Major Facilities for Mathematical Thinking and Understanding.

(3) Logic and deduction.

All professors are brilliant;
Smith is a professor,
therefore, Smith is brilliant.

All professors are brilliant;
Smith is brilliant; therefore,
Smith is a professor.

Mathematics uses logic constantly.

For instance,

let a, b, and c be positive integers. *If* b divides a *and* c divides b, *then* c divides a.

$$\frac{a}{b}, \qquad \frac{b}{c} \Rightarrow \frac{a}{c}.$$

If **a, b, and c are the lengths of the sides of a right angle triangle *and* c > b, c > a,**

then

$$a^2 + b^2 = c^2.$$

a^2 **is *not* negative.**

If $a^2 = 4$,

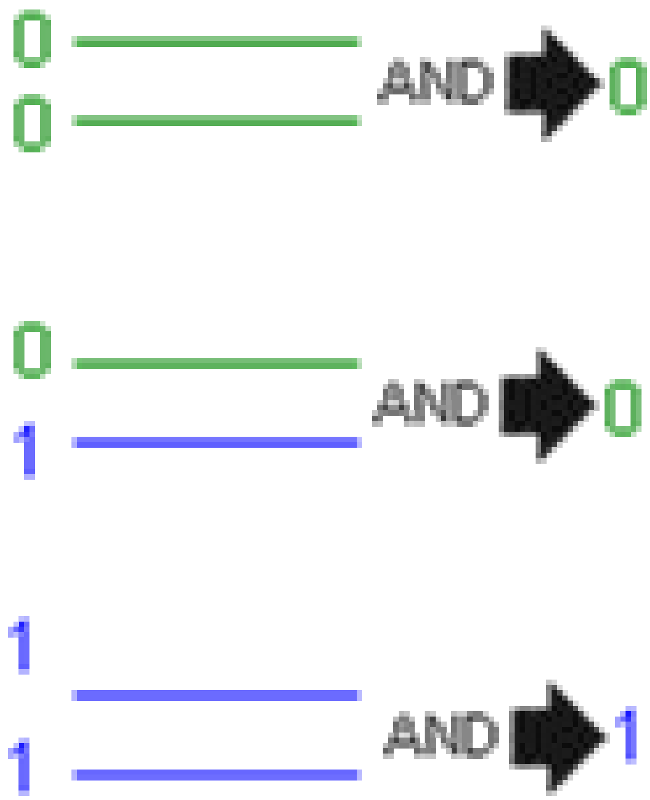
then **a = 2 *or* a = - 2.**

Building blocks of logic:

Truth table for "and":
(statement A <u>and</u> statement B)

A	B	$A \wedge B$
F	**F**	**F**
T	**F**	**F**
F	**T**	**F**
T	**T**	**T**

The “and gates”:



0 – False

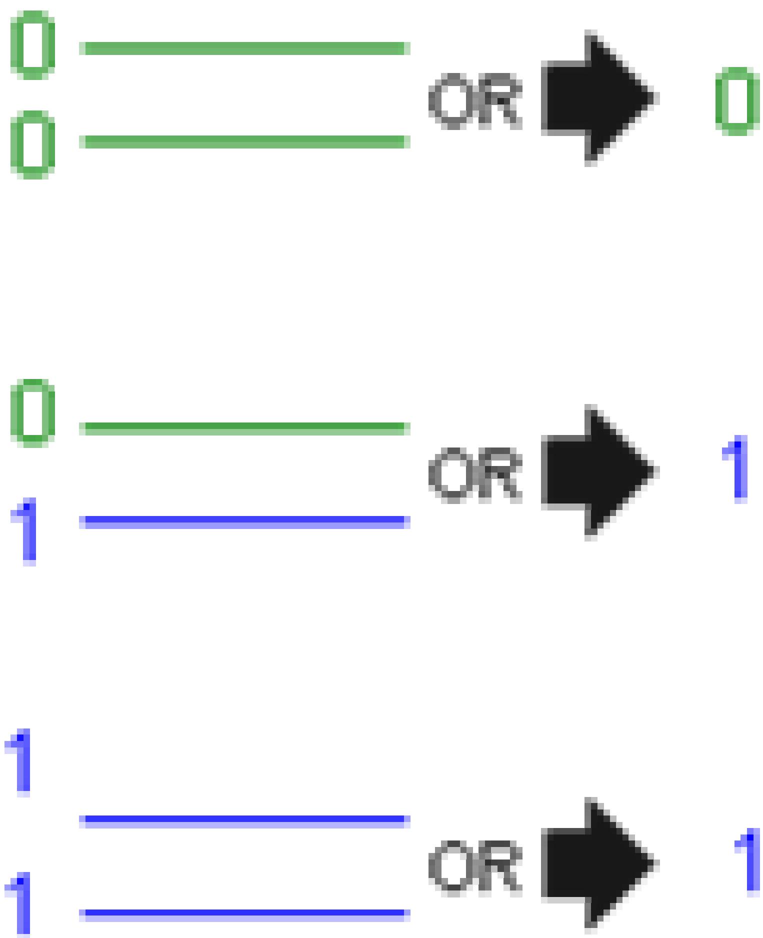
1 - True

Like multiplication: A x B.

Truth table for “or”:
(statement A <u>or</u> statement B)

A	B	$A \vee B$
F	F	F
T	F	T
F	T	T
T	T	T

The “or gates”:



0 – False

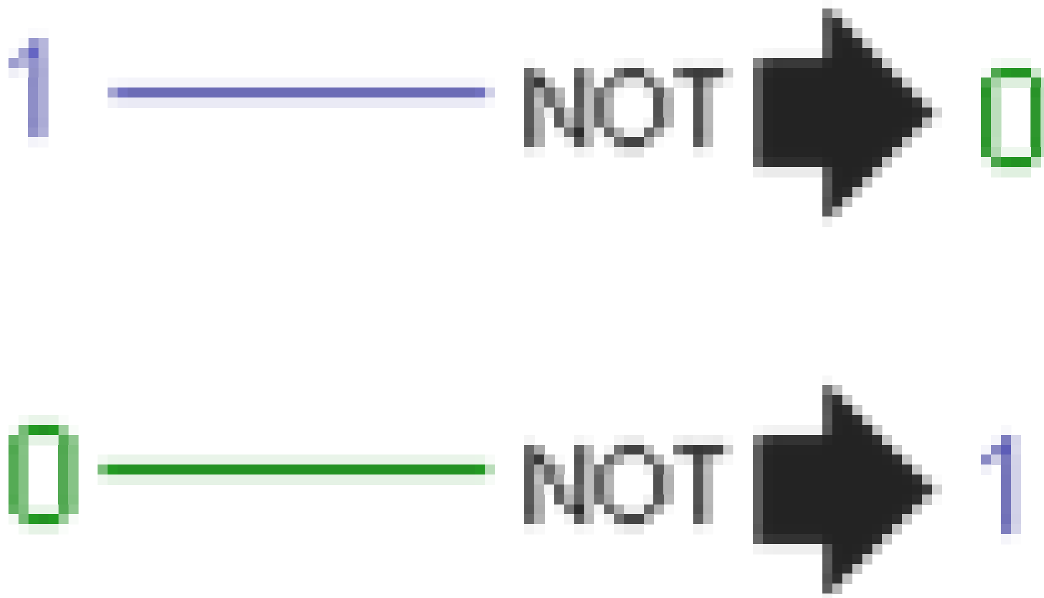
1 - True

Like addition: A + B.

Truth table for "not":
(not statement A)

A	$\neg A$
F	**T**
T	**F**

The “not gates”:



0 – False

1 - True

De Morgans Theorem

$$\neg(A \wedge B) = \neg A \vee \neg B;$$

$$\neg(A \vee B) = \neg A \wedge \neg B.$$

Implication.

If I have 5,000 dollars, I will surely get you a drink.

Assumption: I have 5,000 dollars. ***This may or may not be true.***

Conclusion: I will get you a drink.

I have 5,000 dollars,

(1) and I get you a drink (I'm right);
(2) and I do not get you a drink (I am wrong).

I do not have 5,000 dollars,

(a) and I get you a drink (I'm right);
(b) and I do not get you a drink (I am right).

<u>So</u>: either I do *not* have 5,000 dollars, *or* I get you a drink.

On the other hand, even I get you a drink, it does not imply that I have 5,000 dollars.

Statement A <u>implies</u> statement B

(or: if A, then B)

means *either A is not correct or B is correct.*

$$A \Rightarrow B \qquad \leftrightarrow \qquad \neg A \vee B$$

Truth table of $A \Rightarrow B$ **:**

A	B	$\neg A$	$\neg A \vee B$	$A \Rightarrow B$
T	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**

$$\neg A \vee B \quad \Leftrightarrow \quad B \vee \neg A$$
$$\Leftrightarrow \quad \neg(\neg B) \vee (\neg A)$$

It follows that

$$A \Rightarrow B \quad \Leftrightarrow \quad \neg B \Rightarrow \neg A.$$

For instance,

IF I think this way, I should be able To solve the problem.

