Homomorphisms

Suppose Σ and Γ are two alphabets. Suppose h is a mapping from Σ to Γ^*. Extend h to strings as follows.
$h(\epsilon) = \epsilon$.
$h(aw) = h(a) \cdot h(w)$, for any $a \in \Sigma, w \in \Sigma^*$.
Above h is called a homomorphism.
Show: If L is regular then $h(L) = \{h(w) \mid w \in L\}$ is also regular.

Homomorphisms

By induction on the length of the regular expression M, we give $R(M)$, the regular expression for $h(L(M))$.

$R(\emptyset) = \emptyset$

$R(\epsilon) = \epsilon$

$R(a) = h(a)$, for $a \in \Sigma$

$R(M + N) = R(M) + R(N)$

$R(M \cdot N) = R(M) \cdot R(N)$

$R(M^*) = (R(M))^*$

To see that above works, note that for $M + N$:
$L(R(M + N))$
$= L(R(M) + R(N))$
$= L(R(M)) \cup L(R(N))$
$= h(L(M)) \cup h(L(N))$ (by induction).
Also, $h(L(M + N)) = h(L(M)) \cup h(L(N))$. Thus,
$L(R(M + N)) = h(L(M + N))$.
It can be similarly shown that $L(R(M \cdot N)) = h(L(M \cdot N))$
and $L(R(M^*)) = h(L(M^*))$.