

SECTION A

Answer **all** the questions in this section. Section A carries a total of 60 marks.

Question 1 [20 marks]

- (a) Explain, with the help of examples, the terms ‘axiom’, ‘definition’, ‘theorem’, and ‘proof’.
- (b) Consider replacing the fifth axiom (equivalently, the parallel axiom) in Euclidean geometry by the following postulate.
- (N) *Given a line ℓ_1 and a point P not on the line, there are at least two lines ℓ_2 and ℓ_3 that pass through P and are parallel to ℓ_1 .*

Assuming (N) and the first four axioms of Euclid, show that in this “new geometry” there is a triangle with the sum of the interior angles *less* than 180° . State clearly the assumptions you use.

Question 2 [20 marks]

- (a) Explain the position of ‘conjecture and speculation’ in mathematical thinking. Provide an example to illustrate your discussion.
- (b) Given the formulas

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$$

consider the sum

$$1^4 + 2^4 + \cdots + n^4.$$

In the above, the summations are up to any positive integer n .

- (i) Explain the observation leading to the speculation:

$$1^4 + 2^4 + \cdots + n^4 = n(n+1)(an^3 + bn^2 + cn + d).$$

Here a , b , c and d are unknown numbers to be determined.

- (ii) Provide evidence to support (but *not* proving) the conjecture that $a = 1/5$.
- (iii) Using (i) and (ii), or otherwise, find b , c and d . Justify your answers. (You are *not* required to prove the formula so obtained.)

Question 3 [20 marks]

- (a) Consider the following excerpt from “*The Evolution of Physics*” by Einstein and Infeld.

“Science is not just a collection of laws, a catalogue of unrelated facts. It is a creation of the human mind, with its freely invented ideas and concepts. ‘Three trees’ is something different from ‘two trees’. Again ‘two trees’ is different from ‘two stones’. The concepts of the pure numbers 2, 3, 4, . . . , freed from the objects from which they arose, are creations of the thinking mind which describe the reality of our world.”

With the help of the above, or else, describe the key features in the Idea-Criticism model of mathematics.

- (b) A Pythagorean Tree is constructed by first drawing a *unit* square, then drawing a right angle triangle on top of that square, with the hypotenuse (the longest side) being the top edge of the square, then drawing two squares on top of the triangle, and repeating the procedure for each new square. See the figure for the second step.

Figure not available in the Internet version.

- (i) What is the sum of the areas of the four squares labeled I, II, III and IV in the figure? You may find Pythagoras’ theorem $a^2 + b^2 = c^2$ useful.
- (ii) Conjecture on the total area of the outer squares on each step. Describe the main ideas in supporting your answers.
- (iii) In the Pythagorean tree shown in the above figure, consider the sum of the areas of all the squares and the triangles. What kind of triangles will give you the maximal total area? Provide the key ideas in supporting your answers.

SECTION B

Answer not more than **two** questions in this section. Each question in this section carries 20 marks.

Question 4 [20 marks]

In turtle's graphs on the plane, consider the movements of the turtle:

F means advancing 1 unit in the direction of the turtle's head;

$+/-$ means, respectively, turning the turtle's head to the right/left by d degrees – the position of the turtle remains unchanged.

Take $d = 90^\circ$ and start with $F + F + F + F$ (0 iteration). In the subsequent steps, let the rewriting rule be

$$F \mapsto F - F + F.$$

- (i) Sketch the first two iterations.
- (ii) It is observed that in the first few iterations, the turtle always returns to the same position where it starts. Does this hold for *all* iterations? Present the key ideas critically and carefully to support your answers.

Question 5 [20 marks]

Consider the following excerpts (modified slightly) from the autobiography of Eric Cornell, who received 2001 Nobel Prize in Physics at the age of 40.

“There are relatively few experiments in atomic physics these days that don't involve the use of a laser. One major shortcoming in my graduate education in preparing me for a career in atomic physics research was that I had not learned any laser techniques. I felt my postdoctoral job had better fill in that lacuna. Looking for a postdoc job, I made the usual rounds, visiting Yale, Stanford ... Finally, I went out to Boulder to give a talk... No job offer was forthcoming, but *as luck would have it*, in the audience was a former Wineland-group colleague, Sarah Gilbert. Sarah called her husband, Carl Wieman, who was looking to hire, and suggested that he invite me to visit his laboratory.”

