

## 08—Modal Logic

CS 3234: Logic and Formal Systems

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- 1 Motivation
- 2 Basic Modal Logic
- 3 Logic Engineering

- 1 Motivation
- 2 Basic Modal Logic
- 3 Logic Engineering

# Necessity

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- You are crime investigator and consider different suspects. You know that the victim Ms Smith had called the police.
  - Maybe the cook did it before dinner?
  - Maybe the maid did it after dinner?
- But: “The victim Ms Smith made a phone call *before* she was killed.” is *necessarily* true.
- “Necessarily” means in all possible scenarios (worlds) under consideration.

## Notions of Truth

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- Often, it is not enough to distinguish between “true” and “false”.
- We need to consider *modalities* of truth, such as:
  - necessity (“in all possible scenarios”)
  - morality/law (“in acceptable/legal scenarios”)
  - time (“forever in the future”)
- Modal logic constructs a framework using which modalities can be formalized and reasoning methods can be established.

- 1 Motivation
- 2 **Basic Modal Logic**
  - Syntax
  - Semantics
  - Equivalences
- 3 Logic Engineering

# Syntax of Basic Modal Logic

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$$\begin{aligned} \phi ::= & \top \mid \perp \mid p \mid (\neg\phi) \mid (\phi \wedge \phi) \\ & \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \\ & \mid (\Box\phi) \mid (\Diamond\phi) \end{aligned}$$

## Pronunciation and Examples

### Pronunciation

If we want to keep the meaning open, we simply say “box” and “diamond”.

If we want to appeal to our intuition, we may say “necessarily” and “possibly” (or “forever in the future” and “sometime in the future”)

### Examples

$$(p \wedge \diamond(p \rightarrow \Box \neg r))$$

$$\Box((\diamond q \wedge \neg r) \rightarrow \Box p)$$



# Kripke Models

## Definition

A model  $\mathcal{M}$  of propositional modal logic over a set of propositional atoms  $A$  is specified by three things:

- ① A  $W$  of *worlds*;
- ② a relation  $R$  on  $W$ , meaning  $R \subseteq W \times W$ , called the *accessibility relation*;
- ③ a function  $L : W \rightarrow A \rightarrow \{T, F\}$ , called *labeling function*.

## Who is Kripke?

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How do I know I am not dreaming? Saul Kripke asked himself this question in 1952, at the age of 12. His father told him about the philosopher Descartes.

Modal logic at 17 Kripke's self-studies in philosophy and logic led him to prove a fundamental completeness theorem on modal logic at the age of 17.

Bachelor in Mathematics from Harvard is his only non-honorary degree

At Princeton Kripke taught philosophy from 1977 onwards. Contributions include modal logic, naming, belief, truth, the meaning of "I"

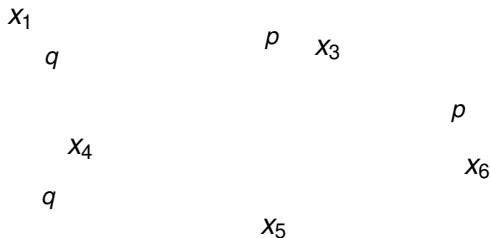
## Example

$$W = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

$$R = \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_4, x_5), (x_5, x_4), (x_5, x_6)\}$$

$$L = \{(x_1, \{q\}), (x_2, \{p, q\}), (x_3, \{p\}), (x_4, \{q\}), (x_5, \{\}), (x_6, \{p\})\}$$

$x_2$   
 $p, q$



## When is a formula true in a possible world?

### Definition

Let  $\mathcal{M} = (W, R, L)$ ,  $x \in W$ , and  $\phi$  a formula in basic modal logic. We define  $x \Vdash \phi$  via structural induction:

- $x \Vdash \top$
- $x \not\Vdash \perp$
- $x \Vdash p$  iff  $L(x)(p) = T$
- $x \Vdash \neg\phi$  iff  $x \not\Vdash \phi$
- $x \Vdash \phi \wedge \psi$  iff  $x \Vdash \phi$  and  $x \Vdash \psi$
- $x \Vdash \phi \vee \psi$  iff  $x \Vdash \phi$  or  $x \Vdash \psi$
- ...