You can't solve the problem, that means you have not thought this way.

Consider the negation

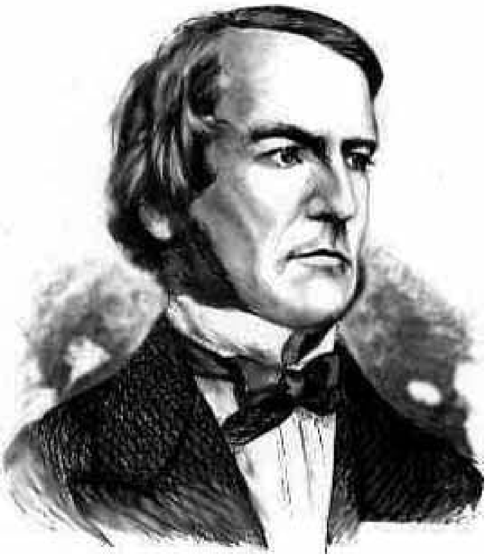
$$\neg(A \Rightarrow B) \quad \leftrightarrow \quad \neg(\neg A \vee B)$$

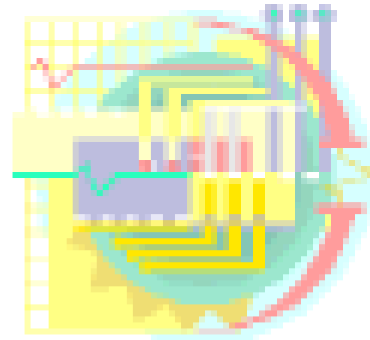
$$\leftrightarrow \quad A \wedge \neg B.$$

Thus to show that an implication is *not* correct, we need to find a situation where A is fulfilled but B is wrong. (Falsification by finding an example.)

If a jean is sold for more than 50 dollars, then it must be a Levis.

The statement is falsified if we can find a jean that is sold for more than 50 dollars, and is not a Levis (may be a Wrangle).





(1815-1864)

George Boole was born in a sordid family whose father was a shoemaker in Lincoln, England.

In 1847, he published *"The Mathematical Analysis of Logic"*, and introduced his early ideas on **Boolean Algebra** to the world. The article demonstrated that logic, as presented and verbalized by Aristotle, could be rendered as algebraic equations, and hence part of Mathematics.

From 1849, he was professor of mathematics at Queen's College in Cork, Ireland.

An (partial) axiomatic system for Boolean Algebra.

Undefined terms:

The operations

(1) And ($\wedge$);

(2) Or ($\vee$);

and the meaning of

(3) Equal (=).

Axioms:

There exists a collection P of elements A, B,…
(think of A, B,… as statements)
such that

(0) There are at least two distinct elements in P.

(1) If A, B are in P, then so do

$$A \wedge B, \qquad A \vee B.$$

(2) There exist elements Z (zero) and I (identity) in P such that

$$A \vee Z = A, \qquad B \wedge I = B$$

for *any* elements A, B in P.

(You may take Z to be the (false) statement 2 = 3; and I the (true) statement 3 = 3.)

(3) $a \vee b = b \vee a$

(4) $a \wedge b = b \wedge a$

(5) $a \vee b \wedge c = (a \vee b) \wedge (a \vee c)$

(6) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

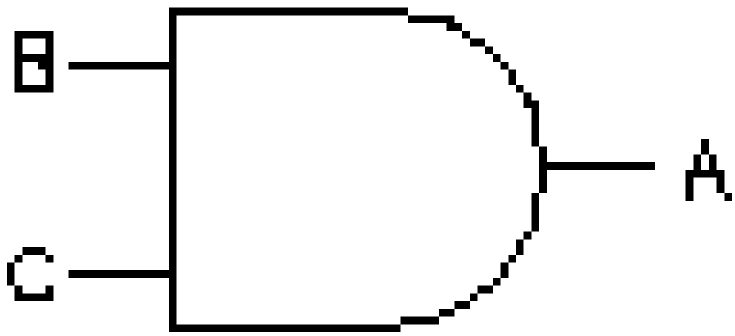
(7) **For any element A in P, there is an element $\neg A$ (called the negation of A) such that**

$$A \wedge \neg A = Z; \qquad A \vee \neg A = I.$$

Claude Shanon
(1916-2001)
Bell Lab./MIT

In 1937, nearly 75 years after Boole's death, Claude Shanon, a student at MIT, recognized [this] connection between electronic circuits and Boolean Algebra. He transferred the two logic states to electronic circuits by assigning different voltage levels to each state. This connection was essential for the design of digital computers.

