CS3231
Tutorial 3

1. Distinguishable state pairs are:

   Base case: $(q0, q4), (q0, q5), (q1, q4), (q1, q5), (q2, q4), (q2, q5), (q3, q4), (q3, q5)$, as in each of these pairs, one is accepting and one rejecting state.

   1st Induction step: $(q0, q3)$ (as $\delta(q0, a) = q1$ and $\delta(q3, a) = q4$ are distinguishable);

   $(q1, q3)$ (as $\delta(q1, a) = q2$ and $\delta(q3, a) = q4$ are distinguishable);

   $(q2, q3)$ (as $\delta(q2, a) = q2$ and $\delta(q3, a) = q4$ are distinguishable).

   There will not be any more iteration, as all letters of alphabet cannot distinguish any more: ($\delta(q4, a) = q0, \delta(q5, a) = q2$ and $\delta(q4, b) = q5, \delta(q5, b) = q4$, $\delta(q0, a) = q1, \delta(q1, a) = q2$ and $\delta(q2, a) = q2$, $\delta(q0, b) = q3, \delta(q1, b) = q3$ and $\delta(q2, b) = q3$, which doesn't allow us to distinguish $(q4, q5)$ or $(q0, q1), (q1, q2), (q0, q2)$.

   Thus, we will have the minimal DFA as:

   $(\{q012, q3, q45\}, \{a, b\}, \delta, q012, \{q45\})$

   With $\delta(q012, a) = q012$, $\delta(q012, b) = q3$, $\delta(q3, a) = q45$, $\delta(q3, b) = q012$, $\delta(q45, a) = q012$, $\delta(q45, b) = q45$.

2. (a) True.
   As A and $\overline{B}$ are regular, let R_A and $R_{\overline{B}}$ be regular expressions for A and $\overline{B}$. Then, $R_A \cdot R_{\overline{B}}$ is a regular expression for $A \cdot \overline{B}$. Thus, $A \cdot \overline{B}$ is regular.

   (b) True.
   Given a regular expression S, we show how to construct S^R, such that $L(S)^R = L(S^R)$.
   Base cases: $a^R = a$, $\epsilon^R = \epsilon$, $\emptyset^R = \emptyset$.
   Induction: $(A + B)^R = A^R + B^R$.
   $(A \cdot B)^R = B^R \cdot A^R$.
   $(A^*)^R = (A^R)^*$.
   Proof that above works is left to the student.
   Second method of proving:
   Suppose $A = (Q, \Sigma, \delta, q_0, F)$ is a DFA for L. Construct an ϵ-NFA for L^R as follows.
   $A_N = (Q \cup \{q_0'\}, \Sigma, \delta_N, q_0', \{q_0\})$, where
   $\delta_N(q_0', \epsilon) = F$, and
   for $q \in Q$, $a \in \Sigma$, $\delta_N(q, a) = \{q' : \delta(q', a) = q\}$.
   Other transitions sets are $\emptyset$.

It can be shown by induction on length of w that, for $q, q' \in Q$, $\hat{\delta}(q, w) = q'$ if and only if $q \in \hat{\delta_N}(q', w^R)$.

Thus, $\hat{\delta}(q_0, w) \in F$ if and only if $q_0 \in \bigcup_{q \in F} \hat{\delta_N}(q, w^R) = \hat{\delta_N}(q'_0, w^R)$.

Thus, A_N accepts L^R.

(c) False. Take $L_1 = \Sigma^*$, and L_2 to be some non-regular subset of Σ^*.

(d) False. Take $L_1 = \Sigma^*$ and L_2 to be some non-regular subset of Σ^*.

(e) True. First note that if A and B are regular then $A \cup B$ is also regular. To see this, note that if R_A is regular expression for A and R_B is regular expression for B, then $R_A + R_B$ is regular expression for $A \cup B$.

As L_1 and L_2 are regular, so are $\overline{L_1}$ and $\overline{L_2}$.

Thus, $\overline{L_1} \cup \overline{L_2}$ is regular.

Thus, $\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$ is regular.

(f) True.

Suppose $(Q, \Sigma, \delta, q_0, F)$ is a DFA accepting L.

Define DFA $A = (Q', \Sigma, \delta', q'_0, F')$ as follows.

$Q' = \{(q, s) : q \in Q, s \in \{0, 1\}\}$.

$q'_0 = (q_0, 0)$

$F' = \{(q, 0) : q \in F\}$

For $q \in Q, a \in \Sigma$,

$\delta'((q, 0), a) = (\delta(q, a), 1)$.

$\delta'((q, 1), a) = (q, 0)$.

It is easy to verify that $\hat{\delta}'((q, 0), ab) = (p, 0)$ if and only if $\delta(q, a) = p$, for all $q, p \in Q$ and $a, b \in \Sigma$.

It follows that A accepts $\{x :$ for some natural number r, $|x| = 2r$ and $x_1 x_3 x_5 \ldots x_{2r-1} \in L\}$.

3. (a) $\{wcw^R \mid w \in \{a, b\}^*\}$.

Ans: Not regular. Suppose by way of contradiction that L is regular. Let n be as in the pumping lemma.

Let $w' = a^n c a^n \in L$.

Let $w' = xyz$ as in the pumping lemma.

Then, $xy \in a^+$ and $z \in a^* c a^n$.

Now, $xy^2 z = a^{n+|y|} c a^n$, which is not in L.

A contradiction. Thus, L is not regular.

(b) $\{ww \mid w \in \{a, b\}^*\}$.

Ans: Not regular. Suppose by way of contradiction that L is regular. Let n be as in the pumping lemma.

Let $w' = a^n b a^n b \in L$.

Let $w' = xyz$ as in the pumping lemma.

Then, $xy \in a^+$ and $z \in a^* b a^n b$.

Now, $xy^2z = a^{n+|y|}ba^nb$, which is not in L (as either its length is odd, or both b's are in the second half of the string).

A contradiction. Thus, L is not regular.

(c) $\{wxw^R \mid w, x \in \{a, b\}^+\}$.

Ans: It is regular.

The regular expression for it is $a(a + b)(a + b)^*a + b(a + b)(a + b)^*b$.

(d) $L = \{a^m : m > 0$ and binary representation of m has even number of bits $\}$.

Ans: Suppose by way of contradiction that L is regular. Let n be as in the pumping lemma, where we can assume without loss of generality that $n > 5$. Pick m such that $n < 2^{2m} - 1$ and $2^{2m} - 1 + n < 2^{2m+1}$.

Let $w = a^{2^{2m}-1}$. Now $2^{2m} - 1$ is a $2m$ bit binary number, thus $w \in L$.

Suppose $w = xyz$, where x, y, z are as in the pumping lemma. Let $y = a^r$, where $1 \leq r \leq n$.

Now, xy^2z should be in L. However, $xy^2z = a^{2^{2m}-1+r}$, is of length l, where $2^{2m} \leq l < 2^{2m+1}$. Thus l has odd number of bits in binary representation. A contradiction.

4. Let $n = 5$. Now consider any string $w \in L$ such that $|w| \geq 5$.

   If $w \in b^*$, then clearly, one can take $x = \epsilon$, $y = b$, and $z = b^{|w|-1}$. Then, $xy^kz \in b^* \subseteq L$ for all k.

   If $w = a^{r+1}b^p$, for some prime p. Then let $x = \epsilon, y = a$ and $z = a^rb^p$. Now, $xy^kz = a^{r+k}b^p$, and thus in L (here, note that if $r = k = 0$, then $a^{r+k}b^p$ is in $b^* \subseteq L$).