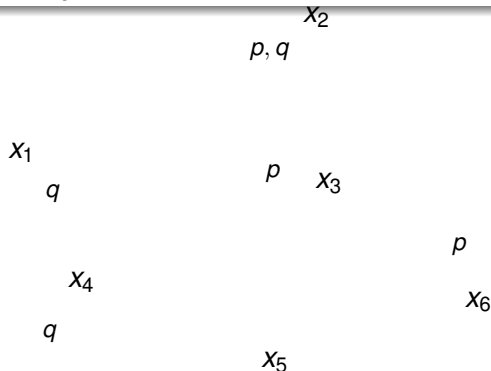
## When is a formula true in a possible world?

Definition (continued)

Let  $\mathcal{M} = (W, R, L)$ ,  $x \in W$ , and  $\phi$  a formula in basic modal logic. We define  $x \Vdash \phi$  via structural induction:

- ...
- $x \Vdash \phi \rightarrow \psi$  iff  $x \Vdash \psi$ , whenever  $x \Vdash \phi$
- $x \Vdash \Box\phi$  iff for each  $y \in W$  with  $R(x, y)$ , we have  $y \Vdash \phi$
- $x \Vdash \Diamond\phi$  iff there is a  $y \in W$  such that  $R(x, y)$  and  $y \Vdash \phi$ .

## Example



- $x_1 \Vdash q$
- $x_1 \Vdash \Diamond q$ ,  $x_1 \not\Vdash \Box q$
- $x_5 \not\Vdash \Box p$ ,  $x_5 \not\Vdash \Box q$ ,  $x_5 \not\Vdash \Box p \vee \Box q$ ,  $x_5 \Vdash \Box(p \vee q)$
- $x_6 \Vdash \Box \phi$  holds for all  $\phi$ , but  $x_6 \not\Vdash \Diamond \phi$  regardless of  $\phi$

# Formula Schemes

## Example

We said  $x_6 \Vdash \Box\phi$  holds for all  $\phi$ , but  $x_6 \not\Vdash \Diamond\phi$  regardless of  $\phi$

## Notation

Greek letters denote formulas, and are not propositional atoms.

## Formula schemes

Terms where Greek letters appear instead of propositional atoms are called *formula schemes*.

# Entailment and Equivalence

## Definition

A set of formulas  $\Gamma$  entails a formula  $\psi$  of basic modal logic if, in any world  $x$  of any model  $\mathcal{M} = (W, R, L)$ , we have  $x \Vdash \psi$  whenever  $x \Vdash \phi$  for all  $\phi \in \Gamma$ . We say  $\Gamma$  entails  $\psi$  and write  $\Gamma \models \psi$ .

## Equivalence

We write  $\phi \equiv \psi$  if  $\phi \models \psi$  and  $\psi \models \phi$ .



## Some Equivalences

- De Morgan rules:  $\neg\Box\phi \equiv \Diamond\neg\phi$ ,  $\neg\Diamond\phi \equiv \Box\neg\phi$ .
- Distributivity of  $\Box$  over  $\wedge$ :

$$\Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$$

- Distributivity of  $\Diamond$  over  $\vee$ :

$$\Diamond(\phi \vee \psi) \equiv \Diamond\phi \vee \Diamond\psi$$

- $\Box\top \equiv \top$ ,  $\Diamond\perp \equiv \perp$

# Validity

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## Definition

A formula  $\phi$  is valid if it is true in every world of every model, i.e. iff  $\models \phi$  holds.

## Examples of Valid Formulas

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- All valid formulas of propositional logic
- $\neg\Box\phi \rightarrow \Diamond\neg\phi$
- $\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$
- $\Diamond(\phi \vee \psi) \rightarrow \Diamond\phi \vee \Diamond\psi$
- Formula *K*:  $\Box(\phi \rightarrow \psi) \rightarrow \Box\phi \rightarrow \Box\psi$ .

- 1 Motivation
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- 3 **Logic Engineering**
  - Valid Formulas wrt Modalities
  - Properties of  $R$
  - Correspondence Theory
  - Preview: Some Modal Logics

## A Range of Modalities

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In a particular context  $\Box\phi$  could mean:

- It is necessarily true that  $\phi$
- It will always be true that  $\phi$
- It ought to be that  $\phi$
- Agent  $Q$  believes that  $\phi$
- Agent  $Q$  knows that  $\phi$
- After any execution of program  $P$ ,  $\phi$  holds.

Since  $\Diamond\phi \equiv \neg\Box\neg\phi$ , we can infer the meaning of  $\Diamond$  in each context.

