given a set of balls B, the alpha complex of B is

$$\mathcal{K}_B = \left\{\mathrm{conv}(z_X) \mid (\bigcup X) \cap \nu_X \neq \emptyset, \nu_X \in V_B\right\}.$$

Intuitively, a Delaunay simplex $\mathrm{conv}(z_X)$ is in $\mathcal{K}_B$ if its corresponding Voronoi cell ν_X intersects with the union of balls $\bigcup X$.[‡] A simplex in an *alpha complex* is referred to as an alpha simplex. The *alpha shape* of B is the underlying space of $\mathcal{K}_B$, which we denote by $|\mathcal{K}_B|$, namely, the union of all the simplices in $\mathcal{K}_B$.

We give a proposition regarding alpha complexes that forms the ideas behind almost every main result found in this paper, especially Theorem 5 in Section 4 and Section 5.

Proposition 2 *A simplex* $\mathrm{conv}(z_X) \in \mathcal{K}_B$ *if and only if there exists a ball in* $\hat{\nu}_X$ *with a negative weight.*

Proof. If $\mathrm{conv}(z_X) \in \mathcal{K}_B$, there exists a point $p \in \nu_X \cap b$ for a ball $b \in X$. Since $\pi_b(p)$ is negative and equal to $\pi_{b'}(p)$ for any $b' \in X$, $(p, \pi_b(p)) \in \hat{\nu}_X$. Conversely, if there is a ball $b \in \hat{\nu}_X$ with the weight w_b is negative, then the point $z_b \in \bigcup X$. Since z_b is also in ν_X, the simplex $\mathrm{conv}(z_X) \in \mathcal{K}_B$. □

With this proposition, we can prove the following lemma about the relationship between the associated orthogonal balls and the alpha simplex for Theorem 4, then Theorem 5.

[‡]Recall that a ball b is viewed as a set of points $\{p \mid \|p - z_b\|^2 < w_b\}$, which excludes the boundary of b. For a simplex $\mathrm{conv}(z_X)$ to be an alpha simplex, the Voronoi region ν_X needs to intersect the interior of some ball in X.

Lemma 3 *Suppose* $\mathrm{conv}(z_X) \in \mathcal{K}_B$*. For every orthogonal ball* $b \in \hat{\nu}_X$*,* $b \cap \mathrm{conv}(z_X) = \emptyset$*.*

Proof. From Proposition 2, there exists a point $p \in \nu_X \cap (\bigcup X)$. The ball $b = (p, w_p) \in \hat{\nu}_X$ has a negative weight and is orthogonal to all balls in $\mathrm{aff}(X)$. This implies that all balls in $\mathrm{aff}(X)$ have positive weights. Any ball b' in $\hat{\nu_X}$ is orthogonal to $\mathrm{aff}(X)$ and this implies that $\mathrm{aff}(z_X) \cap b' = \emptyset$ for $b' \in \hat{\nu_X}$. $\square$

Combining Proposition 1 with Lemma 3, we have the following theorem.

Theorem 4 *Let* b *be an associated orthogonal ball of any* $\nu_X \in V_B$*. Then,* $b \cap |\mathcal{K}_B| = \emptyset$*.*

Proof. Let $b \in \hat{\nu}_X$. For another $\mathrm{conv}(z_{X'}) \in \mathcal{K}_B$, we prove that it does not intersect with b. First, for any weighted points $b' \in X'$, and an associated orthogonal ball $\hat{b'} \in \hat{\nu}_{X'}$, we have $\pi_{\hat{b'}}(b') \leq \pi_b(b')$ from Proposition 1. This implies $\pi_{\hat{b'}}(z_{b'}) \leq \pi_b(z_{b'})$. With Lemma 3, the vertex $z_{b'}$ is also out of b because $\pi_b(z_{b'}) \geq \pi_{\hat{b'}}(z_{b'}) \geq 0$.

So we have $z_{X'} \cap b = \emptyset$. To argue that $\mathrm{conv}(z_{X'})$ is also out of b, we partition the space into

$$
\begin{aligned}
h &= \{p \in \mathbb{R}^d \mid \pi_b(p) \leq \pi_{\hat{b'}}(p)\}, \text{and} \\
h' &= \{p \in \mathbb{R}^d \mid \pi_b(p) \geq \pi_{\hat{b'}}(p)\}.
\end{aligned}
$$

Since $z_{X'} \subseteq h'$, $\mathrm{conv}(z_{X'}) \subseteq h'$. In another word, each $p \in \mathrm{conv}(z_{X'})$ has a larger weighted distance to b than $\hat{b'}$. Since $\pi_{\hat{b'}}(p) \geq 0$ by Lemma 3, $\pi_b(p) \geq 0$. $\square$

4 Subdividing Alpha Complex

In this section we introduce the notion of subdividing alpha complexes. Given a set of polygons in $\mathbb{R}^3$, our goal is to construct a set of weighted points whose alpha shape is the same as the space occupied by the polygons. We assume the input given is in the form of a piecewise linear complex (PLC) which is a set $\mathcal{P}$ of vertices, line segments and polygons.

