03b—Propositional Logic

CS 3234: Logic and Formal Systems

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3 Proof Theory

4 Soundness and Completeness (preview)

Motivation Propositional Atoms Constructing Propositions Syntax of Propositional Logic



- Atoms and Propositions
- Motivation
- Propositional Atoms
- Constructing Propositions
- Syntax of Propositional Logic



Semantics of Propositional Logic

Proof Theory



Soundness and Completeness (preview)

Semantics of Propositional Logic Proof Theory Soundness and Completeness (preview)

Beyond Traditional Logic

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Not just sets

How to express this using traditional logic?

- "The sun is shining today."
- "Earth has more mass than Mars."

Arguments as Propositions

How to formalize a proposition of the form

If p_1 then p_2 ?

Semantics of Propositional Logic Proof Theory Soundness and Completeness (preview)

Atoms

Propositional Atoms Constructing Propositions Syntax of Propositional Logic

Motivation

Anything goes We allow any kind of proposition, for example "The sun is shining today".

Convention

We usually use p, q, p_1 , etc, instead of sentences like "The sun is shining today".

Atoms

More formally, we fix a set A of propositional atoms.

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Meaning of Atoms

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Motivation

Models assign truth values

A model assigns truth values (F or T) to each atom.

More formally

A model for a propositional logic for the set A of atoms is a mapping from A to $\{T, F\}$.

How do you call them?

Models for propositional logic are called *valuations*.

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Motivation

Example

Some valuation Let $A = \{p, q, r\}$. Then a valuation v_1 might assign p to T, q to F and r to T.

More formally $p^{v_1} = T, q^{v_1} = F, r^{v_1} = T.$ write $v_1(p)$ instead of p^{v_1}

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Building Propositions

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We would like to build larger propositions, such as arguments, out of smaller ones, such as propositional atoms. We do this using *operators* that can be applied to propositions, and yield propositions.

Soundness and Completeness (preview)

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Unary Operators

Let *p* be an atom.

All possibilities

The following options exist:

1
$$p^{v} = F$$
: $(op(p))^{v} = F$. $p^{v} = T$: $(op(p))^{v} = F$.
2 $p^{v} = F$: $(op(p))^{v} = T$. $p^{v} = T$: $(op(p))^{v} = T$.
3 $p^{v} = F$: $(op(p))^{v} = F$. $p^{v} = T$: $(op(p))^{v} = T$.
4 $p^{v} = F$: $(op(p))^{v} = T$. $p^{v} = T$: $(op(p))^{v} = F$.

The fourth operator *negates* its argument, *T* becomes *F* and *F* becomes *T*. We call this operator *negation*, and write $\neg p$ (pronounced "not p").

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Nullary Operators are Constants

The constant op

The constant \top always evaluates to *T*, regardless of the valuation.

The constant ot

The constant \perp always evaluates to *F*, regardless of the valuation.

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Binary Operators: 16 choices

р	q	$op_1(p,q)$	$op_2(p,q)$	$op_3(p,q)$	$op_4(p,q)$
F	F	F	F	F	F
F	T	F	F	F	F
T	F	F	F	Т	T
T	Τ	F	Т	F	Т

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Binary Operators: 16 choices (continued)

р	q	$op_5(p,q)$	$op_6(p,q)$	$op_7(p,q)$	$op_8(p,q)$
F	F	F	F	F	F
F	T	Т	Т	Т	T
T	F	F	F	Т	T
T	T	F	Т	F	Т

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Binary Operators: 16 choices (continued)

р	q	$op_9(p,q)$	$op_{10}(p,q)$	$op_{11}(p,q)$	$op_{12}(p,q)$
F	F	Т	Т	Т	Т
F	T	F	F	F	F
T	F	F	F	Т	Т
Τ	T	F	Т	F	Т

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Binary Operators: 16 choices (continued)

р	q	$op_{13}(p,q)$	$op_{14}(p,q)$	$op_{15}(p,q)$	$op_{16}(p,q)$
F	F	Т	Т	Т	Т
F	T	Т	Т	Т	Т
Τ	F	F	F	Т	Т
Τ	Т	F	Т	F	Т

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Three Famous Ones

- $op_2 : op_2(p,q)$ is T when p is T and q is T, and F otherwise. Called *conjunction*, denoted $p \land q$.
- $op_8 : op_8(p,q)$ is T when p is T or q is T, and F otherwise. Called *disjunction*, denoted $p \lor q$.
- $op_{14} : op_{14}(p,q)$ is *T* when *p* is *F* or *q* is *T*, and *F* otherwise. Called *implication*, denoted $p \rightarrow q$.

