03b—Propositional Logic

CS 3234: Logic and Formal Systems

Martin Henz and Aquinas Hobor

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Beyond Traditional Logic

Not just sets

How to express this using traditional logic?

- "$1 + 1 = 3$"
- "The sun is shining today."
- "Earth has more mass than Mars."

Arguments as Propositions

How to formalize a proposition of the form

If p_1 then p_2?

Atoms

Anything goes

We allow any kind of proposition, for example "The sun is shining today".

Convention

We usually use p, q, p_1, etc, instead of sentences like "The sun is shining today".

Atoms

More formally, we fix a set A of propositional atoms.

Meaning of Atoms

Models assign truth values

A *model* assigns truth values (F or T) to each atom.

More formally

A model for a propositional logic for the set A of atoms is a mapping from A to $\{T, F\}$.

How do you call them?

Models for propositional logic are called *valuations*.

Examples

Example

Some valuation Let $A = \{p, q, r\}$. Then a valuation v_1 might assign p to T, q to F and r to T.

More formally

$p^{v_1} = T, q^{v_1} = F, r^{v_1} = T$.
write $v_1(p)$ instead of p^{v_1}

Building Propositions

We would like to build larger propositions, such as arguments, out of smaller ones, such as propositional atoms. We do this using *operators* that can be applied to propositions, and yield propositions.

Unary Operators

Let p be an atom.

All possibilities

The following options exist:

1. $p^v = F$: $(op(p))^v = F$. $p^v = T$: $(op(p))^v = F$.
2. $p^v = F$: $(op(p))^v = T$. $p^v = T$: $(op(p))^v = T$.
3. $p^v = F$: $(op(p))^v = F$. $p^v = T$: $(op(p))^v = T$.
4. $p^v = F$: $(op(p))^v = T$. $p^v = T$: $(op(p))^v = F$.

The fourth operator *negates* its argument, T becomes F and F becomes T. We call this operator *negation*, and write $\neg p$ (pronounced “not p”).

Nullary Operators are Constants

The constant $\top$

The constant $\top$ always evaluates to T, regardless of the valuation.

The constant $\bot$

The constant $\bot$ always evaluates to F, regardless of the valuation.

Binary Operators: 16 choices

p	q	$op_1(p,q)$	$op_2(p,q)$	$op_3(p,q)$	$op_4(p,q)$
F	F	F	F	F	F
F	T	F	F	F	F
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_5(p,q)$	$op_6(p,q)$	$op_7(p,q)$	$op_8(p,q)$
F	F	F	F	F	F
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_9(p,q)$	$op_{10}(p,q)$	$op_{11}(p,q)$	$op_{12}(p,q)$
F	F	T	T	T	T
F	T	F	F	F	F
T	F	F	F	T	T
T	T	F	T	F	T

Binary Operators: 16 choices (continued)

p	q	$op_{13}(p, q)$	$op_{14}(p, q)$	$op_{15}(p, q)$	$op_{16}(p, q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	T	F	T

Three Famous Ones

op_2 : $op_2(p, q)$ is T when p is T *and* q is T, and F otherwise. Called *conjunction*, denoted $p \land q$.

op_8 : $op_8(p, q)$ is T when p is T *or* q is T, and F otherwise. Called *disjunction*, denoted $p \lor q$.

op_{14} : $op_{14}(p, q)$ is T when p is F *or* q is T, and F otherwise. Called *implication*, denoted $p \to q$.

Inductive Definition

Definition

For a given set *A* of propositional atoms, the set of *well-formed formulas in propositional logic* is the least set *F* that fulfills the following rules:

- The constant symbols $\perp$ and $\top$ are in *F*.
- Every element of *A* is in *F*.
- If ϕ is in *F*, then $(\neg\phi)$ is also in *F*.
- If ϕ and ψ are in *F*, then $(\phi \land \psi)$ is also in *F*.
- If ϕ and ψ are in *F*, then $(\phi \lor \psi)$ is also in *F*.
- If ϕ and ψ are in *F*, then $(\phi \to \psi)$ is also in *F*.

Example

$$(((\neg p) \land q) \to (\top \land (q \lor (\neg r))))$$

is a well-formed formula in propositional logic.

More Compact in BNF

$$\phi ::= p \mid \bot \mid \top \mid (\neg\phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \rightarrow \phi)$$

(Backus Naur Form)

Convention

The negation symbol $\neg$ binds more tightly than $\wedge$ and $\vee$, and $\wedge$ and $\vee$ bind more tightly than $\rightarrow$. Moreover, $\rightarrow$ is *right-associative*: The formula $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$.

Example

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$

can be written as

$$\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$$

Negating Truth Values

Definition

Function $\backslash : \{F, T\} \to \{F, T\}$ given in truth table:

B	$\backslash B$
F	T
T	F

Conjunction of Truth Values

Definition

Function $\& : \{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \& B_2$
F	F	F
F	T	F
T	F	F
T	T	T

Disjunction of Truth Values

Definition

Function $\mid$: $\{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \mid B_2$
F	F	F
F	T	F
T	F	F
T	T	T

Implication of Truth Values

Definition

Function $\Rightarrow$: $\{F, T\} \times \{F, T\} \rightarrow \{F, T\}$ given in truth table:

B_1	B_2	$B_1 \Rightarrow B_2$
F	F	T
F	T	T
T	F	F
T	T	T

Evaluation of Formulas

Definition

The result of *evaluating* a well-formed propositional formula ϕ with respect to a valuation v, denoted $v(\phi)$ is defined as follows:

- If ϕ is the constant $\bot$, then $v(\phi) = F$.
- If ϕ is the constant $\top$, then $v(\phi) = T$.
- If ϕ is an propositional atom p, then $v(\phi) = p^v$.
- If ϕ has the form $(\neg\psi)$, then $v(\phi) = \backslash v(\psi)$.
- If ϕ has the form $(\psi \wedge \tau)$, then $v(\phi) = v(\psi) \& v(\tau)$.
- If ϕ has the form $(\psi \vee \tau)$, then $v(\phi) = v(\psi) \mid v(\tau)$.
- If ϕ has the form $(\psi \rightarrow \tau)$, then $v(\phi) = v(\psi) \Rightarrow v(\tau)$.

