

Resource requirements of private quantum channels

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Abstract

Shannon [Sha48, Sha49] in celebrated works had shown that n bits of shared key is necessary and sufficient to transmit n -bit classical information in an information-theoretically secure way. Ambainis, Mosca, Tapp and de Wolf in [AMTdW00] considered a more general setting, referred to as *Private quantum channels*, in which instead of classical information, quantum states are required to be transmitted and only one-way communication is allowed. They show that in this case $2n$ bits of shared key is necessary and sufficient to transmit an n -qubit state. We consider the most general setting in which we allow for all possible combinations i.e. we let the input to be transmitted, the message sent and the shared resources to be classical/quantum. We develop a general framework by which we are able to show tight bounds on communication/shared resources in all but one of these cases and this includes the results of Shannon and Ambainis et al.

1 Introduction

Suppose Alice is required to transmit an n -bit input string to Bob in an *information theoretically secure* way, i.e. without leaking *any information* about her input to an eavesdropper Eve who has complete access to the channel between her and Bob. Shannon in [Sha48, Sha49] had shown that using n bits of shared key and by using *one-time pad* scheme Alice and Bob can accomplish this. He further showed that n bits of shared key are also required by any other scheme which accomplishes the same. Ambainis, Mosca, Tapp and de Wolf [AMTdW00] considered a generalization of this question in which instead of classical input, Alice has quantum input and only one way of quantum communication between Alice to Bob is allowed. They referred to this setting as Private quantum channels (**PQCs**). They showed that in this case the requirement of shared key increases. Their main result was:

Theorem 1.1 *$2n$ bits of shared key are necessary and sufficient to transmit any n -qubit quantum state in an information-theoretically secure way.*

We further generalize the setting by letting the shared resource between Alice and Bob to be quantum. A natural generalization of classical shared keys in the context of quantum communication protocols is a *pure* quantum state $|\psi\rangle^{AB}$ shared between Alice and Bob. This is referred to as *shared entanglement* or simply entanglement. We consider private quantum channels that use entanglement between Alice and Bob to achieve security, and in order to distinguish them from **PQCs** which use classical shared keys, we call them **PQCEs**. We formally define a **PQCE** as follows.

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Definition 1.2 Let S be a subset of pure n -qubit states. Let $|\psi\rangle^{AB}$ be a bi-partite pure state shared between Alice and Bob and let ρ be a quantum state.

1. **Alice's operations:** Alice gets an input pure state $|\phi\rangle$. Alice's operation consists of attaching a few ancilla qubits to her input and her part of the bi-partite state $|\psi\rangle^{AB}$. She then performs a unitary transformation on the combined quantum system of all her qubits and sends a subset of the resulting qubits to Bob. Let \mathcal{A} represent Alice's operations. Let for the input $|\phi\rangle$, $\mathcal{E}(|\phi\rangle)$ represent the (encoded) quantum state of the qubits sent to Bob. We have the following security requirement that $\forall |\phi\rangle \in S, \mathcal{E}(|\phi\rangle) = \rho$.
2. **Bob's operations:** Bob on receiving the quantum message from Alice attaches a few ancilla qubits to the combined system of the received message and his part of the bi-partite state $|\psi\rangle^{AB}$. He then performs a unitary transformation on the combined system of all her qubits and outputs a subset of the resulting qubits. Let \mathcal{B} represent Bob's operations. Let for input state $|\phi\rangle$ to Alice the final (decoded) output of Bob be represented by $\mathcal{D}(|\phi\rangle)$. We have the following correctness requirement that $\forall |\phi\rangle \in S, \mathcal{D}(|\phi\rangle) = |\phi\rangle\langle\phi|$.
Then $[S, \mathcal{A}, \mathcal{B}, |\psi\rangle^{AB}, \rho]$ is called a private quantum channel with entanglement (**PQCE**).

Note:

1. It is clear from the above definition that for one-way communication, the operations of Alice and Bob are as general as possible.
2. If in the above definition of a **PQCE**, we replace the bi-partite shared pure state $|\psi\rangle^{AB}$ with shared random strings between Alice and Bob, we get a **PQC**. We represent a **PQC** by $[S, \mathcal{A}, \mathcal{B}, P, \rho]$, where P is the distribution of the shared random strings between Alice and Bob.
3. A **PQCE/PQC** for S is also **PQCE/PQC** respectively for \tilde{S} which is the closure of S under finite convex combinations.

