Theorem: If L is accepted by a $T(n)$ time bounded k-tape machine then L is accepted by $T(n)logT(n)$ time bounded 2 tape machine.

Proof sketch: We will simulate a k-tape machine M accepting L using a two tape machine M'.

The first tape of M' will use two tracks to simulate each tape of M. The second tape of M' is used as a scratch tape, useful for copying parts of the first tape in the simulation below.

In the proof we show how to simulate one move of M. Let us call this simulation as a basic move of M'. We will give the proof for claimed time bound after the simulation.

Think of the first tape of M' as being divided into blocks, $\ldots, B_{-2}, B_{-1}$, B_0, $B_1, B_2, \ldots$, (see figure below). B_0 consists of one cell. B_i and B_{-i} consist of 2^{i-1} cells. In the presentation below we will assume that the boundaries of the different blocks are marked, though the markers will actually be placed only when the blocks are first used.

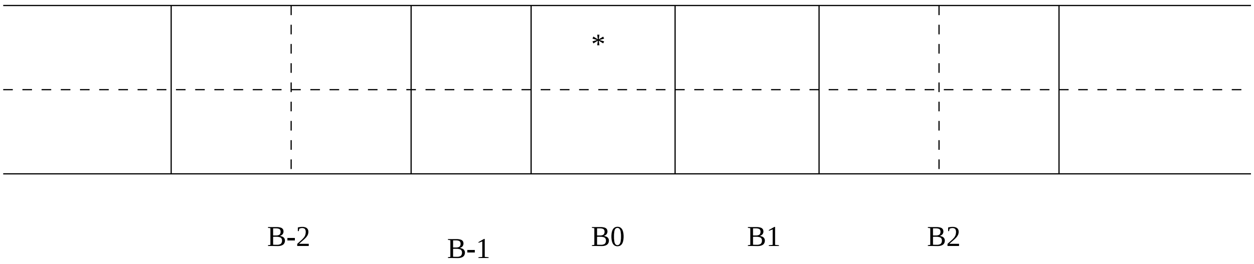

Figure 1: Tape Blocks

Let us concentrate on the simulation of one tape of M by the corresponding 2 tracks on the first tape of M'. The simulation is similar for all the other tapes. M' uses a special symbol called *empty* (this is different from blank).

Suppose at the start of any step the contents of cell being read by M on the tape being simulated is a_0, and the contents of cells on the right are $a_1, a_2, \ldots$, and the contents of the cells on the left are $a_{-1}, a_{-2}, \ldots$ (where some of a_i's may be blank).

At the beginning of any basic step we will have the following invariant.

(A) For any $i > 0$, either

(A1) lower tracks of B_i and B_{-i} are full (i.e. none of the symbols are empty), and upper tracks of B_i and B_{-i} are empty (i.e. all the symbols are empty).

(A2) Both tracks of B_i are full, whereas both tracks of B_{-i} are empty.

(A3) Both tracks of B_{-i} are full, whereas both tracks of B_i are empty.

Note that this invariant implies that for each $i > 0$, B_i and B_{-i} together have exactly 2^i empty symbols and 2^i non-empty symbols.

(B) The lower track of B_0 contains the content of the cell being scanned by M, that is a_0. The head of M' is at B_0.

(C) Suppose we read the contents of the cells to the right of B_0, in the order of their distance from B_0, upper track first and lower track next, then we get $a_1, a_2, \ldots$.

(D) Similarly if we read the contents of the cells to the left of B_0 we get $a_{-1}, a_{-2}, \ldots$.

Initially the upper track of first tape of M' is all empty and the lower track contains $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2,$ where a_0 is in block B_0. Note that the invariant is satisfied in the beginning.

We now show how to do the basic step of M' simulating a step of M and maintain the invariant.

Clearly, M' can determine the symbol being scanned by M (for all the tapes) since they are at B_0, and thus determine the symbol to be written on the cell being scanned and whether the head of M on the corresponding tape moves left or right. Thus, the symbol to be written on the scanned tape can be witten at B_0. We now consider how to simulate the left move of M. The right move can be simulated similarly.

(1) M' first moves to the right until it finds the first block such that at least one of its upper/lower tracks is empty. Let this block be B_i. Note that this implies that the cells, $B_1, B_2, \ldots, B_{i-1}$ have both tracks full. Let the case of B_i having both tracks empty be called Case 1, and B_i having only the upper track empty be called Case 2.

(2) Now we want to rearrange the contents of blocks $B_{-i}, \ldots, B_0, \ldots, B_i$ to maintain the invariant.

Case 1: In this case B_{-i} and $B_1, \ldots, B_{i-1}$ have both tracks full, whereas, B_i, and $B_{-1}, \ldots, B_{-(i-1)}$ have both tracks empty. Thus the symbols in these blocks are $a_{-2*2^{i-1}}, \ldots, a_0, \ldots, a_{2*(2^{i-1}-1)}$.

After the rearrangement, we want: $a_{-2*2^{i-1}}, \ldots, a_{-2}$ to the left of B_0, a_{-1} at B_0 and $a_0, \ldots, a_{2*(2^{i-1}-1)}$ to the right of B_0.

We can do this by first copying these symbols to tape 2 (in sequential order), then copying back to tape 1 by filling lower track of $B_{-i}, \ldots, B_0, \ldots, B_i$. Note that this leaves the tape contents as required above.

Case 2: In this case B_{-i} and B_i have their lower tracks full, upper track empty, $B_1, \ldots, B_{i-1}$ have both tracks full, $B_{-1}, \ldots, B_{-(i-1)}$ have both tracks empty. Thus the symbols in these blocks are $a_{-2^{i-1}}, \ldots, a_0, \ldots, a_{2*(2^{i-1}-1)+2^{i-1}}$.

After the rearrangement we want: $a_{-2^{i-1}}, \ldots, a_{-2}$ to the left of B_0, a_{-1} at B_0 and $a_0, \ldots, a_{2*(2^{i-1}-1)+2^{i-1}}$ to the right of B_0.

We can do this by first copying these symbols to tape 2 (in sequential order), then copying back to tape 1 by filling lower track of $B_{-(i-1)}, \ldots, B_0, \ldots, B_{i-1}$, and then filling both tracks of B_i. Note that this leaves the tape contents as required above.

(3) The head of M' then returns to the block B_0

This completes the basic move of M'. Note that for simulating k tapes we can do the steps 1, 2, and 3 above for each of the tape in a serial fashion (since after the simulation for each tape the head is back at B_0!).

We now consider the time required to simulate each tape. When B_i is selected as in step 1 above, we call the basic step (with respect to the corresponding tape) a B_i-basic step.

Time required to do a B_i-basic step (steps 1 to 3) is proportional to 2^i (where the constant of proportionality is independent of i). Note that a B_i basic step needs to be done at most once every 2^{i-1} steps of M. This is so since, for a B_i-basic step to be done, $B_1, \ldots, B_{i-1}$ must have to be full on both tracks; after B_i-basic step, $B_1, \ldots, B_{i-1}$ are empty in the upper track. It will take at least $2^{i-1} - 1$ steps for them to become full again.) Also the first B_i-basic step cannot be done until atleast 2^{i-1} steps of M are completed since we initially started with all blocks having empty upper track.

Thus the time for simulation is bounded by:

$$k * \sum_{i=1}^{1+\log T(n)} c * 2^i \frac{T(n)}{2^{i-1}}$$

for some constant c. This is bounded by $c'T(n)\log T(n)$, for some constant c'.

This proves the time bound for the simulation. $\blacksquare$