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Inductive Definition

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Definition

For a given set A of propositional atoms, the set of *well-formed* formulas in propositional logic is the least set F that fulfills the following rules:

- The constant symbols \perp and \top are in *F*.
- Every element of A is in F.
- If ϕ is in *F*, then $(\neg \phi)$ is also in *F*.
- If ϕ and ψ are in *F*, then $(\phi \land \psi)$ is also in *F*.
- If ϕ and ψ are in *F*, then $(\phi \lor \psi)$ is also in *F*.
- If ϕ and ψ are in *F*, then ($\phi \rightarrow \psi$) is also in *F*.

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$(((\neg p) \land q) \rightarrow (\top \land (q \lor (\neg r))))$

is a well-formed formula in propositional logic.

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More Compact in BNF

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$\phi ::= p \mid \perp \mid \top \mid (\neg \phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \to \phi)$ (Backus Naur Form)

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Convention

The negation symbol \neg binds more tightly than \land and \lor , and \land and \lor bind more tightly than \rightarrow . Moreover, \rightarrow is *right-associative*: The formula $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$.

Example

$$(((\neg p) \land q) \rightarrow (p \land (q \lor (\neg r))))$$

can be written as

$$\neg p \land q \rightarrow p \land (q \lor \neg r)$$

Operations on Truth Values Evaluation of Formulas



Atoms and Propositions

- 2 Semantics of Propositional Logic
 - Operations on Truth Values
 - Evaluation of Formulas

Proof Theory



Soundness and Completeness (preview)

Operations on Truth Values Evaluation of Formulas

Negating Truth Values

Т

F

Definition Function $\setminus : \{F, T\} \rightarrow \{F, T\}$ given in truth table: $\frac{B \mid \setminus B}{F \mid T}$

Operations on Truth Values Evaluation of Formulas

Conjunction of Truth Values

Definition

Function & : $\{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

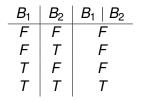


Operations on Truth Values Evaluation of Formulas

Disjunction of Truth Values

Definition

Function $|: \{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:



Operations on Truth Values Evaluation of Formulas

Implication of Truth Values

Definition

Function $\Rightarrow: \{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

Operations on Truth Values Evaluation of Formulas

Evaluation of Formulas

Definition

The result of *evaluating* a well-formed propositional formula ϕ with respect to a valuation v, denoted $v(\phi)$ is defined as follows:

- If ϕ is the constant \bot , then $v(\phi) = F$.
- If ϕ is the constant \top , then $v(\phi) = T$.
- If ϕ is an propositional atom p, then $v(\phi) = p^v$.
- If ϕ has the form $(\neg \psi)$, then $\nu(\phi) = \setminus \nu(\psi)$.
- If ϕ has the form $(\psi \wedge \tau)$, then $v(\phi) = v(\psi) \& v(\tau)$.
- If ϕ has the form $(\psi \lor \tau)$, then $v(\phi) = v(\psi) | v(\tau)$.
- If ϕ has the form $(\psi \to \tau)$, then $v(\phi) = v(\psi) \Rightarrow v(\tau)$.

Operations on Truth Values Evaluation of Formulas

Valid Formulas

Definition A formula is called *valid* if it evaluates to *T* with respect to every possible valuation.

Operations on Truth Values Evaluation of Formulas

Examples

Example

ls

 $(((\neg p) \land q) \rightarrow (\top \land (q \lor (\neg r))))$

valid?

Example

Find a valid formula that contains the propositional atoms p, q, r and w.

Sequents Axioms Derived Rules



Atoms and Propositions



Semantics of Propositional Logic

- 3 Proof Theory
 - Sequents
 - Axioms
 - Our Derived Rules



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Sequents Axioms Derived Rules

Sequents

Definition

A sequent consists of propositional formulas $\phi_1, \phi_2, \ldots, \phi_n$, called *premises*, where $n \ge 0$, and a propositional formula ψ called *conclusion*. We write a sequent as follows:

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

and say " ψ is provable using the premises $\phi_1, \phi_2, \ldots, \phi_n$ ".

