6—Inductive Proofs

CS 3234: Logic and Formal Systems

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Inductive definitions

- Often one wishes to define a set with a collection of rules that determine the elements of that set. Simple examples:
  - Binary trees
  - Natural numbers
  - The syntax of a logic (*e.g.*, propositional logic)
- What does it mean to define a set by a collection of rules?

Example: Binary trees (w/o data at nodes)

- $\bullet$ is a binary tree;
- if l and r are binary trees, then so is $\widehat{l \quad r}$

Examples of binary trees:

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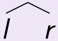
- if l and r are binary trees, then so is 

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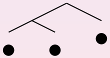
- 

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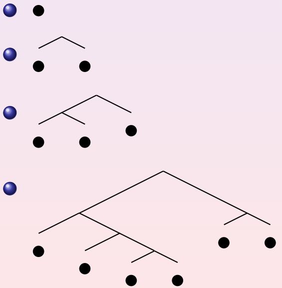

Example 2: Natural numbers in unary (base-1) notation

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$$

It's possible to view naturals as trees, too:

zero	$\equiv$	Z	Z
one	$\equiv$	$S(Z)$	$\begin{array}{c} \mathsf{S} \\ \vert \\ Z \end{array}$
two	$\equiv$	$S(S(Z))$	$\begin{array}{c} \mathsf{S} \\ \vert \\ \mathsf{S} \\ \vert \\ Z \end{array}$

$\dots$

Examples (more formally)

- Binary trees: The set *Tree* is defined by the rules

$$\frac{}{\bullet} \qquad\qquad \frac{t_l \qquad t_r}{\overset{\wedge}{t_l \quad t_r}}$$

- Naturals: The set *Nat* is defined by the rules

$$\frac{}{Z} \qquad\qquad \frac{n}{S(n)}$$

Given a collection of rules, what set does it define?

- What is the set of trees?
- What is the set of naturals?

Do the rules pick out a unique set?

There can be many sets that satisfy a given collection of rules

- $IndNum = \{Z, S(Z), \ldots\}$
- $CoIndNum = \{Z, S(Z), S(S(Z)), \ldots, S(S(S(\ldots)))\}$
- $WeirdNum = MyNum \cup \{\infty, S(\infty), \ldots\}$, where ∞ is an arbitrary symbol.

All three of these different sets satisfy the rules defining numerals.

An inductively defined set is the **least set** for the given rules (*i.e.*, the extremal clause).

Example: $IndNum = \{Z, S(Z), S(S(Z)), \ldots\}$ is the least set that satisfies these rules:

- $Z \in Num$
- if $n \in Num$, then $S(n) \in Num$.

What do we mean by "least"?

Answer: The smallest with respect to the subset ordering on sets.

- Contains no "junk", only what is required by the rules.
- Since $\mathit{CoIndNum} \supsetneq \mathit{IndNum}$, *CoIndNum* is ruled out by the extremal clause.
- Since $\mathit{WeirdNum} \supsetneq \mathit{IndNum}$, *WeirdNum* is ruled out by the extremal clause.
- *IndNum* is "ruled in" because it has no "junk". That is, for any set S satisfying the rules, $S \supset \mathit{IndNum}$

We almost always want to define sets with inductive definitions, and so have some simple notation to do so quickly:

$$S = \text{Constructor}_1(\ldots) \;\mid\; \text{Constructor}_2(\ldots) \;\mid\; \ldots$$

where S can appear in the $\ldots$ on the right hand side (along with other things). The Constructor_i are the names of the different rules (sometimes text, sometimes symbols). This is called a *recursive definition*.