Piecewise Linear Complex. For two elements $\sigma_1, \sigma_2 \in \mathcal{P}$, we say σ_1 is a *face* of σ_2 if $\sigma_1 \subset \sigma_2$. Denote $\partial\sigma$ is the set of all the faces of σ. The *interior* of σ is the space $\sigma - \bigcup \partial\sigma$. The elements of a PLC $\mathcal{P}$ is constrained by the following two conditions:

1. all the faces of an element in $\mathcal{P}$ also belong to $\mathcal{P}$, and,

2. for $\sigma_1, \sigma_2 \in \mathcal{P}$, their intersection is a common face of both or empty

The underlying space of $\mathcal{P}$, denoted by $|\mathcal{P}|$, is the space occupied by $\mathcal{P}$. The $k-$skeleton of $\mathcal{P}$ is

$$\mathcal{P}^{(k)} \quad = \quad \{\sigma \in \mathcal{P} \mid \dim(\sigma) \leq k\}.$$

The *local gap size* [9] is a function $lgs : |\mathcal{P}| \mapsto \mathbb{R}$ where $lgs(x)$ is the radius of the smallest ball centered on x that intersects an element of $\mathcal{P}$ that does not contain x. See Figure 2 for some illustrations. It must be pointed out that lgs is continuous on the interior of each element in $\mathcal{P}$.

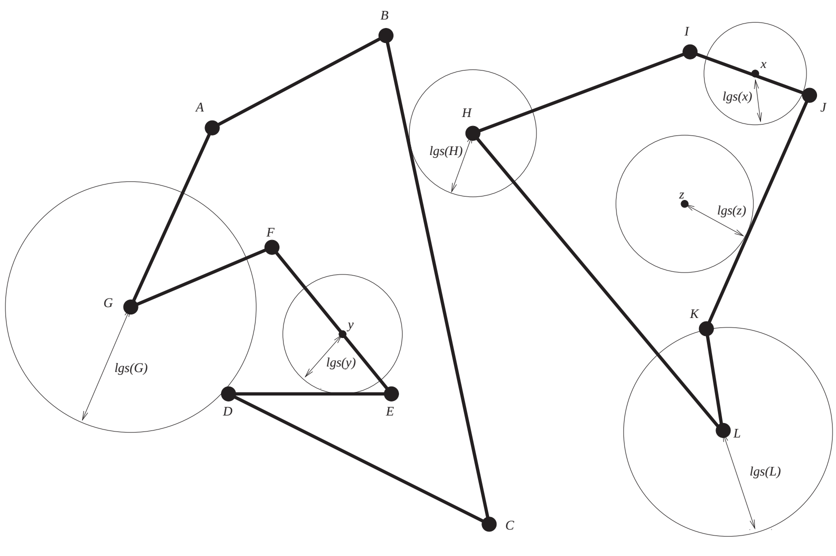

Figure 2: An illustration of the function *lgs*.

4.1 Conditions for Subdividing Alpha Complex

An alpha complex $\mathcal{K}_B$ is said to *subdivide* a piecewise linear complex $\mathcal{P}$ if the following two properties are satisfied.

P1. Every simplex in $\mathcal{K}_B$ is contained in an element in $\mathcal{P}$.

P2. Every element in $\mathcal{P}$ is a union of some simplices in $\mathcal{K}_B$.

We also call $\mathcal{K}_B$ a *subdividing alpha complex*, or in short, an SAC, of $\mathcal{P}$. Furthermore, if all the weights in B are less than a real value ε, then $\mathcal{K}_B$ is called an ε-SAC of $\mathcal{P}$. Note that if $\mathcal{K}_B$ is an SAC of $\mathcal{P}$ then $|\mathcal{K}_B| = |\mathcal{P}|$.

Theorem 5 below will be used to construct the set of balls B that forms the SAC of $\mathcal{P}$.

Theorem 5 *Let $\mathcal{P}$ be a PLC and B be a set of balls. Define $B(\sigma) = \{b \in B \mid b \cap \sigma \neq \emptyset\}$. If B satisfies the following two conditions:*

C1. *For $X \subseteq B$, if $\bigcap X \neq \emptyset$ then $\mathrm{conv}(z_X) \subseteq \sigma$ for some $\sigma \in \mathcal{P}$, and,*

C2. *For each $\sigma \in \mathcal{P}$, $z_{B(\sigma)} \subseteq \sigma \subseteq \bigcup B(\sigma)$,*

then $\mathcal{K}_B$ subdivides $\mathcal{P}$.

A few notes concerning Condition C2: A ball $b \in B(\rho)$ does not intersect another element σ unless ρ is a face of σ. Condition C2 also demands that $B(\rho) \subseteq B(\sigma)$ whenever ρ is a face of σ. The two conditions proposed here are very similar to the notion of protecting balls in computing the conforming Delaunay triangulation [10].

Figure 3 illustrates how Theorem 5 can be used to obtain a set of weighted points whose alpha complex subdivides a certain PLC. In the figure we focus our attention on the segment $\overline{HL}$ which is covered by 11 "white" weighted points. According to Condition C1, none of these white weighted points intersect with weighted points located on the polygon $\overline{ABCDEFG}$. Thus, we avoid creating any extra alpha simplex between the segment $\overline{HL}$ and the polygon $\overline{ABCDEFG}$.

