Distinguishing sets of quantum states

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Abstract
Given two sets finite \( S_0 \) and \( S_1 \) of quantum states. We show necessary and sufficient conditions for distinguishing them by a measurement.

Let there be two finite sets of quantum states \( S_0 = \{\rho_i : 1 \leq i \leq l\} \) and \( S_1 = \{\sigma_i : 1 \leq i \leq l\} \). Please see [NC00] for a good introduction to information theory. By a \( \epsilon \)-separating measurement \( T \) we mean a POVM element \( T \) such that \( \forall \rho \in S_0 \) and \( \forall \sigma \in S_1 \) we have \( \text{Tr} T \rho - \text{Tr} T \sigma \geq \epsilon \) for some constant \( \epsilon \). For distributions \( \mu_0 \) on \( S_0 \) and \( \mu_1 \) on \( S_1 \), let \( \rho_{\mu_0} = E_{i \in \mu_0} [\rho_i] \) and \( \sigma_{\mu_1} = E_{i \in \mu_1} [\sigma_i] \).

We show the following:

**Theorem 0.1** The following statements are equivalent:

1. There exists a \( \epsilon \)-separating measurement \( T \).
2. For all distributions \( \mu_0 \) on \( S_0 \) and \( \mu_1 \) on \( S_1 \), \( \| \rho_{\mu_0} - \sigma_{\mu_1} \|_{\text{tr}} \geq 2\epsilon \).

For our proof we will need the following facts.

**Fact 0.1** ([NC00]) Given quantum states \( \rho, \sigma \),

\[
\max_{\text{Tr}} T: \text{a POVM element} \quad \text{Tr} T \rho - \text{Tr} T \sigma = \frac{1}{2} \| \rho - \sigma \|_{\text{tr}}
\]

We have the following minimax theorem from game theory (see [OR94]).

**Fact 0.2** Let \( A_1, A_2 \) be non-empty, either finite or convex and compact subsets of \( \mathbb{R}^n \). Let \( u : A_1 \times A_2 \to \mathbb{R} \) be a continuous function. Let \( \mu_1, \mu_2 \) be distributions on \( A_1 \) and \( A_2 \) respectively. Then,

\[
\min_{\mu_1} \max_{a_2 \in A_2} E_{\mu_1} [u(a_1, a_2)] = \max_{\mu_2} \min_{a_1 \in A_1} E_{\mu_2} [u(a_1, a_2)]
\]

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Proof:

1) $\Rightarrow$ 2): If $T$ is a separating measurement then from linearity of $\text{Tr}$ operation we see that for all distributions $\mu_0$ on $S_0$ and $\mu_1$ on $S_1$, $\text{Tr}T\rho_{\mu_0} - \text{Tr}T\sigma_{\mu_1} \geq \epsilon$. Fact 0.1 now implies $\|\rho_{\mu_0} - \sigma_{\mu_1}\|_{\text{tr}} \geq 2\epsilon$.

2) $\Rightarrow$ 1): Let us define sets $A_1 = \{ T : T \text{ a POVM element} \}$ and $A_2 = \{ (\rho, \sigma) : \rho \in S_0, \sigma \in S_1 \}$. Let the function $u : A_1 \times A_2 \to \mathbb{R}$ be defined as $u(T, (\rho, \sigma)) = \text{Tr}T\rho - \text{Tr}T\sigma$. For a distribution $\mu$ on $S_0 \times S_1$, let $\mu_0$ be the marginal distribution on $S_0$ and $\mu_1$ be the marginal distribution on $S_1$. From fact 0.2 and the fact that a convex combination of POVM elements is also a POVM element and from linearity of $\text{Tr}$ operation it follows:

\[
\max_T \min_{(\rho,\sigma)\in S_0\times S_1} \text{Tr}T\rho - \text{Tr}T\sigma = \min_\mu \max_T \text{Tr}T\rho_{\mu_0} - \text{Tr}T\sigma_{\mu_1}
\]

From above it is clear that 2) $\Rightarrow$ 1).

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References
