Regular Languages: Properties

Pumping Lemma: Let L be a regular language. Then there exists a constant n (which depends on L) such that for every string w in L satisfying $|w| \geq n$, we can break w into three strings $w = xyz$, such that
(a) $y \neq \epsilon$,
(b) $|xy| \leq n$
(c) For all $k \geq 0$, the string $xy^k z$ is also in L.

Examples:

Let $L = \{a^m b^m \mid m \geq 1\}$.
Then L is not regular.
Proof: Suppose by way of contradiction that L is regular.
Let n be as in the Pumping Lemma.
Let $w = a^n b^n$.
Let $w = xyz$ be as in the Pumping Lemma.
Note that y consists only of as.
Thus, $xy^2z \in L$, however, xy^2z contains more a's than b's.

Examples:

Let $L = \{a^i b^j \mid i < j\}$.
Then L is not regular.
Proof: Suppose by way of contradiction that L is regular.
Let n be as in the Pumping Lemma.
Let $w = a^n b^{n+1}$.
Let $w = xyz$ be as in the Pumping Lemma.
Note that y consists only of as.
Thus, $xy^3z \in L$, however, xy^3z contains more a's than b's.

Examples:

Let $L = \{a^p \mid p$ is prime$\}$.
Then L is not regular.
Proof: Suppose by way of contradiction that L is regular.
Let n be as in the Pumping Lemma.
Let $w = a^p$, where p is prime, and $p > n$.
Let $w = xyz$ be as in the Pumping Lemma.
Thus, $xy^kz \in L$, for all k.
Choose $k = ?$. Thus, $xy^kz = a^r$, where r is not a prime number.

Proof of the Pumping Lemma

Suppose $A = (Q, \Sigma, \delta, q_0, F)$ is a DFA which accepts L.
Let n be the number of states in Q.
Suppose $w = a_1 a_2 \ldots a_n \ldots a_m$ is as given, where $m \geq n$.
For $i \geq 1$, let $q_i = \hat{\delta}(q_0, a_1 \ldots a_i)$.
Then, by Pigeonhole principle, there exists $i, j \leq n$, $i < j$, such that $q_i = q_j$.
Let $x = a_1 \ldots a_i$, $y = a_{i+1} \ldots a_j$, $z = a_{j+1} \ldots a_m$.
As $\hat{\delta}(q_i, y) = q_i$, we have: for all k, $\hat{\delta}(q_i, y^k) = q_i$.
Thus, $\hat{\delta}(q_0, xyz) = \hat{\delta}(q_0, xy^k z)$, for all k.
QED

Closure Properties

- If L_1, L_2 are regular, then so is $L_1 \cup L_2$.
- If L_1, L_2 are regular, then so is $L_1 \cdot L_2$.
- If L is regular, then so is $\overline{L} = \Sigma^* - L$.
- If L_1, L_2 are regular, then so is $L_1 \cap L_2$.
- If L_1, L_2 are regular, then so is $L_1 - L_2$.
- If L is regular, then so is L^R.
- Let h be a homomorphism. If L is regular, then so is $h(L)$.

Homomorphism: $h(a) \in B^*$, where B is an alphabet set.
$h(\epsilon) = \epsilon$.
$h(a_1 a_2 \ldots) = h(a_1)h(a_2)\ldots$.

Decision Problems on Regular Languages

$L = \emptyset$?
$L = \Sigma^*$?
$L(A) = L(A')$?
$w \in L$?