## A Range of Modalities

From the meaning of  $\Box\phi$ , we can conclude the meaning of  $\Diamond\phi$ ,  
since  $\Diamond\phi \equiv \neg\Box\neg\phi$ :

$\Box\phi$	$\Diamond\phi$
It is necessarily true that $\phi$	It is possibly true that $\phi$
It will always be true that $\phi$	Sometime in the future $\phi$
It ought to be that $\phi$	It is permitted to be that $\phi$
Agent $Q$ believes that $\phi$	$\phi$ is consistent with $Q$ 's beliefs
Agent $Q$ knows that $\phi$	For all $Q$ knows, $\phi$
After any run of $P$ , $\phi$ holds.	After some run of $P$ , $\phi$ holds

# Formula Schemes that hold wrt some Modalities

$\Box\phi$	$\Box\phi \rightarrow \phi$	$\Box\phi \rightarrow \Box\phi$	$\Box\phi \rightarrow \Box\Box\phi$	$\Box\phi \rightarrow \Box\Diamond\phi$	$\Box\phi \rightarrow \Diamond\phi$	$\Box\phi \vee \Box\neg\phi$	$\Box(\phi \rightarrow \psi) \wedge \Box\phi \rightarrow \Box\psi$	$\Diamond\phi \wedge \Diamond\psi \rightarrow \Diamond(\phi \wedge \psi)$
It is necessary that $\phi$	✓	✓	✓	✓	✓	×	✓	×
It will always be that $\phi$	×	✓	×	×	×	×	✓	×
It ought to be that $\phi$	×	×	×	✓	✓	×	✓	×
Agent Q believes that $\phi$	×	✓	✓	✓	✓	×	✓	×
Agent Q knows that $\phi$	✓	✓	✓	✓	✓	×	✓	×
After running P, $\phi$	×	×	×	×	×	×	✓	×

## Modalities lead to Interpretations of $R$

$\Box\phi$	$R(x, y)$
It is necessarily true that $\phi$	$y$ is possible world according to info at $x$
It will always be true that $\phi$	$y$ is a future world of $x$
It ought to be that $\phi$	$y$ is an acceptable world according to the information at $x$
Agent $Q$ believes that $\phi$	$y$ could be the actual world according to $Q$ 's beliefs at $x$
Agent $Q$ knows that $\phi$	$y$ could be the actual world according to $Q$ 's knowledge at $x$
After any execution of $P$ , $\phi$ holds	$y$ is a possible resulting state after execution of $P$ at $x$



## Possible Properties of $R$

- reflexive: for every  $w \in W$ , we have  $R(x, x)$ .
- symmetric: for every  $x, y \in W$ , we have  $R(x, y)$  implies  $R(y, x)$ .
- serial: for every  $x$  there is a  $y$  such that  $R(x, y)$ .
- transitive: for every  $x, y, z \in W$ , we have  $R(x, y)$  and  $R(y, z)$  imply  $R(x, z)$ .
- Euclidean: for every  $x, y, z \in W$  with  $R(x, y)$  and  $R(x, z)$ , we have  $R(y, z)$ .
- functional: for each  $x$  there is a unique  $y$  such that  $R(x, y)$ .
- linear: for every  $x, y, z \in W$  with  $R(x, y)$  and  $R(x, z)$ , we have  $R(y, z)$  or  $y = z$  or  $R(z, y)$ .
- total: for every  $x, y \in W$ , we have  $R(x, y)$  and  $R(y, x)$ .
- equivalence: reflexive, symmetric and transitive.

## Example

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Consider the modality in which  $\Box\phi$  means  
“it ought to be that  $\phi$ ”.

- Should  $R$  be reflexive?
- Should  $R$  be serial?

# Necessarily true and Reflexivity

Guess

$R$  is reflexive if and only if  $\Box\phi \rightarrow \phi$  is valid.

# Motivation

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- We would like to establish that some formulas hold whenever  $R$  has a particular property.
- Ignore  $L$ , and only consider the  $(W, R)$  part of a model, called *frame*.
- Establish formula schemes based on properties of frames.