Examples:

- Binary trees: $\tau = \bullet \;\mid\; \widehat{\tau \quad \tau}$
- Naturals: $\mathbb{N} = Z \;\mid\; S(\mathbb{N})$

There is a close connection between a recursive definition and a definition by rules:

- Binary trees: $\tau = \bullet \mid \overset{\frown}{\tau \quad \tau}$

$$\frac{}{\bullet} \qquad\qquad \frac{t_l \quad t_r}{\overset{\frown}{t_l \quad t_r}}$$

- Naturals: $\mathbb{N} = Z \mid S(\mathbb{N})$

$$\frac{}{Z} \qquad\qquad \frac{n}{S(n)}$$

A definition written in "recursive definition style" is assumed to be the least set satisfying the rules; that is, the notation means that

CoInductive sets

What about the other two choices? Is there any value in them?

- $CoIndNum = \{Z, S(Z), S(S(Z)), \ldots, S(S(S(\ldots)))\}$
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As a rule, there is no point at all to *WeirdNum*: it is just a set that we don't want—and if we do, we can define it inductively by $WeirdNum = Z \mid \infty \mid S(WeirdNum)$.

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But there is value to the set CoIndNum. This is the *greatest* set that can be defined using a set of rules without adding junk like ∞. Such a set is called *co-inductively* defined, and is useful for reasoning about infinitely-long objects such as streams.

What's the Big Deal with inductively defined sets?

Inductively defined sets "come with" an *induction principle*.
Suppose I is inductively defined by rules R.

- To show that every $x \in I$ has property P, it is enough to show that regardless of which rule is used to "build" x, P holds; this is called *taking cases* or *inversion*.
- Note that one can take cases also on co-inductively defined sets like *CoIndNum*—but not on sets like *WeirdNum*.
- Sometimes, taking cases is not enough; in that case we can attempt a more complicated proof where we show that P is preserved by each of the rules of R; this is called *structural induction* or *rule induction*. We need to have an inductively defined set; *we cannot do induction over coinductive sets*.

Example: Sign of a Natural

Consider the following definition:

- The natural Z has sign $\mathbf{0}$.
- For any natural n, the natural $S(n)$ has sign $\mathbf{1}$.

Let P be the following property: Every natural has sign $\mathbf{0}$ or $\mathbf{1}$.

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Let P be the following property: Every natural has sign $\mathbf{0}$ or $\mathbf{1}$.

Does P satisfy the rules $\dfrac{}{Z}$ $\dfrac{n}{S(n)}$?

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To show that every $n \in Nat$ has property P, it is enough to show:

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- For any n, $S(n)$ has sign $\mathbf{1}$, so P holds for any $S(n)$. $\surd$

Thus, P holds for all naturals.

Example: Even and Odd Naturals

- The natural Z has parity $\mathbf{0}$.
- If n is a natural with parity $\mathbf{0}$, then $S(n)$ has parity $\mathbf{1}$.
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Can we prove this by taking cases?

Taking cases

We need to show $P =$ "Every natural has parity $\mathbf{0}$ or parity $\mathbf{1}$.",

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Where parity is defined by

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We are stuck! We need an extra fact about n's parity...

Induction hypothesis

This fact is called an *induction hypothesis*. To get such an induction hypothesis we do *induction*, which is a more powerful way to take cases. To show that every $n \in Num$ has property P, we must show that every rule preserves P; that is:

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Note that for the naturals, structural induction is just ordinary mathematical induction!

Using induction to fix our proof

Every natural has parity **0** or parity **1**.

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Proof. We **do induction on the structure of n** as follows:

- Z has parity **0**, so P holds for Z. $\surd$
- Given an n such that P holds on n, show that P holds on $S(n)$. Since P holds on n, the parity of n is **0** or **1**. If the parity of n is **0**, then the parity of $S(n)$ is **1**. If the parity of n is **1**, then the parity of $S(n)$ is **0**. In either case, the parity of $S(n)$ is **0** or **1**, so if P holds on n then P holds on $S(n)$. $\surd$

Thus, P holds for an natural n.

