Depth-Independent Lower bounds on Communication Complexity of Read-Once Boolean Functions

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Abstract

We show new depth independent lower bounds on the classical and quantum communication complexity of read-once boolean functions. Our results complement the recent results of Leonardos and Saks [LS09] and Jayram, Kopparty, Raghavendra [JKR09], where they present lower bounds on the classical communication of read-once boolean functions which depend on the depth of the corresponding canonical AND-OR trees.

We obtain our result by 'embedding' either the Disjointness problem or its complement in any given read-once boolean function.

A *read-once boolean function* $f : \{0,1\}^n \to \{0,1\}$ is a function which can be represented by a *boolean formula* involving AND and OR such that each variable appears, possibly negated, at most once in the formula. An *alternating* AND-OR *tree* is a layered tree in which each internal node is labeled either AND or OR and the leaves are labeled by variables. Each path from the root to the any leaf alternates between AND and OR labeled nodes. It is well known (see eg. [HW91]) that given a read-once boolean function $f : \{0,1\}^n \to \{0,1\}$ there exists a unique alternating AND-OR tree, denoted T_f, with n leaves labeled by input boolean variables $z_1, \ldots, z_n$, such that the output at the root, when the tree is evaluated according to the labels of the internal nodes, is equal to $f(z_1 \ldots z_n)$. Given an alternating AND-OR tree T, let f_T denote the corresponding read-once boolean function evaluated by T.

Let $x, y \in \{0,1\}^n$ and let $x \wedge y, x \vee y$ represent the *bit-wise* AND, OR of the strings x and y respectively. For $f : \{0,1\}^n \to \{0,1\}$, let $f^\wedge : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be given by $f^\wedge(x,y) = f(x \wedge y)$. Similarly let $f^\vee : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be given by $f^\vee(x,y) = f(x \vee y)$. Recently Leonardos and Saks [LS09] (see [JKR09] for related results), investigated the *two-party randomized communication complexity* (denoted $\mathsf{R}(\cdot)$) of $f^\wedge, f^\vee$ and showed the following (please refer to [KN97] for familiarity with basic definitions in communication complexity):

Fact 1 ([LS09]) *Let $f : \{0,1\}^n \to \{0,1\}$ be a read-once boolean function such that T_f has depth d. Then*

$$\max\{\mathsf{R}(f^\wedge), \mathsf{R}(f^\vee)\} \geq \Omega(n/8^d).$$

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It follows from results of Snir [Sni85] and Saks and Wigderson [SW86] (via a generic simulation of trees by communication protocols) that for read-once boolean functions with their canonical trees being *complete binary* AND-OR trees, the randomized communication complexity is $O(n^{0.753\ldots})$. However in this situation, the results of [LS09, JKR09] do not provide any lower bound since $d = \log n$ for complete binary trees. We complement their result by giving lower bounds that do not depend on the depth (below $\mathsf{Q}(\cdot)$ represent the two-party *quantum communication complexity*).

Theorem 1 *Let $f : \{0,1\}^n \to \{0,1\}$ be a read-once boolean function. Then,*

$$\max\{\mathsf{R}(f^\wedge), \mathsf{R}(f^\vee)\} \geq \Omega(\sqrt{n}).$$
$$\max\{\mathsf{Q}(f^\wedge), \mathsf{Q}(f^\vee)\} \geq \Omega(n^{1/4}).$$

Remark: Note that the maximum in the above result is necessary since for example if f is the AND of the n input bits then it is easily seen that $\mathsf{R}(f^\wedge)$ is constant.

Proof of Thm. 1: We start with the following definition.

Definition 1 (Embedding) *We say that a function $g_1 : \{0,1\}^r \times \{0,1\}^r \to \{0,1\}$ can be* embedded *into a function $g_2 : \{0,1\}^t \times \{0,1\}^t \to \{0,1\}$, if there exist maps $h_a : \{0,1\}^r \to \{0,1\}^t$ and $h_b : \{0,1\}^r \to \{0,1\}^t$ such that $\forall x, y \in \{0,1\}^r,\ g_1(x,y) = g_2(h_a(x), h_b(y))$.*

It is easily seen that if g_1 can be embedded into g_2 then the communication complexity of g_2 is at least as large as that of g_1.