PQCEs were also considered by Leung [Leu02] by the name of *Quantum Vernam Cipher* who considered issues like security of key recycling and reliability of message transfer. In this paper we are primarily concerned with bounds on communication and entanglement requirements of **PQCEs**. We consider the following notion of the amount of entanglement.

Definition 1.3 (Amt. of entanglement) For a bi-partite pure state $|\psi\rangle^{AB}$, consider its Schmidt decomposition, $|\psi\rangle^{AB} = \sum_{i=1}^k \sqrt{\lambda_i} |a_i\rangle \otimes |b_i\rangle$, where $\{|a_i\rangle\}$ is an orthonormal set and so is $\{|b_i\rangle\}$, $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. The amount of entanglement of $|\psi\rangle^{AB}$ is defined to be $E(|\psi\rangle^{AB}) \triangleq -\sum_i \lambda_i \log \lambda_i$.

We consider all possible cases i.e. when the input to Alice, the message sent by Alice and the shared resource between Alice and Bob is either classical or quantum. We develop a general argument by which we are able to show tight bounds simultaneously on communication and shared resource usage in all the above cases except the case in which all the three quantities are quantum. Following is a compilation of all the results we obtain due to our analysis. Below when we say the “x,y,z” case (e.g. classical, quantum, classical case) we mean, Alice gets n -(qu)bits of x input, the communication is y and the shared resource is z .

- Result 1.4**
1. *In the classical, classical, classical case, n bits of communication and n bits of shared key is required. The one-time pad scheme hence is simultaneously optimal in both communication and shared key usage. This is basically Shannon's result [Sha48, Sha49].*
 2. *In the classical, quantum, classical case, n qubits of communication and n bits of shared key is required. Hence here again simultaneously optimal upper bound is achieved by the one-time pad scheme.*
 3. *In the classical, classical, quantum case, n bits of communication and n ebits of entanglement is required. Hence here again simultaneously optimal upper bound is achieved by the one-time pad scheme.*
 4. *In the classical, quantum, quantum case, $n/2$ qubits of communication and $n/2$ ebits of entanglement is required. The simultaneously optimal upper bound here is achieved by the standard protocol for super-dense coding [BW92, NC00] which is a **PQCE**. In it, Alice transfers n bits of classical input in an information-theoretically secure manner to Bob using $n/2$ qubits of communication and $n/2$ **EPR** pairs [NC00] shared between them. In this case the message of Alice is always in the maximally mixed state independent of her input.*
 5. *The quantum, classical, classical case is impossible with finite communication.*
 6. *In the quantum, quantum, classical case, n qubits of communication and $2n$ bits of shared key is required. This is the main result of Ambainis et al. [AMTdW00]. In the same paper they have exhibited a **PQC** which transfers an n -qubit state with n -qubits of communication and $2n$ bits of shared randomness and is therefore simultaneously optimal in both communication and shared randomness.*
 7. *In the quantum, classical, quantum case, $2n$ bits of communication and n ebits of entanglement is required. Here the simultaneously optimal scheme is the standard protocol for teleportation [BGC⁺93, NC00] which is a **PQCE** (pointed to us by de Wolf in personal communication). In this protocol Alice can transfer n -qubits to Bob in an information theoretically secure way by using $2n$ bits of communication and using n **EPR** pairs between them. In this case the message of Alice always has uniform distribution independent of her input.*
 8. *In the quantum, quantum, quantum case, n qubits of communication and n ebits of entanglement is required. However this is the only case in which we do not know of a simultaneously optimal scheme or whether the lower bounds are needed to be improved.*

Finally we discuss two-way multiple round **PQCs** (**MPQCs**) and **PQCEs** (**MPQCEs**). We show that an **MPQC** which can transfer an n -qubit state must use n -bits of classical shared keys. Also an **MPQCE** which can transfer an n -qubit state must use $\Omega(n)$ ebits of entanglement. Hence there is not much saving even when multiple rounds are allowed.

2 Organization of the paper

In the next section we make a few definitions and state a few facts which we will be using later in our proofs. In Section 4 we present the proofs of all the parts of Result 1.4. In Section 5

we discuss two-way multiple round private quantum channels and finally conclude with a few open questions in Section 6.