Extending case analysis and structural induction to trees

Case analysis: to show that every tree has property P, prove that

- $\bullet$ has property P.
- for all τ_1 and τ_2, $\begin{array}{c} \diagup\!\diagdown \\ \tau_1 \quad \tau_2 \end{array}$ has property P.

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Note that we do not require that τ_1 and τ_2 be the same height!

Structural induction vs. induction on naturals

You are probably familiar with regular mathematical induction: to prove something for any natural n, first prove it is true about 0 and then show that if it is true about n then it is true about $n + 1$.

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- For regular mathematical induction on the height of trees:
  if τ_1 and τ_2 **are trees of height n** and have property P, then $\underset{\tau_1 \quad \tau_2}{\bigwedge}$ **is a tree of height $\mathbf{n+1}$** and has property P.

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- Remember that I is (by definition) the smallest set satisfying the rules in R.
- Hence if P satisfies (is preserved by) the rules of R, then $P \supseteq I$.
- This is why the extremal clause matters so much!

Example: Height of a Tree

- To show: Every tree has a height, defined as follows:
  - The height of $\bullet$ is 0.
  - If the tree l has height h_l and the tree r has height h_r, then the tree $\overset{\frown}{l \quad r}$ has height $1 + max(h_l, h_r)$.
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- Clearly, every tree has at most one height, but does it have any height at all?
- It may seem obvious that every tree has a height, but notice that the justification relies on structural induction!
  - An "infinite tree" does not have a height!
  - But the extremal clause rules out the infinite tree!

Example: height

- Formally, we prove that for every tree t, there exists a number h satisfying the specification of height.
- Proceed by induction **on the structure of trees**, showing that the property "there exists a height h for t" satisfies (is preserved by) these rules.

Example: height

- Rule 1: $\bullet$ is a tree.
  Does there exist h such that h is the height of *Empty*?
  Yes! Take h=0.
- Rule 2: $\overset{\frown}{l \quad r}$ is a tree if l and r are trees.
  Suppose that there exists h_l and h_r, the heights of l and r, respectively (*the induction hypothesis*).
  Does there exist h such that h is the height of $Node(l, r)$?
  Yes! Take $h = 1 + max(h_l, h_r)$.

Thus, we have proved that all trees have a height.

Please see the Coq script.

Extension: the syntax of propositional logic

We have already seen a major example of a recursive definition in class: the syntax of propositional logic!

$$F = \text{Atom}(\alpha) \;\mid\; \neg F \;\mid\; F \vee F \;\mid\; F \wedge F \;\mid\; F \rightarrow F$$

It is perfectly reasonable to do case analysis and structural induction on the syntax of a formula ϕ. In fact, we will see an example of this shortly!

Extension: the structure of a natural deduction proof

We have seen another important kind of tree-like structure in class already: natural deduction proofs! In homework 1, you did proofs using a "3 column" style; in homework 2, you will do a few proofs using the graphical tree-style, such as this proof of $p \wedge q \vdash q \wedge p$:

$$\dfrac{\dfrac{p \wedge q}{q}[\wedge e2] \qquad \dfrac{p \wedge q}{p}[\wedge e1]}{q \wedge p}[\wedge i]$$

It is *also* reasonable to do structural induction on the structure of a formal proof. We will see an example of this shortly, too!

- An inductively defined set is the least set closed under a collection of rules.
- Rules have the form:
  "If $x_1 \in X$ and $\ldots$ and $x_n \in X$, then $x \in X$."
- Notation:
  $$\frac{x_1 \qquad \cdots \qquad x_n}{x}$$
- Notation: sometimes we can define the entire set easily with a recursive definition: $S = \mathrm{C}_1(\ldots) \mid \mathrm{C}_2(\ldots) \mid \ldots$

- Inductively defined sets admit proofs by rule induction.
- For each rule

$$\frac{x_1 \qquad \cdots \qquad x_n}{x}$$

  assume that $x_1 \in P$, ..., $x_n \in P$, and show that $x \in P$.
- Conclude that every element of the set is in P.