Furthermore, only the "white" weighted points intersect the segment $\overline{HL}$ and their centers are all located along the segment $\overline{HL}$(Condition C2). Since they cover the whole segment, the white weighted points will form some alpha simplices that partition the segment $\overline{HL}$.

We divide the proof into two lemmas. Lemma 6 states that Condition C1 implies property P1. Lemma 9 states that Conditions C1 and C2 imply property P2.

Lemma 6 *If B satisfies Condition C1, then every simplex in $\mathcal{K}_B$ is contained in an element in $\mathcal{P}$, that is, property P1.*

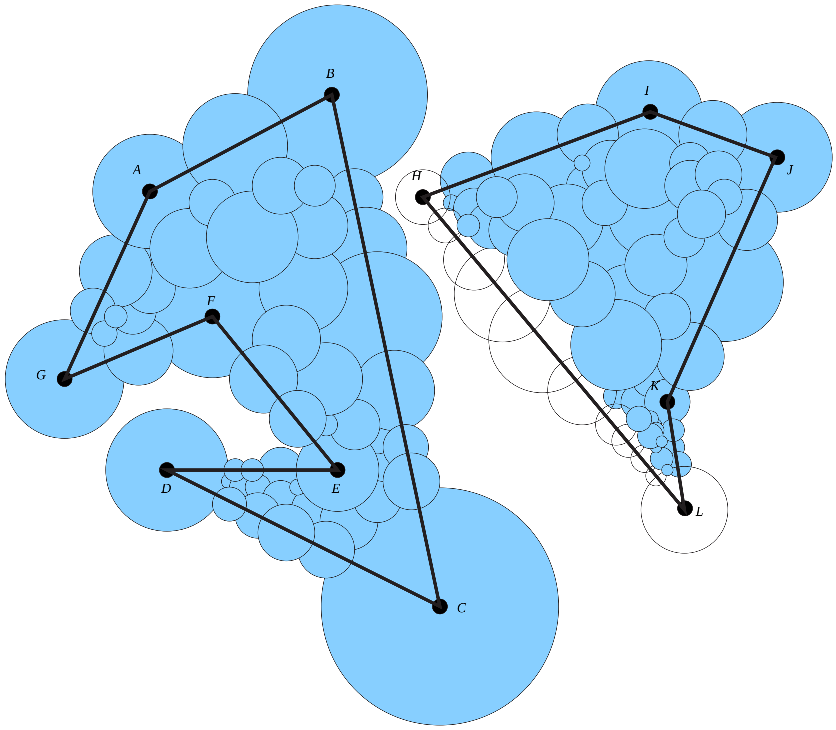

Figure 3: An illustrated example of Theorem 5.

Proof. It is immediate that every vertex in $\mathcal{K}_B$ is inside an element in $\mathcal{P}$. Let $\mathrm{conv}(z_X)$ be a simplex in $\mathcal{K}_B$. By the remark in the definition of alpha complex, $\bigcap X \neq \emptyset$. Then, by Condition C1, there is an element $\sigma \in \mathcal{P}$ such that $\mathrm{conv}(z_X) \subseteq \sigma$. $\square$

Before we proceed to prove the second part of the theorem, we give a lemma to assist the proof.

Lemma 7 *If B satisfies Conditions C1 and C2, for $b \in B$ and $\sigma \in \mathcal{P}$, we have*

$$\nu_b \cap \sigma \neq \emptyset \quad \Rightarrow \quad z_b \in \sigma.$$

Proof. Given $\nu_b \cap \sigma \neq \emptyset$, assume that z_b is not in σ. For a point $p \in \nu_b \cap \sigma$, p is inside $\cup B(\sigma)$ and there exists a ball $b' \in B(\sigma)$ such that $\pi'_b(p) < 0$. However, $p \in \nu_b$ implies $\pi_b(p) \leq \pi'_b(p) < 0$. This contradicts the Condition C2 because p is in both σ and b. $\square$

In other words, this lemma states that the Voronoi region of b can intersect with an element $\sigma \in \mathcal{P}$ only if the center of b is in σ. We claim that the converse is true, which is part of the following claim.

Claim 8 *If B satisfies Conditions C1 and C2, for $b \in B$ and $\sigma \in \mathcal{P}^{(d)}$, we have*

1. $z_b \in \sigma \Rightarrow \nu_b \cap \sigma \neq \emptyset$,
2. $|K_{B(\sigma)}| = \sigma$, *and*
3. *For a simplex* $\mathrm{conv}(z_X) \in K_{B(\sigma)}$, $\nu_X \cap \sigma \neq \emptyset$.

Proof. We will prove this claim by induction on $d \geq 0$. Indeed, Claim 8 is true for $d = 0$ because for each vertex $\sigma \in \mathcal{P}$, $B(\sigma)$ consists of only one ball b with center on σ. The vertex σ is only contained in b and so $\sigma \in \nu_b$[§]. The set of balls $\bigcup_{\dim(\sigma)=0} B(\sigma)$ forms the alpha complex that is the same as $\mathcal{P}^{(0)}$.

To simplify the notation, we denote by $B_i = \bigcup_{\dim(\sigma)=i} B(\sigma)$ and K_i, the alpha complex of B_i. Assuming the claim is true for $d = i - 1$. By item 3 of our claim, if a simplex $\mathrm{conv}(z_X)$ is an alpha simplex, the intersection $\nu_X \cap \mathcal{P}^{(i-1)} \neq \emptyset$. Condition C2 implies every ball in $B_i - B_{i-1}$ does not intersect $\mathcal{P}^{(i-1)}$. Thus, $K_{i-1} \subseteq K_i$.

