Push Down Automata

$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$.

- Q: Finite set of states; q_0 is the start state;
  F is the set of final/accepting states.
- Σ: Alphabet set; Γ: Stack alphabet
- Z_0 is the (only) initial symbol on the stack.
- δ: transition function.

δ takes as input a state q, an input letter a (or ϵ), and a stack symbol (top of stack) X. $\delta(q, a, X)$ is then a finite subset of $Q \times \Gamma^*$.

$(p, \gamma) \in \delta(q, a, X)$ denotes that when in state q, reading symbol a (or ϵ), with top of stack being X, the machine's new state is p, X at the top of stack is popped and γ is pushed to the stack. (By convention, if $\gamma = RS$, then S is pushed first, and then R is pushed on the stack).

Instantaneous Descriptions

(q, w, α): denotes that current state is q, input left to read is w, and α is on the stack (first symbol of α is top of stack).

$(q, aw, X\alpha) \vdash (p, w, \beta\alpha)$, if $(p, \beta) \in \delta(q, a, X)$ (here a can be ϵ).

One can similarly define $\vdash_P^*$ (or simply $\vdash^*$, where P is understood).

1. $I \vdash^* I$
2. $I \vdash^* J$ and $J \vdash K$, then $I \vdash^* K$

Language accepted by PDA

Acceptance by final state.
$\{w \mid (q_0, w, Z_0) \vdash_P^* (q_f, \epsilon, \alpha)$, for some $q_f \in F\}$.

Acceptance by empty stack.
$\{w \mid (q_0, w, Z_0) \vdash_P^* (q, \epsilon, \epsilon)$, for some $q \in Q\}$.

From Acceptance using empty stack to Acceptance using Final State

Intuition: Initially put a special symbol onto the stack.
If ever see the top of stack as that symbol, then go to final state.

$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$.
$P_F = (Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p_f\})$.
1. $\delta_F(p_0, \epsilon, X_0) = \{(q_0, Z_0 X_0)\}$.
2. For all $Z \in \Gamma$, $a \in \Sigma \cup \{\epsilon\}$: $\delta_F(p, a, Z)$ contains all (q, γ) which are in $\delta(p, a, Z)$.
3. $\delta_F(p, \epsilon, X_0)$ contains (p_f, ϵ), for all $p \in Q$.

From Acceptance using final state to Acceptance using empty Stack

Place a transition from final state to a special state which empties the stack.

$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$.

$P_E = (Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_E, p_0, X_0, \{p_f\})$.

1. $\delta_E(p_0, \epsilon, X_0) = \{(q_0, Z_0 X_0)\}$.
2. $\delta_E(p, a, Z)$ contains all (q, γ) which are in $\delta(p, a, Z)$, for all $Z \in \Gamma$ and $a \in \Sigma \cup \{\epsilon\}$.
3. $\delta_E(p, \epsilon, Z)$ contains (p_f, ϵ), for all $p \in F$, and $Z \in \Gamma \cup \{X_0\}$.
4. $\delta_E(p_f, \epsilon, Z)$ contains (p_f, ϵ), for all $Z \in \Gamma \cup \{X_0\}$.

Equivalence of CFGs and PDAs

First we show how to accept a CFG.
We use the accepting by empty stack model.
Intuitively, do left-most derivation. Use stack to keep track of "what is left to derive". Each time there is a non-terminal on the top of stack, guess a production to be used and push it on the stack. Terminal symbols can be matched as it is.

Details:
$G = (V, T, P, S)$.
Then, construct $PDA = (\{q_0\}, \Sigma, \Gamma, \delta, q_0, S, F)$,
where, $\Sigma = T$,
$\Gamma = V \cup T$.
For all $a \in \Sigma$, $\delta(q_0, a, a) = \{(q_0, \epsilon)\}$
For all $A \in V$, $\delta(q_0, \epsilon, A) = \{(q_0, \gamma) : A \to \gamma \text{ in } P\}$.

Now we show that each language accepted by a PDA (using empty stack) can be generated by a CFG.
Suppose PDA is $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$.
We define grammar $G = (V, \Sigma, R, S)$ as follows.
$V = \{S\} \cup \{[qZp] : q, p \in Q, Z \in \Gamma\}$.
$S \to [q_0 Z_0 p]$, for each $p \in Q$.
If $\delta(q, a, X)$ contains $(r, Y_1 \ldots Y_k)$, then we have productions of the form:
$[qXr_k] \to a[rY_1r_1][r_1Y_2r_2] \ldots [r_{k-1}Y_kr_k]$,
for all $r_1, r_2, \ldots, r_k \in Q$.
Here, if $k = 0$, then $[qXr] \to a$ (think of $r = r_0$).
Intuitively, $[qXr]$ generates w iff $(q, w, X) \vdash^* (r, \epsilon, \epsilon)$.
By induction on number of steps of PDA/derivation in the CFG.

Deterministic PDA

1. For all $a \in \Sigma \cup \{\epsilon\}, Z \in \Gamma$ and $q \in Q$, there is at most one element in $\delta(q, a, Z)$.
2. If $\delta(q, \epsilon, X)$ is non-empty, then $\delta(q, a, X)$ is empty for all $a \in \Sigma$.

Theorem: There exists a language which is accepted by PDA (NPDA) but not by any DPDA.

Deterministic PDA

Theorem: If we consider acceptance by final state, then every regular language can be accepted by a DPDA.

Theorem: If we consider acceptance by empty stack, then $\{a, aa\}$ is not accepted by a DPDA.
Any language accepted by a DPDA accepting by empty stack has prefix property: For every x, y in the language, x is not a prefix of y.