Soundness and Completeness (preview)

Introducing \top

Sequents Axioms Derived Rules

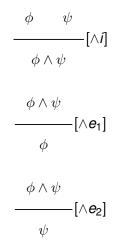
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Sequents Axioms Derived Rules

Soundness and Completeness (preview)

Rules for Conjunction



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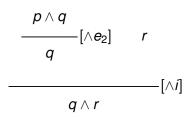
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Sequents Axioms Derived Rules

Example

 $p \land q, r \vdash q \land r$

Proof (graphical notation):



Example

Sequents Axioms Derived Rules

 $p \land q, r \vdash q \land r$

Proof (text-based notation):

1	$(p \wedge q)$	premise
2	q	∧e 1
3	r	premise
4	$\boldsymbol{q}\wedge \boldsymbol{r}$	∧i 2,3

Sequents Axioms Derived Rules

Soundness and Completeness (preview)

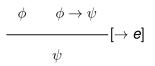
Double Negation Elimination



Sequents Axioms Derived Rules

Soundness and Completeness (preview)

Implication Elimination



Sequents Axioms Derived Rules

Soundness and Completeness (preview)

We would like...

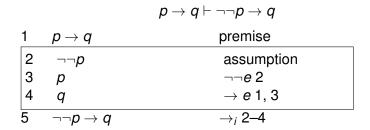
...to be able to prove:

$$p
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Sequents Axioms Derived Rules

Soundness and Completeness (preview)

A proof should look like this

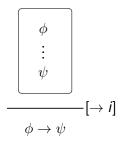


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Soundness and Completeness (preview)

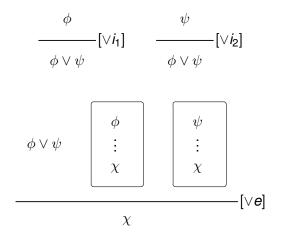
Implication Elimination



Sequents Axioms Derived Rules

Soundness and Completeness (preview)

Rules for Disjunction

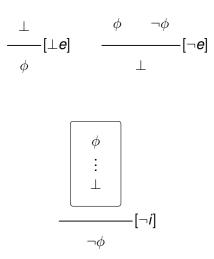


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Sequents Axioms Derived Rules

Soundness and Completeness (preview)

Axioms for \perp and Negation



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Double Negation Introduction

Lemma $(\neg \neg i)$

The following sequent holds for any formula ϕ :

 $\phi \vdash \neg \neg \phi$

Proof:

1	ϕ	premise
2	$\neg \phi$	assumption
3	\perp	¬e 1,2
4	$\neg \neg \phi$	_i 2–3

Sequents Axioms Derived Rules

Double Negation Introduction

Lemma $(\neg \neg i)$

The following sequent holds for any formula ϕ :

 $\phi \vdash \neg \neg \phi$

can be written like an axiom:



Soundness and Completeness (preview)

Sequents Axioms Derived Rules

Law of Excluded Middle

Lemma (LEM)

$$------[LEM]$$

$$\phi \lor \neg \phi$$

.

Entailment Soundness and Completeness







Semantics of Propositional Logic

Proof Theory



Soundness and Completeness (preview)

- Entailment
- Soundness and Completeness

Entailment

Entailment Soundness and Completeness

Definition

If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to T, the formula ψ evaluates to T as well, we say that $\phi_1, \phi_2, \ldots, \phi_n$ semantically entail ψ , written:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

Entailment Soundness and Completeness

Soundness and Completeness

Theorem (Soundness of Propositional Logic)

Let $\phi_1, \phi_2, \dots, \phi_n$ and ψ be propositional formulas. If $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$, then $\phi_1, \phi_2, \dots, \phi_n \models \psi$.

Theorem (Completeness of Propositional Logic) Let $\phi_1, \phi_2, \dots, \phi_n$ and ψ be propositional formulas. If $\phi_1, \phi_2, \dots, \phi_n \models \psi$, then $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$.

Entailment Soundness and Completeness

Admin

- Coq Homework 1: due 27/8, 9:30pm
- Assignment 2: out on module homepage; due 2/9, 11:00am
- Monday, Wednesday: Office hours
- Tuesday: Tutorials (Assignments 1 and 2)
- Wednesday: Labs (Coq Homework 1; Quiz 1)
- Thursday: Lecture on Predicate Logic