CS3231
Tutorial 1

1. For $n = 0$, the result clearly holds.

   Suppose it holds for $n = k$.

   Then, $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(2k^2+k+6k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$.

   Thus, the statement holds for $n = k+1$. Hence, by induction, the statement holds for all values of n.

2. For first part $|\Sigma|^n$.

   For second part, If $|\Sigma| = 1$, then $n+1$;

   If $|\Sigma| \neq 1$, then $\frac{|\Sigma|^{n+1}-1}{|\Sigma|-1}$.

3. (a) We first show that: $A \cdot (\bigcup_{i=1}^{\infty} B_i) \subseteq \bigcup_{i=1}^{\infty} A \cdot B_i$.

   Suppose $w \in A \cdot (\bigcup_{i=1}^{\infty} B_i)$. Then, there exist w_1, w_2 such that $w = w_1 w_2$, $w_1 \in A$ and $w_2 \in \bigcup_{i=1}^{\infty} B_i$. Let j be such that $w_2 \in B_j$. Then, $w_1 w_2 \in A \cdot B_j$, and thus, $w \in \bigcup_{i=1}^{\infty} A \cdot B_i$.

   We now show that $\bigcup_{i=1}^{\infty} A \cdot B_i \subseteq A \cdot (\bigcup_{i=1}^{\infty} B_i)$.

   Suppose $w \in (\bigcup_{i=1}^{\infty} A \cdot B_i)$. Let j be such that $w \in A \cdot B_j$. Then, there exist w_1, w_2 such that $w = w_1 w_2$ and $w_1 \in A$ and $w_2 \in B_j$. Thus, $w_2 \in \bigcup_{i=1}^{\infty} B_i$. Thus, $w \in A \cdot (\bigcup_{i=1}^{\infty} B_i)$.

   It follows that $A \cdot (\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty} A \cdot B_i$.

   (b) Note that $\epsilon \in A^*$. Thus, $(A^*)^+ = (A^*)^*$.

   Also, if $B \subseteq C$, then $B^* \subseteq C^*$. Thus, we immediately have $A^* \subseteq (A^+)^* \subseteq (A^*)^*$.

   Hence, it suffices to show that $(A^*)^* \subseteq A^*$.

   Suppose $w \in (A^*)^*$

   Let $w_1, w_2, \ldots, w_k$ be such that $w = w_1 w_2 \ldots w_k$ and each $w_i \in A^*$.

   For each i, let $w_{i,1}, w_{i,2} \ldots w_{i,r_i}$ be such that $w_i = w_{i,1} w_{i,2} \ldots w_{i,r_i}$ and each $w_{i,j} \in A$.

   Thus, we have that $w = w_{1,1} w_{1,2} \ldots w_{1,r_1} w_{2,1} \ldots w_{2,r_2} \ldots w_{k,1} \ldots w_{k,r_k}$, where each $w_{i,j} \in A$. Thus, $w \in A^*$.

4. (a) Consider DFA $(Q, \{a, b\}, \delta, q_0, \{q_1\})$, where

   $Q = \{q_0, q_1, q_2\}$, and

   $\delta(q_0, b) = q_0$, $\delta(q_1, b) = q_1$, $\delta(q_2, b) = q_2$, and $\delta(q_0, a) = q_1$, $\delta(q_1, a) = q_2$, $\delta(q_2, a) = q_0$,

(b) Consider DFA $(Q, \{a, b\}, \delta, q_0, \{q_5\})$, where

$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$

$\delta(q_0, a) = q_1$

$\delta(q_1, b) = q_2$

$\delta(q_2, a) = q_3$

$\delta(q_3, a) = q_4$

$\delta(q_4, b) = q_5$

$\delta(q_5, a) = \delta(q_5, b) = q_5$

$\delta(q_0, b) = q_0$

$\delta(q_1, a) = q_1$

$\delta(q_2, b) = q_0$

$\delta(q_3, b) = q_2$

$\delta(q_4, a) = q_1$

(c) $A = (\{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}, \{a, b\}, \delta, q_0, \{q_0, q_1, q_2, q_3, q_4, q_5\})$.

$\delta(q_0, a) = q_1$

$\delta(q_1, b) = q_2$

$\delta(q_2, a) = q_3$

$\delta(q_3, b) = q_4$

$\delta(q_4, b) = q_5$

$\delta(q_5, a) = q_6$

$\delta(q_6, a) = \delta(q_6, b) = q_6$.

$\delta(q_0, b) = q_0$

$\delta(q_1, a) = q_1$

$\delta(q_2, b) = q_0$

$\delta(q_3, a) = q_1$

$\delta(q_4, a) = q_3$

$\delta(q_5, b) = q_0$

5. (Odd number of a's and no b's) or (any number of a's followed by a b followed by even number of a's).

6. The DFA accepts any string which satisfies the following requirements:

   – it ends in a b, and

   – it has at least one a, and

   – after the first a, there are odd number of b's and all these b's after the first a (except the last b), appear in pairs.

(That is the string is of the form $b...baa^{i_1}bba^{i_2}bba^{i_3}bb\ldots a^{i_k}b$, where $k \geq 1$, and $i_1, i_2, \ldots, i_k \geq 0$.)

In terms of regular expression (to be taught later), this is same as: $b^*a(a^*bba^*)^*b$.