Reduction in number of tapes

Space: We can use only 1 work tape, without change in space used. (Note that for space complexity we have assumed that input tape is read only).

We now consider Time.

Theorem: If L is accepted by a $T(n)$ time bounded k-tape machine then L is accepted by a $O(T(n)^2)$ time bounded 1 tape machine.

Theorem: If L is accepted by a $T(n)$ time bounded k-tape machine then L is accepted by $T(n)logT(n)$ time bounded 2 tape machine.
Problem was that heads of the multitape TM are all over the place.

Instead think of tapes moving left/right rather than the heads moving.

That makes the heads stationary, we just need to figure out how to move the tape.

Show how to simulate one tape using 2 tapes.
This may look odd, as it is trivial.
However, the technique we give will be easy to generalize to simulate k tapes using 2 tapes.

Main tape (1st tape) of the simulating machine is used to mimic the contents of the original TM tape.
Two tracks in the Main tape.
The scratch tape (2nd tape) is used in simulating one step of the original TM.

$$\ldots \mid B_{-2} \mid B_{-1} \mid B_0 \mid B_1 \mid B_2 \mid \ldots$$

Size of block B_i, B_{-i} (for $i \geq 1$) is 2^{i-1} (and thus contains $2^{i-1} * 2$ symbols in the two tracks).

TM uses an empty symbol (different from blank)

For each $i > 0$, one of the following holds:

A1: lower tracks of B_i, B_{-i} are full, and upper track are empty,

A2: both tracks of B_i are full, but both tracks of B_{-i} are empty,

A3: both tracks of B_{-i} are full, but both tracks of B_i are empty.

Note that the invariants imply that B_i and B_{-i} together have exactly 2^i empty and 2^i non-empty symbols.
Initially, upper tracks empty, lower tracks full.

At any time, in the original machine tape, suppose the head is at a_0, with the symbols to the right being $a_1, a_2, \ldots$ and to the left being $a_{-1}, a_{-2}, \ldots$.

Then in the simulating machine:
a_0 is at the block B_0 (in the top track), special symbol $*$ in the lower track of B_0.
If you read to the right, ignoring empty symbols, top track first and bottom track next in each cell, then one gets $a_1, a_2, \ldots$.
If you read to the left, ignoring empty symbols, top track first and bottom track next in each cell, then one gets $a_{-1}, a_{-2}, \ldots$.

Left Move:
Go to the right, to find the first B_i which has either upper or both tracks empty.
So $B_1, B_2, \ldots, B_{i-1}$ have both tracks full (correspondingly $B_1, B_2, \ldots, B_{i-1}$ have both tracks empty).
The total number of elements in upper track of these is:
$2^0 + 2^1 + \ldots + 2^{i-2} = 2^{i-1} - 1$.
When we add one more symbol to the right, we have exactly to find space for $2^{i-1} = 2^i/2$ symbols.
This is exactly the space in one of the tracks in block B_i.
(The situation on the left side is reverse of above: we need to remove one symbol there from B_{-i} and spread the remaining to $B_{-1}, \ldots B_{-i+1}$.)

We call the above a B_i-step.
We can use the scratch tape (second tape) to implement the above step, by copying to the scratch tape and then copying back.

B_i step takes $O(2^i)$ steps.

B_i step is taken at most once every 2^{i-1} steps (including from start).

Thus, total time complexity is:
$\Sigma_{i=1}^{1+\log T(n)} c * 2^i T(n)/2^{i-1} = O(T(n) \log T(n))$