We consider the set $B(\sigma)$, where $\sigma \in \mathcal{P}^{(i)}$ and $\dim(\sigma) = i$. Since the balls in $B(\sigma)$ are all centered on σ, we focus only on the space $\mathrm{aff}(\sigma)$. That is, when we say ν_b, we mean the Voronoi region of b restricted to $\mathrm{aff}(\sigma)$.

Notice that $D_{B(\sigma)}$ forms a constrained Delaunay triangulation of σ because each $\rho \in \partial\sigma$ is partitioned by the Voronoi cells of $B(\rho)$ and $K_{B(\rho)}$ remains in K_i. Furthermore, for a ball b whose center is in the interior of σ, its Voronoi cell does not touch any boundary element of σ because of Lemma 7. However, the Voronoi cell of b is not outside σ, otherwise, we can find an orthogonal ball $\hat{b}$ that has its center out of σ but intersecting b. This contradicts Theorem 4 because the orthogonal ball also intersects the boundary of σ, which is partitioned into some alpha simplices of K_{i-1}. So, we have $\nu_b \subset \sigma$, in particular, $\nu_b \cap \sigma \neq \emptyset$.

For the second item of the claim, it is equivalent to say that every element in $D_{B(\sigma)}$ with dimension i and within σ, remains in $K_{B(\sigma)}$. For any ball b whose center is in the interior of σ, it is connected locally as a topological

[§]Here we abuse the notation. To be more precise, we should write that $\sigma = \{p\}$ and p is contained only in b. So, $p \in \nu_b$.

disk because $\nu_b \subset \sigma$. It means any such interior ball does not contribute to the boundary of $K_{B(\sigma)}$. If there exist a boundary element in $K_{B(\sigma)}$ within σ, it has an $(i-1)$-dimensional element that connects two balls from two faces of σ, which contradicts Conditions C1 and C2. Therefore, it is either that $K_{B(\sigma)}$ covers σ, or it is only the boundary of σ. Together with the interior vertices of σ, the second case is false. Therefore $|K_{B(\sigma)}| = \sigma$.

For the simplices in $K_{B(\sigma)}$ that are on the boundary of σ, the item 3 of this claim remains true. Otherwise, a simplex $\mathrm{conv}(z_X)$ has a vertex z_b that is not on the boundary of σ. The Voronoi region ν_b is inside σ, thus, $\nu_X \subset \sigma$. $\square$

Statement 3 of the claim above immediately implies that if $\mathrm{conv}(z_X) \in K_{B(\sigma)}$ then $\nu_X \cap \sigma \subseteq \bigcup X$, thus, $\nu_X \cap \sigma \cap (\bigcup X) \neq \emptyset$. This is because Condition C2 demands that σ is covered by $\bigcup B(\sigma)$, thus, the points in $\nu_X \cap \sigma$ must be covered by balls in X.

Moreover, all the balls $B - B(\sigma)$ do not intersect with σ. Thus, adding them will not effect the alpha simplices in $K_{B(\sigma)}$. Therefore, $K_{B(\sigma)} \subseteq K_B$. Combining this with statement 2 of the Claim above, we have:

Lemma 9 $|\mathcal{P}| \subseteq |\mathcal{K}_B|$.

4.2 The Algorithm

In this subsection we describe our algorithm to construct the ε-SAC of a given piecewise linear complex $\mathcal{P}$. The aim is to construct a set of balls B that satisfies Conditions C1 and C2 in Theorem 5 and at the same time all the weights of the balls are bounded above by an input real number $\varepsilon > 0$.

First, we fix a constant real number $0 < \gamma < 0.5$. Then we construct the set of balls $B(\sigma)$ for each $\sigma \in \mathcal{P}$, starting with those of dimension 0, then dimension 1 and ending with those of dimension 2.

The construction of $B(\sigma)$ where $\dim(\sigma) = 0$ is trivial. For each vertex v in $\mathcal{P}$, we add a ball with center v and radius $r = \min(\gamma \cdot lgs(v), \sqrt{\varepsilon})$. So, $B(v) = \{(v, r^2)\}$.

To describe the construction of $B(\sigma)$ when σ is of dimension 1 or 2, we need the notation of *restricted Voronoi complex.* The restricted Voronoi complex of a set of balls X on $\sigma \in \mathcal{P}$, denoted by $V_X(\sigma)$, is the complex which consists of $\nu_X \cap \sigma$, for all $\nu_X \in V_X$. A restricted Voronoi vertex u in $V_X(\sigma)$ is called *positive* if $\pi_b(u) > 0$, for all $b \in X$. Note that such a restricted vertex is outside every ball in X. To determine whether a restricted vertex is positive, it suffices to compute $\pi_{b'}(u)$ where u is the restricted Voronoi vertex in the restricted Voronoi cell $\nu_{b'}(\sigma)$.