## Reflexivity and Transitivity

### Theorem 1

Let  $\mathcal{F} = (W, R)$  be a frame. The following statements are equivalent:

- $R$  is reflexive;
- $\mathcal{F}$  satisfies  $\Box\phi \rightarrow \phi$ ;
- $\mathcal{F}$  satisfies  $\Box p \rightarrow p$  for any atom  $p$

### Theorem 2

The following statements are equivalent:

- $R$  is transitive;
- $\mathcal{F}$  satisfies  $\Box\phi \rightarrow \Box\Box\phi$ ;
- $\mathcal{F}$  satisfies  $\Box p \rightarrow \Box\Box p$  for any atom  $p$

## Proof of Theorem 1

Let  $\mathcal{F} = (W, R)$  be a frame. The following statements are equivalent:

- 1  $R$  is reflexive;
- 2  $\mathcal{F}$  satisfies  $\Box\phi \rightarrow \phi$ ;
- 3  $\mathcal{F}$  satisfies  $\Box p \rightarrow p$  for any atom  $p$

1  $\Rightarrow$  2: Let  $R$  be reflexive. Let  $L$  be any labeling function;  
 $\mathcal{M} = (W, R, L)$ . Need to show for any  $x$ :  
 $x \Vdash \Box\phi \rightarrow \phi$  Suppose  $x \Vdash \Box\phi$ .  
Since  $R$  is reflexive, we have  $x \Vdash \phi$ .  
Using the semantics of  $\rightarrow$ :  $x \Vdash \Box\phi \rightarrow \phi$

# Proof of Theorem 1

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Let  $\mathcal{F} = (W, R)$  be a frame. The following statements are equivalent:

- ①  $R$  is reflexive;
- ②  $\mathcal{F}$  satisfies  $\Box\phi \rightarrow \phi$ ;
- ③  $\mathcal{F}$  satisfies  $\Box p \rightarrow p$  for any atom  $p$

2  $\Rightarrow$  3: Just set  $\phi$  to be  $p$

## Proof of Theorem 1

Let  $\mathcal{F} = (W, R)$  be a frame. The following statements are equivalent:

- ①  $R$  is reflexive;
- ②  $\mathcal{F}$  satisfies  $\Box\phi \rightarrow \phi$ ;
- ③  $\mathcal{F}$  satisfies  $\Box p \rightarrow p$  for any atom  $p$

③  $\Rightarrow$  ①: Suppose the frame satisfies  $\Box p \rightarrow p$ .

Take any world  $x$  from  $W$ .

Choose a labeling function  $L$  such that

$L(x)(p) = F$ , but  $L(y)(p) = T$  for all  $y$  with  $y \neq x$

Proof by contradiction: Assume  $(x, x) \notin R$ . Then we would have  $x \Vdash \Box p$ , but not  $x \Vdash p$ .

Contradiction!



# Formula Schemes and Properties of $R$

name	formula scheme	property of $R$
T	$\Box\phi \rightarrow \phi$	reflexive
B	$\phi \rightarrow \Box\Diamond\phi$	symmetric
D	$\Box\phi \rightarrow \Diamond\phi$	serial
4	$\Box\phi \rightarrow \Box\Box\phi$	transitive
5	$\Diamond\phi \rightarrow \Box\Diamond\phi$	Euclidean
	$\Box\phi \rightarrow \Diamond\phi \wedge \Diamond\phi \rightarrow \Box\phi$	functional
	$\Box(\phi \wedge \Box\phi \rightarrow \psi) \vee \Box(\psi \wedge \Box\psi \rightarrow \phi)$	linear

## Which Formula Schemes to Choose?

### Definition

Let  $\mathcal{L}$  be a set of formula schemes and  $\Gamma \cup \{\psi\}$  a set of formulas of basic modal logic.

- A set of formula schemes is said to be *closed* iff it contains all substitution instances of its elements.
- Let  $\mathcal{L}_c$  be the smallest closed superset of  $\mathcal{L}$ .
- $\Gamma$  entails  $\psi$  in  $\mathcal{L}$  iff  $\Gamma \cup \mathcal{L}_c$  semantically entails  $\psi$ . We say  $\Gamma \models_{\mathcal{L}} \psi$ .

## Examples of Modal Logics: K

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K is the weakest modal logic,  $\mathcal{L} = \emptyset$ .

## Examples of Modal Logics: KT45

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$$\mathcal{L} = \{T, 4, 5\}$$

Used for reasoning about knowledge.

- T: Truth: agent  $Q$  only knows true things.
- 4: Positive introspection: If  $Q$  knows something, he knows that he knows it.
- 5: Negative introspection: If  $Q$  doesn't know something, he knows that he doesn't know it.

## Next Week

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- Examples of Modal Logic
- Natural deduction in modal logic
- Modal logic in Coq