Question 5 (continued)

“(In Carl’s lab.) My attention was immediately drawn to his smaller, laser cooling experiment. In contrast to the other laser cooling experiments I had seen, which took up the better part of a room, Carl’s experiment could have fit on a card table. There was just one graduate student working on the project, and this *impressed* me as well - if a single student could make it work, how hard could it be? It was clear to me that I could learn how to make a fun little laser-cooling set up like Carl’s, and, looking ahead, it also seemed to me that I could duplicate such an experiment as an assistant professor without much trouble. It would be sufficiently easy to construct that I would have energy, time and money left over to use the cold atoms in turn to study something else.”

“In October of 1990 I arrived in Boulder. I found working with Carl to be a very congenial experience. Carl and I share very similar tastes in what makes for an interesting physics experiment, and I was happy to assimilate a fraction of his seemingly endless bag of technological ideas. *Carl taught me to decide what part of the experimental apparatus really mattered, and then to spare no effort improving that part. Conversely, Carl emphasized that one needs to recognize where ‘good enough’ was indeed good enough, and to waste no time worrying about it.*”

“During those early years in Boulder, I spent a lot of time trying to *imagine* what a Bose-Einstein condensate (BEC) would be like, if we could ever make one. Would it be superfluid, like liquid helium? Would it be coherent, like a laser? What do “superfluid” and “coherent” really mean? I understood these words in the context of the experiments the words had been invented to describe, or at least I thought I did, but it seemed to me that to understand how these words applied to a dramatically different physical system, one had to have a much deeper understanding. Superfluidity and lasing were two of my favorite topics in physics, but each was surrounded by a vast thicket of lore and literature. *It was hard to step off of the well-worn paths through these thickets, hard for a newcomer to get a fresh look at the underlying phenomena. If one could make a gas-phase condensate, one would have a less brambled system against which to test one’s own physical intuition.* Meditations along these lines converted me from BEC dabbler to true believer.”

With the help of the above, discuss the functions of (i) **personal use of imagination**, (ii) **connoisseurship**, (iii) **conviviality**, and (iv) **serendipity** in scientific and mathematical discoveries.

Question 6 [20 marks]

For the first few prime numbers $p = 2, 3, 5$, we have the expansions

$$\begin{aligned}(x + 1)^2 &= x^2 + 2x + 1, \\(x + 1)^3 &= x^3 + 3x^2 + 3x + 1, \\(x + 1)^5 &= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1.\end{aligned}$$

In each case, all the coefficients, except the first and the last ones, are divisible by the power on the left hand side. For an integer $n \geq 3$, consider the binomial expansion

$$(x + 1)^n - x^n - 1 = \sum_{k=1}^{n-1} C_k^n x^k,$$

where k is a positive integer and

$$C_1^n = n, \quad C_2^n = \frac{n(n-1)}{2 \cdot 1}, \quad C_k^n = \frac{n(n-1)(n-2) \cdots [n-(k-1)]}{k(k-1)(k-2) \cdots 1}, \quad n \geq k \geq 3.$$

It is known that C_k^n are integers.

- (i) Using the binomial expansion, or otherwise, show that all the coefficients in the expansion of

$$(x + 1)^7 - x^7 - 1$$

are divisible by 7. Show also that there is a coefficient in the expansion of

$$(x + 1)^8 - x^8 - 1$$

that is *not* divisible by 8.

- (ii) Let $p > 3$ be a prime number. Explain why p can divide

$$\frac{p(p-1)(p-2) \cdots [p-(k-1)]}{k(k-1)(k-2) \cdots 1}, \quad p > k \geq 3.$$

(*Hint:* Recall that C_k^n are integers.) Conclude that if $p > 1$ is a prime number, then p can divide all the coefficients in the expansion of

$$(x + 1)^p - x^p - 1.$$

Question 6 (continued)

- (iii) Let $n = p^r m$, where r and m are positive integers with $m > 1$, and $p > 1$ is a prime number which does not divide m . Explain why p does not divide any one of the numbers

$$(n - 1), (n - 2), \dots, [n - (p - 1)].$$

Conclude that n does not divide the number

$$\frac{n(n - 1)(n - 2) \cdots [n - (p - 1)]}{p(p - 1)(p - 2) \cdots 1}.$$

[END OF PAPER]