Algorithm 1 describes the construction of $B(\sigma)$ where $\dim(\sigma) = 1, 2$ The basic idea is to add a ball centered on a positive restricted Voronoi vertex in an edge (or, a polygon) until it is covered by the balls. To avoid unwanted elements, we set the radius of every ball to be less than both $\sqrt{\varepsilon}$ and γ times the *lgs* of the ball center.

Figure 4 illustrates some steps of Algorithm 1 when $\dim(\sigma) = 1$. In the beginning we have the set $X = B(H) \cup B(L) = \{b_1, b_2\}$, since $B(H) = \{b_1\}$ and $B(L) = \{b_2\}$. The algorithm computes the restricted Voronoi complex $V_X(\overline{HL})$. The restricted Voronoi vertex $\nu_{b_1,b_2}(\overline{HL})$ is positive, so we add the ball b_3, centered on $\nu_{b_1,b_2}(\overline{HL})$, to X. Then we recompute $V_X(\overline{HL})$. The restricted Voronoi vertex $\nu_X(\overline{HL})$ is positive. So we add the ball b_4, centered on $\nu_{b_1,b_3}(\overline{HL})$, to X. We repeat the whole process until there is no more positive restricted Voronoi vertex in $V_X(\overline{HL})$.

Algorithm 1 To construct $B(\sigma)$ for all $\sigma \in \mathcal{P}$

```
for i = 1, 2 do
    for all σ ∈ P and dim(σ) = i do
        X := ⋃ B(∂σ)
        while there exists a positive restricted Voronoi vertex u in V_X and
        u ∈ σ  do
            r := min(γ · lgs(u), √ε)
            X := X ∪ {(u, r^2)}
        end while
        B(σ) := X
    end for
end for
```

We claim that our algorithms terminate and the output $B = \bigcup_{\sigma \in \mathcal{P}} B(\sigma)$ satisfies both Conditions C1 and C2. It should be clear that all weights in B are at most ε. Since every ball with center p has radius less than $0.5 \times lgs(p)$, it is obvious that Condition C1 is satisfied. Condition C2 follows from Proposition 10 below. Lemma 12 establishes the termination of our algorithm.

Proposition 10 *Let X be a set of balls. Suppose $z_X \subseteq \sigma$. Then $\sigma \subseteq \bigcup X$ if and only if there is no positive restricted Voronoi vertex in $\mathrm{V}_X(\sigma)$.*

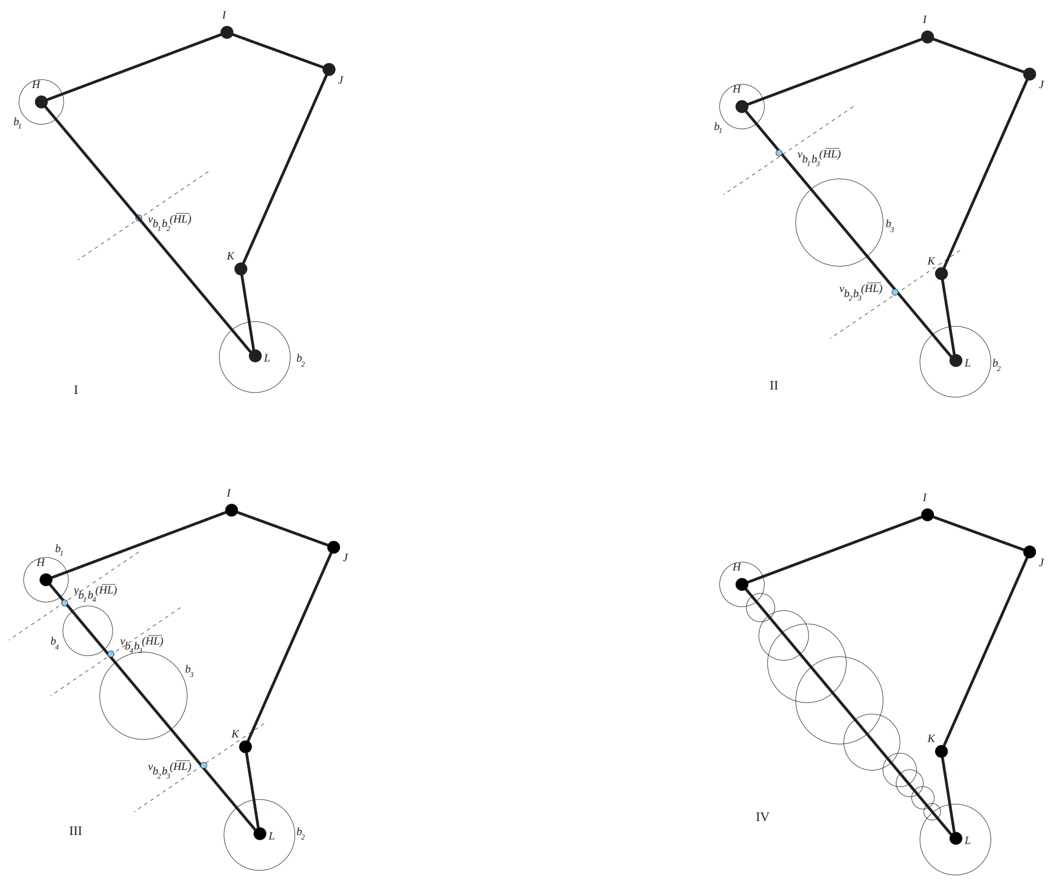

Figure 4: An illustrated example of Algorithm 1 on the segment $\overline{HL}$.