Valid Formulas

Definition

A formula is called *valid* if it evaluates to T with respect to every possible valuation.

Examples

Example

Is

$$(((\neg p) \wedge q) \rightarrow (\top \wedge (q \vee (\neg r))))$$

valid?

Example

Find a valid formula that contains the propositional atoms p, q, r *and* w.

1. Atoms and Propositions
2. Semantics of Propositional Logic
3. **Proof Theory**
   - Sequents
   - Axioms
   - Derived Rules
4. Soundness and Completeness (preview)

Sequents

Definition

A sequent consists of propositional formulas $\phi_1, \phi_2, \ldots, \phi_n$, called *premises*, where $n \geq 0$, and a propositional formula ψ called *conclusion*. We write a sequent as follows:

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

and say "ψ is provable using the premises $\phi_1, \phi_2, \ldots, \phi_n$".

Introducing $\top$

$$\frac{}{\top}[\top i]$$

Rules for Conjunction

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}[\wedge i]$$

$$\frac{\phi \wedge \psi}{\phi}[\wedge e_1]$$

$$\frac{\phi \wedge \psi}{\psi}[\wedge e_2]$$

Example

$$p \land q, r \vdash q \land r$$

Proof (graphical notation):

$$\dfrac{\dfrac{p \land q}{q}\,[\land e_2] \qquad r}{q \land r}\,[\land i]$$

Example

$$p \wedge q, r \vdash q \wedge r$$

Proof (text-based notation):

1	$(p \wedge q)$	premise
2	q	$\wedge$e 1
3	r	premise
4	$q \wedge r$	$\wedge$i 2,3

Double Negation Elimination

$$\frac{\neg\neg\phi}{\phi}\,[\neg\neg i]$$

Implication Elimination

$$\frac{\phi \qquad \phi \to \psi}{\psi}\,[\to e]$$

We would like...

...to be able to prove:

$$p \to q \vdash \neg\neg p \to \neg q$$

A proof should look like this

$$p \rightarrow q \vdash \neg\neg p \rightarrow q$$

1	$p \rightarrow q$	premise
2	$\neg\neg p$	assumption
3	p	$\neg\neg e$ 2
4	q	$\rightarrow e$ 1, 3
5	$\neg\neg p \rightarrow q$	$\rightarrow_i$ 2–4

Implication Elimination

$$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \to \psi}\,[\to i]$$

Rules for Disjunction

$$\frac{\phi}{\phi \vee \psi}\,[\vee i_1] \qquad \frac{\psi}{\phi \vee \psi}\,[\vee i_2]$$

$$\frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\,[\vee e]$$

Axioms for ⊥ and Negation

$$\frac{\bot}{\phi}\,[\bot e] \qquad \frac{\phi \qquad \neg\phi}{\bot}\,[\neg e]$$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\,[\neg i]$$

Double Negation Introduction

Lemma ($\neg\neg i$)

The following sequent holds for any formula ϕ:

$$\phi \vdash \neg\neg\phi$$

Proof:

1	ϕ	premise
2	$\neg\phi$	assumption
3	$\perp$	$\neg$e 1,2
4	$\neg\neg\phi$	$\neg$i 2–3

Double Negation Introduction

Lemma ($\neg\neg i$)

The following sequent holds for any formula ϕ:

$$\phi \vdash \neg\neg\phi$$

can be written like an axiom:

$$\frac{\phi}{\neg\neg\phi}\,[\neg\neg i]$$

Law of Excluded Middle

Lemma (LEM)

$$\frac{}{\phi \vee \neg\phi}\,[LEM]$$

1. Atoms and Propositions
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3. Proof Theory
4. Soundness and Completeness (preview)
   - Entailment
   - Soundness and Completeness

Entailment

Definition

If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to $\mathtt{T}$, the formula ψ evaluates to $\mathtt{T}$ as well, we say that $\phi_1, \phi_2, \ldots, \phi_n$ semantically entail ψ, written:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

Soundness and Completeness

Theorem (Soundness of Propositional Logic)

Let $\phi_1, \phi_2, \ldots, \phi_n$ *and* ψ *be propositional formulas.*
If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, *then* $\phi_1, \phi_2, \ldots, \phi_n \models \psi$.

Theorem (Completeness of Propositional Logic)

Let $\phi_1, \phi_2, \ldots, \phi_n$ *and* ψ *be propositional formulas.*
If $\phi_1, \phi_2, \ldots, \phi_n \models \psi$, *then* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.

Admin

- Coq Homework 1: due 27/8, 9:30pm
- Assignment 2: out on module homepage; due 2/9, 11:00am
- Monday, Wednesday: Office hours
- Tuesday: Tutorials (Assignments 1 and 2)
- Wednesday: Labs (Coq Homework 1; Quiz 1)
- Thursday: Lecture on Predicate Logic