

Regular Languages: Properties

Pumping Lemma: Let L be a regular language. Then there exists a constant n (which depends on L) such that for every string w in L satisfying $|w| \geq n$, we can break w into three strings $w = xyz$, such that

(a) $y \neq \epsilon$,

(b) $|xy| \leq n$

(c) For all $k \geq 0$, the string xy^kz is also in L .

Examples:

Let $L = \{a^m b^m \mid m \geq 1\}$.

Then L is not regular.

Proof: Let n be as in Pumping Lemma.

Let $w = a^n b^n$.

Let $w = xyz$ be as in Pumping Lemma.

Thus, $xy^2z \in L$, however, xy^2z contains more a 's than b 's.

Let $L = \{a^i b^j \mid i < j\}$.

Then L is not regular.

Proof: Let n be as in Pumping Lemma.

Let $w = a^n b^{n+1}$.

Let $w = xyz$ be as in Pumping Lemma.

Thus, $xy^3z \in L$, however, xy^3z contains more a 's than b 's.

Let $L = \{a^p \mid p \text{ is prime}\}$.

Then L is not regular.

Proof: Let n be as in Pumping Lemma.

Let $w = a^p$, where p is prime, and $p > n$.

Let $w = xyz$ be as in Pumping Lemma.

Thus, $xy^kz \in L$, for all k .

Choose $k = 2$. Thus, $xy^kz = a^r$, where r is not a prime number.

Proof of Pumping Lemma

Suppose $A = (Q, \Sigma, \delta, q_0, F)$ is a DFA which accepts L .

Let n be number of states in Q .

Suppose $w = a_1 a_2 \dots a_n \dots a_m$ is as given, where $m \geq n$.

For $i \geq 1$, let $q_i = \hat{\delta}(q_0, a_1 \dots a_i)$.

Then, by Pigeonhole principle, there exists $i, j \leq n, i \neq j$, such that

$q_i = q_j$.

Let $x = a_1 \dots a_i, y = a_{i+1} \dots a_j, z = a_{j+1} \dots a_m$.

As $\hat{\delta}(q_i, y) = q_i$, we have: for all $k, \hat{\delta}(q_i, y^k) = q_i$.

Thus, $\hat{\delta}(q_0, xyz) = \hat{\delta}(q_0, xy^kz)$, for all k .

QED

Closure Properties

- If L_1, L_2 are regular, then so is $L_1 \cup L_2$.
- If L_1, L_2 are regular, then so is $L_1 \cdot L_2$.
- If L is regular, then so is $\bar{L} = \Sigma^* - L$.
- If L_1, L_2 are regular, then so is $L_1 \cap L_2$.
- If L_1, L_2 are regular, then so is $L_1 - L_2$.
- If L is regular, then so is L^R .
- Let h be a homomorphism. If L is regular, then so is $h(L)$.

Homomorphism: $h(a) \in B^*$, where B is an alphabet set.

$$h(\epsilon) = \epsilon.$$

$$h(a_1 a_2 \dots) = h(a_1) h(a_2) \dots$$

Decision Problems on Regular Languages

$L = \emptyset?$

$L = \Sigma^*?$

$w \in L?$