Proof. The "only if" part is immediate. We will show the "if" part. Suppose there is no positive restricted Voronoi vertex in $V_X(\sigma)$. We claim that $\nu_b(\sigma) \subseteq b$ for all $b \in X$. This claim follows from the fact that $\nu_b(\sigma)$ is the convex hull of its Voronoi vertices and bounded. Thus, by our assumption that all the restricted Voronoi vertices are not positive, it is immediate that $\nu_b(\sigma) \subseteq b$ for any $b \in X$. Since σ is partitioned into $\nu_b(\sigma)$ for all $b \in X$, it follows that $\sigma \subseteq \bigcup X$. □

To establish the termination of the algorithm, we observe the following remark.

Remark 11 Let $\sigma \in \mathcal{P}$ and let $\Gamma \subset \sigma$ be a closed region such that it does not intersect the boundary of σ. Then there exists a constant $c > 0$ such that for every point $p \in \Gamma$, $lgs(p) > c$.

The reasoning is as follows. We observe that lgs is a continuous function on Γ. Moreover, Γ is compact. Thus, there exists $p_0 \in \Gamma$ such that $lgs(p_0) = \min_{p \in \Gamma} lgs(p)$. The value $lgs(p_0) \neq 0$ since p_0 is in the interior of σ. Thus, we can choose $\frac{1}{2} lgs(p_0)$ as the value for c.

Lemma 12 *Algorithm 1 terminates.*

Proof. It suffices to show that for each σ the `while`-loop does not iterate infinitely many times. We concern ourselves only with $\dim(\sigma) = 1$. The case for $\dim(\sigma) = 2$ is similar, thus, omitted. The proof is by contradiction and it follows from the fact that each element ρ in $\mathcal{P}$ is compact.

Assume to the contrary that for some edge $\sigma = (v_1, v_2) \in \mathcal{P}$ the `while`-loop iterates infinitely many times. That is, it inserts infinitely many balls to $B(\sigma)$ whose centers are in the region $\sigma - (b_1 \cup b_2)$ where $b_i \in B(v_i)$ for $i = 1, 2$. The region $\sigma - (b_1 \cup b_2)$ is a closed region which does not intersect with the boundary of σ. By Remark 11, there exists a constant $c > 0$ such that all the radii of the balls are greater than c.

Moreover, $\sigma - (b_1 \cup b_2)$ is compact, so if $B(\sigma)$ contains infinitely many balls, then there are two balls b and b' whose centers are at the distance less than c. Without loss of generality, we assume that b was inserted before b'. This is impossible, because at the time b' was inserted, its center would be a negative restricted Voronoi vertex. Therefore, the `while`-loop iterates only finitely many times. $\square$

Readers may concern about the number of balls used in creating the ε-SAC, which depends on the function lgs and input ε. For a good approximation of the object, we assume the user may set a value for ε that is smaller than the local gap size in general. If we assume the local gap size dominates, each polygon with area a is covered by approximately $O(\frac{a}{\varepsilon^2})$ balls. If the totally surface area of $\mathcal{O}$ is A, the number of balls is $O(\frac{A}{\varepsilon^2})$.

5 Approximating Polyhedral Object with a Union of Balls

We define a *polyhedral object* to be $\mathcal{O} \subseteq \mathbb{R}^3$ such that: $\mathcal{O}$ is a 3-dimensional compact manifold and its boundary, denoted by $\partial\mathcal{O}$, is decomposable into a PLC, namely $\mathcal{P}$. We assume that $|\mathcal{P}|$ is a 2-manifold without boundary.

Our method in approximating $\mathcal{O}$ with a union of balls can be summarized as follows.

1. Construct a set of balls B such that $\mathcal{K}_B$ is an ε^2-SAC of $\mathcal{P}$.
2. Compute the Voronoi complex of B.
3. Let T be the set of Voronoi vertices in V_B which are located inside the object $\mathcal{O}$.
4. Let $B^\perp$ be the set of all associated orthogonal balls of $\nu_X \in T$.
5. Output $B^\perp$.

We will show that $\bigcup B^\perp$ approximates the object $\mathcal{O}$ well, in the sense, that the Hausdorff distance between $\partial \bigcup B^\perp$ and $\partial\mathcal{O}$ is less than ε and they are homeomorphic. The approach suggested here is very similar to the power crust method proposed by Amenta. et. al. [1]. As analogy, we can view the ε-SAC as the sample points of the object $\mathcal{O}$ and $B^\perp$ as the "polar" balls, defined in [1]

We give here the definition of Hausdorff distance. The Hausdorff distance from a set A to a set B is $d(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|$. The Hausdorff distance between two sets A and B is the larger value between $d(A, B)$ and $d(B, A)$.

First, we prove that the Hausdorff distance between $\partial \bigcup B^\perp$ and $\partial\mathcal{O}$ is less than ε. By Theorem 4, $\bigcup B^\perp \cap \partial\mathcal{O}$ is empty, and by Claim 8, every center of $B^\perp$ lies within $\mathcal{O}$. Therefore, $\bigcup B^\perp$ is contained in $\mathcal{O}$ and so is $\partial \bigcup B^\perp$. With the following lemma, we can prove that $\partial \bigcup B^\perp \subset \bigcup B$. In this lemma, the convex hull of a set of balls is similar to the definition of the affine hull of a set of balls in Equation (1) except that all the coefficients $\lambda_i \geq 1$.