3 Preliminaries

Let \mathcal{H}_k represent the Hilbert space of dimension k . Let \mathcal{C}_k represent the set of quantum states corresponding to the standard basis of \mathcal{H}_k , also referred to as the *classical states*. Let I_k represent the identity transformation in a k dimensional space.

For a quantum state ρ with eigenvalues λ_i the *von-Neumann entropy* of it is defined as $S(\rho) \triangleq -\sum_i \lambda_i \log \lambda_i$. Given a joint quantum system AB , the mutual information between them is defined as $I(A : B) \triangleq S(A) + S(B) - S(AB)$. Relative entropy between two states ρ and σ is defined as $S(\rho||\sigma) \triangleq \text{Tr} \rho(\log \rho - \log \sigma)$. We require the following properties of von-Neumann entropy, relative entropy and mutual information. Please refer to [NC00] for a good introduction to quantum information theory.

- Fact 3.1**
1. $S(A) + S(B) - S(AB) \geq 0$. This is called as *sub-additivity property of von-Neumann entropy*. This implies $I(A : B) \geq 0$.
 2. $S(ABC) + S(A) \leq S(AB) + S(AC)$. This is called the *strong sub-additivity property*. This implies $I(\mathcal{E}(A) : B) \leq I(A : B)$, where \mathcal{E} is a quantum operation.
 3. We have the following *chain rule of mutual-information*, $I(A : BC) = I(A : B) + I(AB : C) - I(B : C)$, which follows easily from definition.
 4. $S(AB) \geq |S(A) - S(B)|$. This is called as *Araki-Lieb inequality*.
 5. Given a bi-partite system ρ^{AB} , $I(A : B) = S(\rho^{AB} || \rho^A \otimes \rho^B)$, where ρ^A, ρ^B are the states of the systems A and B respectively.
 6. Given a joint system AB with A being a classical system, $S(AB) \geq \max\{S(A), S(B)\}$.

We will need the following theorem.

Theorem 3.2 (Local transition theorem [May97, LH97, LH98]) *Let \mathcal{K}, \mathcal{H} be Hilbert spaces. Let ρ be a quantum state in \mathcal{K} . Let $|\phi_1\rangle$ and $|\phi_2\rangle$ be two purification of ρ in $\mathcal{H} \otimes \mathcal{K}$. Then there is a local unitary transformation U acting on \mathcal{H} such that $(U \otimes I)|\phi_1\rangle = |\phi_2\rangle$.*

We will also need the following *Substate* theorem from [JRS02].

Fact 3.3 *Let ρ, σ be quantum state. If $S(\rho||\sigma) \leq k$ then,*

$$\sigma - \frac{\rho'}{2^{\mathcal{O}(k)}} \geq 0$$

where $\text{Tr}|\rho' - \rho| \leq 0.1$.

4 Resource bounds

We first derive a few lemmas which will finally lead us to our results. In [AMTdW00] it is shown that a **PQC** which can transmit n -qubit quantum states can be converted into a **PQC** which uses the same amount of shared classical randomness to transmit any $2n$ bit classical state. We note that this proof works for **PQCE**'s as well. We only give a sketch of the proof here.

Lemma 4.1 *If there exists a **PQCE**, $[\mathcal{H}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, \rho]$ then there exists a **PQCE**, $[\mathcal{C}_{2^{2n}}, \mathcal{A}', \mathcal{B}', |\psi^{AB}\rangle, I_{2^n} \otimes \rho]$ which uses the same bi-partite state as the shared entanglement between Alice and Bob and uses extra n -qubits of communication.*

Proof: \mathcal{A}' maps every state of $\mathcal{C}_{2^{2n}}$ to a tensor product of n -Bell states [NC00] by mapping pairs of input bits on to one of the four Bell states. The second half of the Bell states are then encrypted using \mathcal{A} . Because of the entanglement between the pairs of qubits in the Bell states, the resulting state is $I_{2^n} \otimes \rho$. The decryption operation \mathcal{B}' of Bob corresponds to first decrypting the second half of the received qubits using \mathcal{B} and then recovering the input classical state by making measurement on the n -Bell states. ■

Below we show a similar lemma which implies that a **PQC/PQCE** which transmits any n -qubit quantum state can be converted into a **PQC/PQCE** which uses the same communication and extra n ebits of entanglement to transmit any $2n$ bit classical state. We sketch the proof for **PQCE**s and a similar proof holds for **PQCs** as well.

