Linear algebra in less than 30 minutes
Eigenvalues & Eigenvectors

- **Eigenvectors** (for a square $m \times m$ matrix $S$)
  
  \[ S\mathbf{v} = \lambda \mathbf{v} \]

  (right) eigenvector \quad eigenvalue

  \[ \mathbf{v} \in \mathbb{R}^m \neq 0 \quad \lambda \in \mathbb{R} \]

- **How many eigenvalues** are there at most?

  \[ S\mathbf{v} = \lambda \mathbf{v} \iff (S - \lambda I) \mathbf{v} = 0 \]

  only has a non-zero solution if \[ |S - \lambda I| = 0 \]

  this is a $m$-th order equation in $\lambda$ which can have **at most $m$ distinct solutions** (roots of the characteristic polynomial) - can be complex even though $S$ is real.

Example

\[
\begin{pmatrix} 6 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]
Illustration of Eigenvectors

Matrix \( A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \)
describes an affine transformation \( x \mapsto Ax \)

Eigenvector \( x_1 = (0.52 \ 0.85)^T \)
for Eigenvalue \( \lambda_1 = 3.62 \)

Eigenvector \( x_2 = (0.85 \ -0.52)^T \)
for Eigenvalue \( \lambda_2 = 1.38 \)
Matrix-vector multiplication

\[ S = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

has eigenvalues 3, 2, 0 with corresponding eigenvectors

\[ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

On each eigenvector, \( S \) acts as a multiple of the identity matrix: but as a different multiple on each.

Any vector (say \( x = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \)) can be viewed as a combination of the eigenvectors:

\[ x = 2v_1 + 4v_2 + 6v_3 \]
Matrix vector multiplication

• Thus a matrix-vector multiplication such as $Sx$ ($S$, $x$ as in the previous slide) can be rewritten in terms of the eigenvalues/vectors:

$$Sx = S(2v_1 + 4v_2 + 6v_3)$$
$$Sx = 2Sv_1 + 4Sv_2 + 6Sv_3 = 2\lambda_1 v_1 + 4\lambda_2 v_2 + 6\lambda_3 v_3$$

• Even though $x$ is an arbitrary vector, the action of $S$ on $x$ is determined by the eigenvalues/vectors.
• Suggestion: the effect of “small” eigenvalues is small.
For symmetric matrices, eigenvectors for distinct eigenvalues are **orthogonal**

$$Sv_{\{1,2\}} = \lambda_{\{1,2\}} v_{\{1,2\}}, \text{ and } \lambda_1 \neq \lambda_2 \Rightarrow v_1 \cdot v_2 = 0$$

All eigenvalues of a real symmetric matrix are **real**.

for complex $\lambda$, if $|S - \lambda I| = 0$ and $S = S^T \Rightarrow \lambda \in \mathbb{R}$

All eigenvalues of a positive semidefinite matrix are **non-negative**

$$\forall w \in \mathbb{R}^n, w^T Sw \geq 0, \text{ then if } Sv = \lambda v \Rightarrow \lambda \geq 0$$
Example

• Let \( S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \)  
  \( \text{Real, symmetric.} \)

• Then \( S - \lambda I = \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} \) \( \Rightarrow (2 - \lambda)^2 - 1 = 0. \)

• The eigenvalues are 1 and 3 (nonnegative, real).
• The eigenvectors are orthogonal (and real):

\[
\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Plug in these values and solve for eigenvectors.