Let us define the *Disjointness problem* $\mathsf{DISJ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ as $\mathsf{DISJ}_n(x,y) = \bigwedge_{i=1,\ldots,n} x_i \vee y_i$ (where the usual negation of the variables is left out for notational simplicity). Similarly the *Non-Disjointness problem* $\mathsf{NDISJ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is defined as $\mathsf{NDISJ}_n(x,y) = \bigvee_{i=1,\ldots,n} x_i \wedge y_i$.

Recall that for the given read-once boolean function $f : \{0,1\}^n \to \{0,1\}$ its the canonical tree is denoted T_f. We have the following lemma which we prove later.

Lemma 2
1. *Let T_f have its last layer consisting of* AND *gates. Let m_0 be the largest integer such that DISJ_{m_0} can be embedded into $f^\vee$ and m_1 be the largest integer such that NDISJ_{m_1} can be embedded into $f^\vee$. Then $m_0 m_1 \geq n$.*
2. *Let T_f have its last layer consisting of* OR *gates. Let m_0 be the largest integer such that DISJ_{m_0} can be embedded into $f^\wedge$ and m_1 be the largest integer such that NDISJ_{m_1} can be embedded into $f^\wedge$. Then $m_0 m_1 \geq n$.*

The lower bounds on $\max\{\mathsf{R}(f^\wedge), \mathsf{R}(f^\vee)\}$ and $\max\{\mathsf{Q}(f^\wedge), \mathsf{Q}(f^\vee)\}$ now follow immediately from the above lemma and the following well known lower bounds :

Fact 2 ([KS92, Raz92]) $\mathsf{R}(\mathsf{DISJ}_n) = \Omega(n), \mathsf{R}(\mathsf{NDISJ}_n) = \Omega(n)$.

Fact 3 ([Raz03]) $\mathsf{Q}(\mathsf{DISJ}_n) = \Omega(\sqrt{n}), \mathsf{Q}(\mathsf{NDISJ}_n) = \Omega(\sqrt{n})$.

This finishes the proof of Thm. 1. ■

Proof of Lem. 2: We prove 1. and the proof of 2. follows similarly. We have the following claim which we prove later.

Claim 1 *For an alternating* AND-OR *tree T such that all its internal nodes just above the leaves have exactly two children (denoted the x child and the y child), let $s(T)$ denote the number of such nodes directly above the leaves not counting the root. For example if T has a single internal node (which is also the root) with two leaves then we have $s(T) = 0$. Let $m_0(T)$ be the largest integer such that DISJ_{m_0} that can be embedded into f_T and $m_1(T)$ be the largest integer such that NDISJ_{m_1} can be embedded into f_T. Then $m_0(T)m_1(T) \geq s(T)$.*

Now let us view $f^\vee : \{0,1\}^{2n} \to \{0,1\}$ as a read-once boolean function and let $T_{f^\vee}$ be the corresponding canonical tree. Note that in this case $T_{f^\vee}$ satisfies the conditions of the above claim and $s(T_{f^\vee}) = n$ (the correspondence between the x, y children of fan-in 2 internal nodes just above the leaves and the inputs x, y of $f^\vee$ is clear). Hence the proof of 1. finishes. ■

Proof of Claim 1: The proof is by induction over depth d of T. The case $d = 1$ is true since then $m_0(T) = m_1(T) = s(T) = 0$.

Base Case $d = 2$: In this case T consists either of the root labeled AND with $s(T)$ (fan-in 2) children labeled ORs or it consists of the root labeled OR with $s(T)$ (fan-in 2) children labeled ANDs. We consider the former case and the latter follows similarly. In the former case f_T is clearly $\mathsf{DISJ}_{s(T)}$ and hence $m_0(T) = s(T)$. Also $m_1(T) = 1$ as follows. Let us choose the first two children v_1, v_2 of the root. Further choose the x child of v_1 and the y child of v_2 which are kept free and the values of all other input variables are set to 0. It is easily seen that the function (of input bits x, y) now evaluated is NDISJ_1. Hence $m_0(T)m_1(T) = s(T)$.