Lemma 4.2 *If there exists a **PQCE**, $[\mathcal{H}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, \rho]$ then there exists a **PQCE**, $[\mathcal{C}_{2^{2n}}, \mathcal{A}', \mathcal{B}', |\psi^{AB}\rangle \otimes (\frac{|00\rangle+|11\rangle}{\sqrt{2}})^{\otimes n}, \rho]$ which uses the same communication and extra n -EPR pairs.*

Proof: In $[\mathcal{C}_{2^{2n}}, \mathcal{A}', \mathcal{B}', |\psi^{AB}\rangle \otimes (\frac{|00\rangle+|11\rangle}{\sqrt{2}})^{\otimes n}, \rho]$, for each pair of her input bits, Alice applies one of the *Pauli matrices* [NC00] to her part of one of the extra **EPR** pairs and then encodes them using the encoding procedure of the earlier **PQCE** $[\mathcal{H}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, \rho]$, and sends the the resulting qubits to Bob. The security property of $[\mathcal{H}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, \rho]$ implies the security property of $[\mathcal{C}_{2^{2n}}, \mathcal{A}', \mathcal{B}', |\psi^{AB}\rangle \otimes (\frac{|00\rangle+|11\rangle}{\sqrt{2}})^{\otimes n}, \rho]$. On receiving Alice's message, Bob first applies the decoding procedure of $[\mathcal{H}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, \rho]$, and gets one of the four Bell states for each pair of the classical input of Alice from which he is able to recover Alice's input. ■

We will need the following lemma.

Lemma 4.3 *Let ABX be a tripartite system. Then,*

1. $S(AX) + S(BX) - S(ABX) - S(X) \leq \min\{2S(A), 2S(B)\}$.
2. *If AX is a classical system then we have the stronger inequality $S(AX) + S(BX) - S(ABX) - S(X) \leq \min\{S(A), S(B)\}$.*
3. $I(A : B) \leq \min\{2S(A), 2S(B)\}$.

Proof:

1.

$$\begin{aligned} S(AX) - S(ABX) + S(BX) - S(X) &\leq S(AX) - S(ABX) + S(B) \\ &\leq S(B) + S(B) = 2S(B) \end{aligned}$$

Above first inequality comes from part (1) and second inequality comes from part (4) of Fact 3.1. Similarly we get $S(AX) + S(BX) - S(ABX) - S(X) \leq 2S(A)$.

2.

$$S(AX) - S(ABX) + S(BX) - S(X) \leq S(BX) - S(X) \leq S(B)$$

Above first inequality arises from part (6), since AX is a classical system, and the second inequality comes from part (1) of Fact 3.1. Again, since A is a classical system, we get

$$S(AX) - S(X) + S(BX) - S(ABX) \leq S(AX) - S(X) \leq S(A)$$

Above the first inequality comes from part (6) and the second inequality comes from part (1) of Fact 3.1.

3.

$$I(A : B) = S(A) + S(B) - S(AB) \leq S(A) + S(A) = 2S(A)$$

The inequality above follows from part (4) of Fact 3.1. ■

We now have the following theorem.

Theorem 4.4 *If $[\mathcal{C}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, \rho]$ is a **PQCE** then,*

1. $S(\sigma^B) \geq n/2$, where σ^B is the quantum state corresponding to Bob's part of $|\psi^{AB}\rangle$. We note from definitions that $S(\sigma^B) = E(|\psi\rangle^{AB})$.
2. $S(\rho) \geq n/2$.

Proof: Let X be a random variable which takes values in $\{1, 2, \dots, 2^n\}$ uniformly and through the **PQCE** Alice is able to communicate X to Bob. We can assume that the operations of Alice are *safe* on X which means that at the beginning Alice makes a copy of X (since it is a classical state) and then her subsequent operations do not touch the original copy of X . Let M_1 be the quantum state corresponding to the message of Alice and let M_2 be the quantum state corresponding to Bob's part of $|\psi\rangle^{AB}$. Then from Fact 3.1,

$$\begin{aligned} n &= H(X) = I(\mathcal{D}(X) : X) = I(\mathcal{B}(M_1 M_2 \otimes |0\rangle\langle 0|_{\text{ancilla}}) : X) \\ &\leq I(M_1 M_2 \otimes |0\rangle\langle 0|_{\text{ancilla}} : X) = I(M_1 M_2 : X) \\ &= I(M_1 : X) + I(M_2 : M_1 X) - I(M_1 : M_2) \\ &= I(M_1 : X) + I(M_2 : X) + I(M_2 X : M_1) - I(M_1 : X) - I(M_1 : M_2) \\ &\leq 0 + 0 + I(M_2 X : M_1) - I(M_1 : X) \\ &= S(M_2 X) + S(M_1 X) - S(M_1 M_2 X) - S(X) \\ &\leq \min\{2S(M_2), 2S(M_1)\} \end{aligned}$$