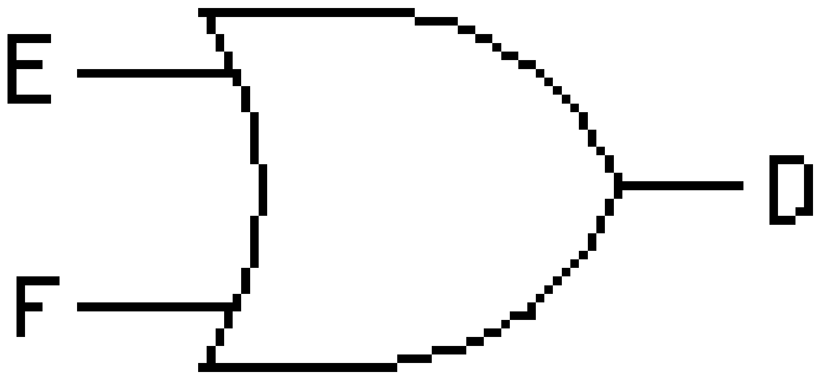
Circuit Diagrams:



Diagrams:

A = B * C

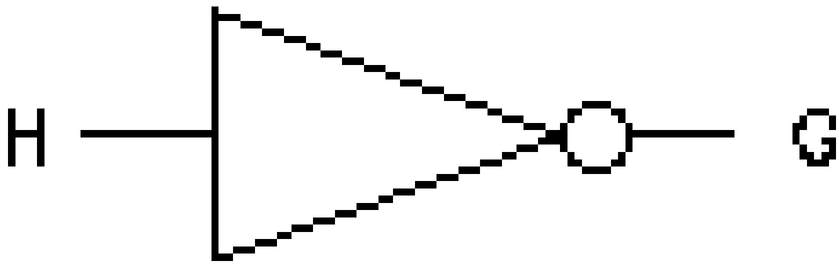
$$A = B \wedge C$$



Diagrams:

D = E + F

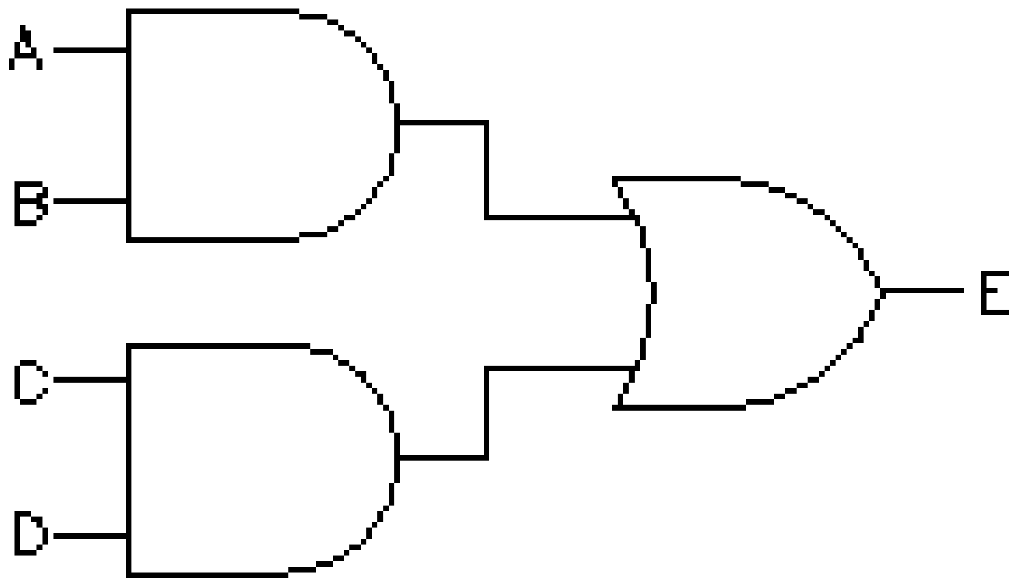
$$D = E \lor F$$



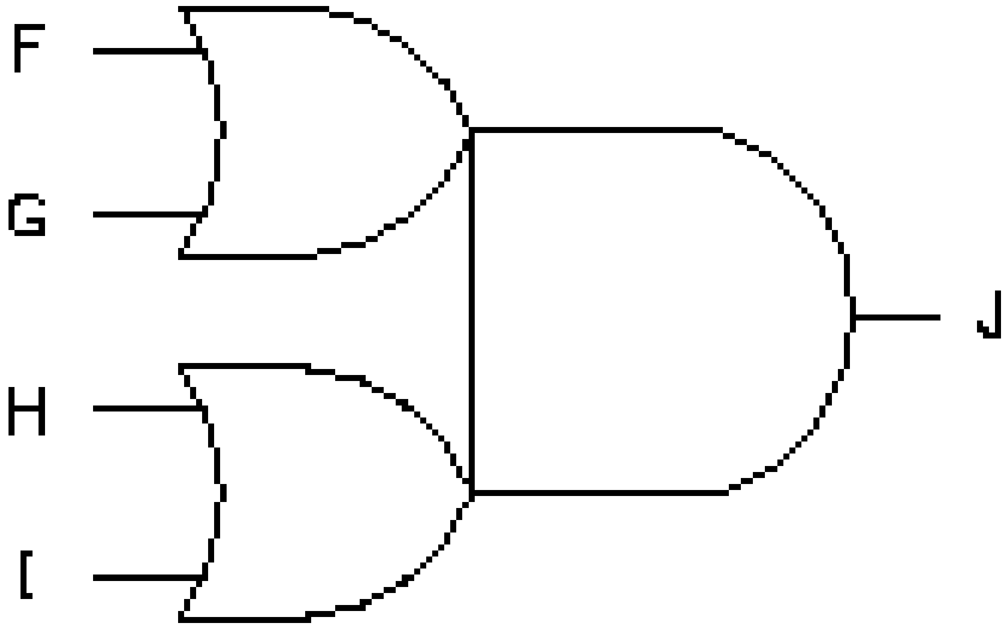
Diagrams:

$G = \overline{H}$

$$G = \neg H$$

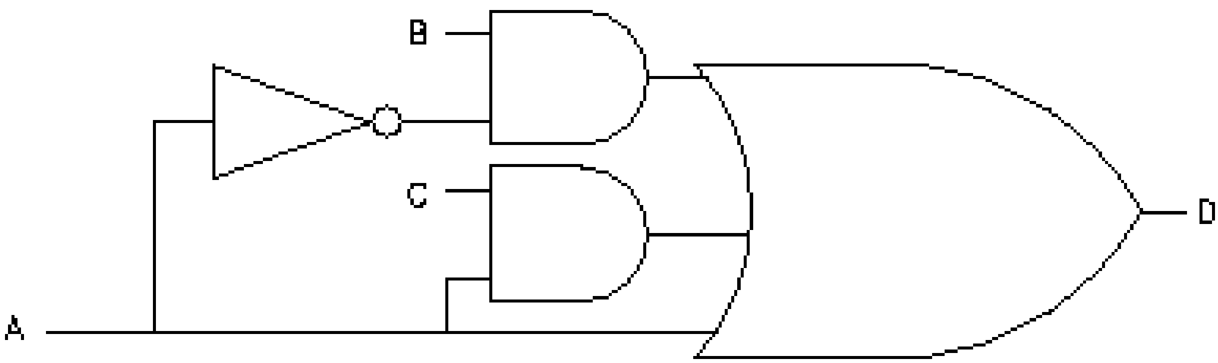


$$E = (A \wedge B) \vee (C \wedge D)$$



$$J = (F \vee G) \wedge (H \vee D = I)$$

Diagrams:



With its application in light switches and computers, Boolean Algebra has become a much more important part of mathematics. Circuits, the fundamental building blocks of computers, as well as sets and combinations, are dependent upon Boolean Algebra and have caused its usefulness to rise.

One must note with fascination (again) that a seemingly mundane math. could be an integral part of what is regarded as perhaps the most significant technology of the 20th century. It makes one wonder and speculate what other aspects of math. will be brought back to life to serve some useful purpose.

Some interesting articles:

(1) The Calculus of Logic, available at

http://www.maths.tcd.ie/pub/HistMath/People/Boole/CalcLogic/CalcLogic.pdf

(2) Boole meet Gates, A Look at the History of Computers, and the Role of Boolean Algebra

http://personal.nbnet.nb.ca/michaels/boole.htm