Induction Step $d > 2$: Assume the root is labeled AND (the case when the root is labeled OR follows similarly). Let the root have r children $v_1, \ldots, v_r$ which are labeled OR and let the corresponding subtrees be $T_1, \ldots, T_r$ rooted at $v_1, \ldots, v_r$ respectively. Let w.l.o.g the first r' (with $0 \le r' \le r$) of these trees be of depth 1 in which case the corresponding $s(\cdot) = 0$. It is easily seen that

$$\left(\sum_{i=1}^{r} s(T_i)\right) + r' = s(T) \ .$$

For $i \in [r]$, we have from the induction hypothesis that $m_1(T_i)m_0(T_i) \ge s(T_i)$. Note that if $s(T_i) = 0$ then $m_0(T_i) = 1$ and $m_1(T_i) = 0$.

It is clear that $m_0(T) \ge \sum_{i=1}^{r} m_0(T_i)$, since we can simply combine the disjointness instances of the subtrees. Also we have $m_1(T) \ge \max\{m_1(T_1), \ldots, m_1(T_r), 1\}$, because we can either take any one of the subtree instances (and set all other inputs to 0), or at the very least can pick a pair of x, y leaves (as in the base case above) and fix the remaining variables appropriately to realize a single AND gate which amounts to embedding NDISJ_1. Now,

$$\begin{aligned} m_0(T)m_1(T) &\ge \left(\sum_{i=1}^{r} m_0(T_i)\right) \cdot (\max\{m_1(T_1), \ldots, m_1(T_r), 1\}) \\ &\ge \left(\sum_{i=1}^{r} m_0(T_i)m_1(T_i)\right) + r' \\ &\ge \left(\sum_{i=1}^{r} s(T_i)\right) + r' = s(T) \ . \end{aligned}$$

■

Remarks:

1. The randomized communication complexity varies between $\Theta(n)$ for the Tribes_n function (a read-once boolean function whose canonical tree has depth 2) and $O(n^{0.753\ldots})$ for functions corresponding to completely balanced AND-OR trees (which have depth $\log n$). It will probably be hard to prove a generic lower bound much larger than $\sqrt{n}$ for all read-once boolean functions $f : \{0,1\}^n \to \{0,1\}$, since the best known lower bound on the randomized query complexity of every read-once boolean function is $\Omega(n^{.51})$ [HW91] and communication complexity lower bounds immediately imply slightly weaker query complexity lower bounds (via the generic simulation of trees by communication protocols).

2. Ambainis et al. [ACR$^+$07] show how to evaluate any alternating AND-OR tree with n leaves by a quantum query algorithm with slightly more than $\sqrt{n}$ queries. On the other hand, it is well known [**?**] that the *parity* of n bits can be computed by a formula of size $O(n^2)$ involving $\mathsf{AND}, \mathsf{OR}$. Therefore it is easy to show that the function *Inner Product modulo* 2 i.e. the function $\mathsf{IP}_m : \{0,1\}^m \times \{0,1\}^m \to \{0,1\}$ given by $\mathsf{IP}_m(x,y) = \sum_{i=1}^m x_i y_i \bmod 2$, with $m = \sqrt{n}$ can be reduced to the evaluation of an alternating AND-OR tree of size $O(n)$ and logarithmic depth. However it is known that $\mathsf{Q}(\mathsf{IP}_{\sqrt{n}}) = \Omega(\sqrt{n})$ [CvDNT99] and hence we get an example of an alternating AND-OR tree T with n leaves such that $\mathsf{Q}(f_T) = \Omega(\sqrt{n})$. Hence $\Theta(\sqrt{n})$ might turn out to be the correct bound on $\mathsf{Q}(f_T)$ for all alternating AND-OR trees T with n leaves regardless of the depth.

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