Above, first inequality comes from part (2) of Fact 3.1. $I(M_1 : X) = 0$ because of the privacy property of the channel. $I(M_2 : X) = 0$ because they were independent to begin with and Alice's operations are safe on X . The last inequality follows from part (1) of lemma 4.3. ■

We note in the proof of Theorem 4.4, due to part (2) of lemma 4.3, that if either M_2 is a classical system (as in a **PQC**) or if M_1 is a classical system, then we get $n \leq \min\{S(M_2), S(M_1)\}$. Therefore we have the following corollary:

Corollary 4.5 1. If $[\mathcal{C}_{2^n}, \mathcal{A}, \mathcal{B}, P, \rho]$ is a **PQC** then, $S(P) \geq n$ and $S(\rho) \geq n$.

2. If $[\mathcal{C}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, P]$ is a **PQCE** with classical communication then, $S(\sigma) \geq n$, where σ is Bob's part of $|\psi^{AB}\rangle$, and $S(P) \geq n$.

We are now set to show various parts of Result 1.4.

Proof:

1. Follows from part (1) of corollary 4.5.
2. Follows from part (1) of corollary 4.5.
3. Follows from part (2) of corollary 4.5.
4. Follows from theorem 4.4.
5. Follows from the fact that quantum states cannot be encoded as finite classical distributions and faithfully recovered.
6. From **PQC** $[\mathcal{H}_{2^n}, \mathcal{A}, \mathcal{B}, P, \rho]$, using lemma 4.1 we get a **PQC** $[\mathcal{C}_{2^{2n}}, \mathcal{A}', \mathcal{B}', P, I_{2^n} \otimes \rho]$. Part (1) of corollary 4.5 now implies $S(P) \geq 2n$. Part (1) of corollary 4.5 also implies $S(I_{2^n} \otimes \rho) \geq 2n$ which implies $S(\rho) \geq n$.
7. Lower bound on communication follows from lemma 4.2 and corollary 4.5. Lower bound on entanglement follows from the fact that a **PQCE** for \mathcal{H}_{2^n} is also a **PQCE** for \mathcal{C}_{2^n} and corollary 4.5.
8. From $[\mathcal{H}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, \rho]$ using lemma 4.1 we get a **PQCE** $[\mathcal{C}_{2^{2n}}, \mathcal{A}', \mathcal{B}', |\psi^{AB}\rangle, I_{2^n} \otimes \rho]$. Theorem 4.4 now implies $E(|\psi^{AB}\rangle) \geq n$. Also from $[\mathcal{H}_{2^n}, \mathcal{A}, \mathcal{B}, |\psi^{AB}\rangle, \rho]$, using the standard protocol for teleportation we get a **PQCE** $[\mathcal{H}_{2^n}, \mathcal{A}'', \mathcal{B}'', |\psi_1^{AB}\rangle, P]$ where P is a classical distribution on twice the number of bits as the number of qubits of ρ . Now from part 7 we get that $S(P) \geq 2n$. Hence the number of qubits of ρ must be $\geq n$.

■

5 Multiple round private quantum channels

When we consider two-way multiple round **PQCs** (denoted **MPQC**) or multiple round **PQCEs** (**MPQCE**), we note that keeping the privacy of *individual messages* cannot be the only criteria. For example let us consider a protocol in which in the first message Alice transfers **EPR** pairs followed by a junk message of Bob and then Alice transfers her quantum state privately using the earlier sent **EPR** pairs. In this protocol none of the individual messages give any information about the transferred state but it does not mean that Eve,

who can access the channel in all rounds, cannot get any information about the transferred state. We note that while considering multiple round *private classical channels*, in which the communication is classical, we can consider all the messages of Alice and Bob together and put the requirement that they together should not reveal any information about Alice's input state. Hence we get exactly the same resource requirements as for one round private classical channels. But we cannot do this in case of quantum communication since all the messages together do not make any sense.