Lemma 13 *Let $X = \{b_1, b_2, b_3, b_4\}$ and b a ball such that $\{b\} \perp X$. Then,*

$$\mathrm{conv}(z_X) - \bigcup X \subseteq b.$$

Proof. Let $p \in \mathrm{conv}(z_X) - \bigcup X$ and $w \in \mathbb{R}$ such that $(p, w) \in \mathrm{conv}(X)$. Since $\bigcup X = \bigcup \mathrm{conv}(X)$, the weight $w < 0$. Furthermore, (p, w) is orthogonal to b. Thus, $\|z_b - p\|^2 - w_b = w < 0$. Therefore, $p \in b$. □

This lemma implies that no point in $\partial \bigcup B^\perp$ lies in $\mathcal{O} - \bigcup B$. Thus, $\partial \bigcup B^\perp \subset \bigcup B$ and we have the following Theorem.

Theorem 14 *The Hausdorff distance between $\partial \bigcup B^{\perp}$ and $\partial \mathcal{O} \leq \varepsilon$.*

For the homeomorphism proof, we leave it together with the skin approximation in the end of next section, namely, Theorem 19.

6 Approximating Polyhedral Object with the Skin Surface

In this section we discuss our method in obtaining a skin surface which approximates a given polyhedral object. We start by reviewing the basic definition of skin surface in Subsection 6.1, to be followed by the construction of the approximation and the proofs of homeomorphism and Hausdorff distance guarantee.

6.1 The Skin Surface

The skin surface was first defined by Edelsbrunner [12] based on an algebraic structure of balls. Readers interested in a detailed treatment of the algebra of balls may find the text by Pedoe [16] useful.

With the additions and scalar multiplication of balls in Section 2.1, we can define the *convex hull* of a set of balls $B = \{b_1, \ldots, b_n\}$ as

$$\mathrm{conv}(B) = \Big\{ \sum_{i=1}^{n} \gamma_i b_i \mid \sum_{i=1}^{n} \gamma_i = 1 \text{ and } \gamma_i \geq 0 \text{ for all } i = 1, \ldots, n \Big\}.$$

It must also be noted that $\bigcup B = \bigcup \mathrm{conv}(B)$[12]. For a ball b and $t \in \mathbb{R}$,we define $b^t = (z_b, t w_b)$. For a set of balls B, B^t is defined as $B^t = \{b^t \mid b \in B\}$.

For $0 \leq t \leq 1$, the skin body of a set of balls B is defined as

$$\mathrm{body}^t(B) = \bigcup \mathrm{conv}(B)^t,$$

that is, the set of points obtained by shrinking all balls in the convex combination of B. The skin surface is the boundary of the skin body of B, denoted by

$$\mathrm{skin}^t(B) = \partial \mathrm{body}^t(B).$$

It is known that $\mathrm{skin}^t(B)$ is a smooth surface for $0 < t < 1$.

Note that $\bigcup B = \mathrm{body}^1(B)$ and $\mathrm{body}^s(B) \subseteq \mathrm{body}^t(B)$ for $0 \leq s < t \leq 1$. We cite here an important relation between a union of balls $\bigcup B$ and the skin body that it generates.

Theorem 15 *The union of balls* $\bigcup B$ *is homeomorphic to* $\text{body}^t(B)$*, for* $0 < t \leq 1$[12].

At this point, we highlight a rather obvious, but important, fact concerning the skin body and surface. Though very trivial in nature, this is the main idea of the proof in the next section.

Remark 16 Let B be a set of balls and $b \in \text{conv}(B)$. If $w_b > 0$ then the point z_b is in the interior of the $\text{body}^t(B)$, for all $0 < t \leq 1$.

6.2 Approximation by the Skin Surface

For each simplex $\text{conv}(z_X) \in D_B$, we define the orthogonal ball of $\text{conv}(z_X)$ as $b_X^\perp = (z_X^\perp, w_X^\perp)$ such that

$$\begin{aligned} z_X^\perp &= \text{aff}(z_X) \cap \nu_X, \\ w_X^\perp &= \|z_X^\perp z_i\|^2 - w_i, \end{aligned}$$

for all $b_i \in X$ such that $b_X^\perp \perp b_i$. We pick all the simplices inside $\mathcal{O}$ and define the orthogonal balls of these simplices as

$$B^* = \{b_X^\perp \mid \text{conv}(z_X) \subseteq \mathcal{O}, \text{conv}(z_X) \in K_B, z_X^\perp \neq \emptyset\} \cup B^\perp.$$

We claim that:

Lemma 17 $\mathcal{O} - \bigcup B = \text{body}^0(B^*)$.

Proof. First, we show that $\mathcal{O} - \bigcup B \subseteq \text{body}^0(B^*)$. Let $p \in \mathcal{O} - \bigcup B$ and it is in a Voronoi cell ν_{b_1} in the Voronoi complex of B for some $b_1 \in B$. The goal of this proof is to show that there exists a ball $b_p = (p, w_p)$ in $\text{conv}(B^*)$ such that $w_p \geq 0$, and it implies $p \in \text{body}^0(B^*)$.

