A1. Let $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}$; Starting state: q_0; Accepting states: $\{q_3, q_5\}$; $\Sigma = \{a, b\}$.

$\delta(q_0, a) = q_1$, $\delta(q_0, b) = q_4$

$\delta(q_1, a) = q_2$, $\delta(q_1, b) = q_4$

$\delta(q_2, a) = q_3$, $\delta(q_2, b) = q_4$

$\delta(q_3, a) = q_3$, $\delta(q_3, b) = q_5$

$\delta(q_4, a) = q_1$, $\delta(q_4, b) = q_6$

$\delta(q_5, a) = q_3$, $\delta(q_5, b) = q_6$

$\delta(q_6, a) = \delta(q_6, b) = q_6$

Intuitively, q_6 denotes the dead state, which is reached whenever the automaton sees two consecutive b's.

Following applies if two consecutive b's are not yet seen in the input:

The automaton is in state q_3 (respectively q_5) if it has seen three consecutive a's in the past, and the latest input character seen is an a (respectively b).

If the automaton hasn't seen three consecutive a's, then being state q_4 denotes that the latest input character seen is a b. Otherwise being in state q_1 (respectively q_2) denotes that the latest input character seen is an a and just before that there was a b or start of input (respectively an a).

q_0 is the starting state when no input has been seen.

A2. False.

Let $B = \{\epsilon\}$. Let $A = \{a\} \cup \{a^p : p \text{ is a prime }\}$.

Note that $A^* = a^*$ is regular, and B is regular. However, $A \cdot B = A$ is not regular. To see that A is not regular, suppose otherwise. Then $A \cap \{a^m : m \geq 2\} = \{a^p : p \text{ is a prime}\}$ must be regular. A contradiction to result done in class.

A3. Let $B = \{\{q'_0, q'_1, q'_2\}, \Sigma, \delta', q'_0, \{q'_2\}\}$, where $\delta'(q'_i, a) = \delta'(q'_i, b) = q'_{i+1 \mod 3}$.

Let $C = \{\{q''_0, q''_1, q''_2, q''_3, q''_4\}, \Sigma, \delta'', q''_0, \{q''_3\}\}$, where $\delta''(q''_i, a) = \delta''(q''_i, b) = q''_{i+1 \mod 5}$.

Note that B is a DFA for accepting $\{w : |w| \mod 3 = 2\}$ and C is a DFA for accepting $\{w : |w| \mod 5 = 3\}$.

Let D be a DFA for accepting $Lang(B) \cup Lang(C)$.

Note that D is a DFA for the language containing all strings which have length $3i + 2$ or length $5i + 3$ for some natural number i.

Algorithm:

1. Input DFA A.
2. Construct a DFA E for $Lang(A) \cap Lang(D)$, using the method done in class.
3. Check whether $Lang(E) = \emptyset$ using the method done in class.
   If so, then output NO. Else output YES.

End

Note that E above gives the strings in $Lang(A)$ which have length $3i+2$ or $5i+3$, for some natural number i. Thus, we just need to check if $Lang(E)$ is emptyset or not to get our answer.

A4: True.

Suppose $(Q, \Sigma, \delta, q_0, F)$ is a DFA for A.

Then, construct an ϵ-NFA $B = (Q, \Sigma, \delta', q_0, F)$, where

$\delta'(q, a) = \delta(q, a)$, for $a \in \Sigma$, $q \in Q$.

$\delta'(q, \epsilon) = \bigcup_{a \in \Sigma}\{\delta(q, a)\}$, for $q \in Q$.

We claim that B acceptes $\{x : (\exists y \in A)[x$ is a subsequence of $y]\}$.

(A) Suppose $x \in Lang(B)$.

Consider the accepting path/transitions taken by B on input x: $q'_0 b_1 q'_1 b_2 q'_2 b_3 \ldots b_m q'_m$, where $b_i \in \Sigma \cup \{\epsilon\}$ and $q'_m \in F$, $q'_0 = q_0$ and $x = b_1 \cdot b_2 \cdots b_m$.

Then, consider the string $y = b'_1 b'_2 \ldots b'_m$, where

$b'_i = b_i$, if $b_i \in \Sigma$, and

$b'_i = c \in \Sigma$ such that $\delta(q'_{i-1}, c) = q'_i$, if $b_i = \epsilon$.

Then, clearly, $y \in Lang(A)$, and x is a subsequence of y.

(B) Suppose $x = a_1 a_2 \ldots a_n$, is a subsequence of $y = b_1 b_2 \ldots b_m$, where $a_i, b_j \in \Sigma$, $y \in Lang(A)$, and $j_1 < j_2 < \ldots < j_n$ are such that $a_i = b_{j_i}$, for $1 \leq i \leq n$. Suppose the accepting path for y was $q'_0 b_1 q'_1 b_2 q'_2 b_3 \ldots b_m q'_m$, where $q'_m \in F$, $q'_0 = q_0$.

Then, consider the transitions in B where we take the path $q'_0 b'_1 q'_1 b'_2 q'_2 b'_3 \ldots b'_m q'_m$, where $b'_k = b_k$, if $k = j_i$ for some i; otherwise it is ϵ transition corresponding to b_k. It is easy to verify that B would accept in this path.

It follows that B is ϵ-NFA for $\{x : (\exists y \in A)[x$ is a subsequence of $y]\}$.

A5: Suppose by way of contradiction that L_5 is regular. Then, Let n be as in the pumping lemma for L_5.

Note that:

(a) $i^2 - 5i - 33 \geq 0$ and is an increasing function for $i \geq 10$.

(b) For all $j < 100$ and $i \geq 100$, $j^2 - 5j - 33 < i^2 - 5i - 33$.

(c) $(i+1)^2 - 5(i+1) - 33 - [i^2 - 5i - 33] = 2i + 1 - 5 \geq n + 1$, for $i \geq n + 100$.

Let $m = (n + 100)^2 - 5(n + 100) - 33$.

Let $w = a^m$. Clearly, $w \in L$.

Let $w = xyz$ as in the pumping lemma.

Note that $y \in a^+$, and $|y| \leq n$.

Thus, $m = (n + 100)^2 - 5(n + 100) - 33 < |xy^2z| = m + |y| \leq m + n < m + n + 1 \leq (n + 101)^2 - 5(n + 101) - 33$. (Using property (c) mentioned above).

Thus, xy^2z is not in L, as $|xy^2z|$ is not of the form $i^2 - 5i - 33$ for any natural number i. (Using properties, (a), (b) mentioned above).

A contradiction.

Thus L is not regular.