We therefore consider two possible definitions of **MPQCs** and **MPQCEs**. We define **MPQCs** and **MPQCEs** are similar with only shared randomness replaced by shared entanglement.

1. **MPQCs without abort:** In this case Alice and Bob never abort the protocol but satisfy the following:
 - Any interfering Eve gets no information about the input state of Alice.
 - If Eve is not interfering then the input state is faithfully transferred to Bob.
2. **MPQCs with abort:** In this case Alice can abort the protocol any time but satisfy the following:
 - Before abort any interfering Eve gets no information about the input state of Alice.
 - If there is no abort then the input state is faithfully transferred to Bob.

Remark: Consider an implementation of a private quantum channel in which Alice and Bob first use *quantum key distribution* (QED) protocols like BB84 for key generation and then use these keys to transfer quantum states privately. However it is not strictly an **MPQC** according to our definition, because current implementations of QEDs require the existence of a classical broadcast channel which is unjam able by Eve. Also such a protocol would not be perfectly secure and there would still be a small amount of information that Eve can obtain even in case Alice does not abort the protocol.

Below we discuss the resource requirements of **MPQCs** and **MPQCEs**. The *cheating strategies* of Eve discussed below work in both type of protocols, with and without abort.

Lemma 5.1 *Let σ be the distribution of the shared random strings between Alice and Bob in an MPQC for \mathcal{C}_{2^n} . Then $S(\sigma) \geq n$.*

Proof: Consider an attack of Eve where she starts acting like Bob. She guesses the random string which has highest probability, say p of occurring. The probability that her guessed string is equal to Alice's random string is at least p . In the event that she guesses Alice's random string correct, she gets to know Alice's input state faithfully at the end of the protocol and Alice does not abort the protocol in this case. Hence from the security criterion, $p \leq 2^{-n}$. This implies $S(\sigma) \geq n$. ■

We show a similar statement for **MPQCEs**.

Lemma 5.2 *Let $|\psi\rangle^{AB}$ be the prior shared pure state between Alice and Bob in an MPQCEs for \mathcal{C}_{2^n} . Let $\sigma^{AB} = |\psi\rangle\langle\psi|$. Let σ^A and σ^B denote state of Alice's and Bob's parts respectively in σ^{AB} . Then $E(|\psi\rangle^{AB}) = S(\sigma^A) = S(\sigma^B) = \Omega(n)$.*

Proof: Let $S(\sigma^B) = k$. Similar to above, let us consider a cheating strategy of Eve in which she starts acting like Bob. She starts with the state σ^B in the register which holds Bob's part of the entanglement. Let M_1 and M_2 represent Alice and Bob's parts in σ^{AB} . Then, from lemma 4.3 we get,

$$S(\sigma^{AB} || \sigma^A \otimes \sigma^B) = I(M_1 : M_2) \leq 2S(\sigma_B) = 2k$$

From substate theorem,

$$\sigma^A \otimes \sigma^B - \frac{\sigma'^{AB}}{2^{O(k)}} \geq 0$$

where $\text{Tr}|\sigma'^{AB} - \sigma^{AB}| \leq 0.1$

This implies that Eve with probability $2^{-O(k)}$ gets the same state created with her when Alice and Bob start with σ'^{AB} as the prior entangled state. Because $\text{Tr}|\sigma'^{AB} - \sigma^{AB}| \leq 0.1$, Alice's probability of abort ≤ 0.1 . Hence the state created with Eve will be the same as the input state of Alice with probability at least $(0.8)2^{-O(k)}$. Because of the security criterion $(0.8)2^{-O(k)} \leq 2^{-n} \Rightarrow k = \Omega(n)$. ■

6 Conclusion

We have considered private quantum channels of all possible kinds and in almost all cases shown optimal resource requirements. The interesting questions that arise are:

1. What is the optimal situation when all the three; Alice's input, communication and the shared resource are allowed to be quantum? Does the lower bounds improve or does there exist a scheme which is simultaneously optimal with current lower bounds?
2. If Eve is allowed arbitrary access to the channel, we see that there is not much saving on prior entanglement/shared randomness that one gets by allowing two-way communication. However, by allowing a classical broadcast channel between Alice and Bob, unjamable by Eve, saving is possible on prior entanglement/shared randomness by using QKD protocols. Is there a weaker assumption we can make for saving on prior entanglement/shared randomness?

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