Consider the set of tetrahedra

$$\Phi_1 = \{\text{conv}(\{z_{\{b_1\}}^\perp, z_{X_1}^\perp, z_{X_2}^\perp, z_{X_3}^\perp\}) \mid z_{X_i}^\perp \in B^* \text{ and } b_1 \in X_i\}.$$

Note that the union of all tetrahedra in Φ_1 contains $\mathcal{O} \cap \nu_{b_1}$ because the tetrahedra in Φ_1 contain all the intersection of Voronoi edges of ν_{b_1} with $\partial\mathcal{O}$ and Voronoi vertices of ν_{b_1} inside $\mathcal{O}$. This implies the existence of a tetrahedron $\text{conv}(\{z_{b_1}^\perp, z_{X_1}^\perp, z_{X_2}^\perp, z_{X_3}^\perp\} \in \Phi$ such that it contains p.

Let $b_p = (p, w_p) \in \text{conv}(\{b^{\perp}_{\{b_1\}}, b^{\perp}_{X_1}, b^{\perp}_{X_2}, b^{\perp}_{X_3}\})$. Because $b_1 \perp b^{\perp}_{\{b_1\}}$ and $b_1 \perp b^{\perp}_{X_i}$ for $i = 1$ to 3, we have $b_1 \perp b_p$. Thus, $w_p \geq 0$ if p is not in b_1. Since b_p is also in $\text{conv}(B^*)$ and it implies $p \in \text{body}^0(B^*)$.

For $\mathcal{O} - \bigcup B \supseteq \text{body}^0(B^*)$, first we cite the result in Cheng's thesis [3] that if b^* is in the convex hull of B^* such that $b^* = \sum_i \lambda_i b_i$ for $b_i \in B^*$, the weighted distance between b^* and another ball b is

$$\pi_b(b^*) \;\; = \;\; \sum \lambda_i \pi_b(b_i).$$

For every $b \in B$ and $b_i \in B^*$, $\pi_b(b_i) \geq 0$ because B^* is a subset of all the associated orthogonal balls. Thus, $\pi_b(b^*) = \|z_b z_{b^*}\|^2 - w_b - w_{b^*} \geq 0$ if $b \in \text{conv}(B^*)$. If the center of b^* is inside $\bigcup B$, $\|z_b z_{b^*}\|^2 < w_b$ and $w_{b^*} < 0$. Therefore, the interior of $\bigcup B$ does not touch $\text{body}^0(B^*)$ because it is the union of all centers of the balls in B^* which have non-negative weights. $\square$

This lemma immediately implies the Hausdorff distance between $\text{skin}^t(B^*)$ and $\partial\mathcal{O}$ is less than ε because the surface $\text{skin}^t(B^*)$, for $0 < t < 1$, is located in between the surface $\partial \bigcup B$ within $\mathcal{O}$ and the surface $\partial\mathcal{O}$.

The homeomorphism between $\text{skin}^t(B^*)$ and $\partial\mathcal{O}$ can be established via the smooth deformation contraction from the boundary of $\bigcup B$ within $\mathcal{O}$, i.e. $\text{skin}^0(B^*)$ to $|\mathcal{P}|$[11]¶. Therefore, we have the following theorem.

Theorem 18 $\text{skin}^t(B^*)$ *is homeomorphic to* $\partial\mathcal{O}$ *and the Hausdorff distance between* $\text{body}^t(B^*)$ *and* $\partial\mathcal{O}$ *is less than* ε.

All the balls in $B^* - B^{\perp}$ have negative weights. Thus, $\bigcup B^{\perp} = \bigcup B^* = \text{skin}^1(B^*)$, and the above theorem, we also have:

Theorem 19 $\bigcup B^{\perp}$ *is homeomorphic to* $\mathcal{O}$.

7 Conclusion

In this paper we propose a method to approximate a given polyhedral object with a union of balls (Theorems 14 and 19), as well as, with the skin surface (Theorem 18). By representing polyhedral objects with a union of balls and the skin surface, we hope to be able to perform deformations between objects. Moreover, we would also like to apply the same idea to obtain an

¶Recall that $\mathcal{P}$ is the decomposition of $\partial\mathcal{O}$ into a PLC.

approximation of smooth object with the skin surface. Such representation will enable a deformation to be performed between smooth objects. The other main result is Theorem 5, together with the algorithm to compute the subdividing alpha complex. Although these are all in $\mathbb{R}^3$, the proof is able to extend for objects in arbitrary dimensions. Also, other than using the local gap size for Condition C1, we may also use the *protecting cells* in the earlier work of the authors [7].

One possible future direction is to implement the same idea in approximating smooth objects with skin surfaces. Amenta et.al [1] showed that given a sufficiently dense sample points on a smooth surface, the set of polar balls obtained can be used to approximate the object well. There is an analogy between such approach with our method here. We can view the ε-SAC constructed as the sample points and $B^{\perp}$ as the polar balls.

By appropriately assigning certain weights to the sample points and taking the polar balls, we hope to be able to approximate the smooth object by a skin surface. At this point, the usefulness of this idea is still under investigation.

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