Answer sketches

A1(a).

True.

$NTIME(100+20n^3) \subseteq \bigcup_{c>0} DTIME(c^{100+20n^3})$ by a result done in class.

$\bigcup_{c>0} DTIME(c^{100+20n^3}) \subseteq DTIME(2^{n^{3.1}})$ as for any c, for large enough n, $2^{n^{3.1}} \geq c^{100+20n^3}$.

$DTIME(2^{n^{3.1}}) \subset DTIME(2^{n^4})$ by time hierarchy theorem as

$\lim_{n\to\infty} \frac{n^{3.1}2^{n^{3.1}}}{2^{n^4}} = 0$

$DTIME(2^{n^4}) \subseteq NTIME(2^{n^4})$ by definition.

Thus, the claim follows.

A1(b).

True.

By Gap theorem, there exists an increasing $T'(n) \geq n$ such that

$DTIME(T'(n)) = DTIME((T'(n)^2)^{(T'(n)^2)})$.

Let $T(n) = T'(n)^2$.

Now, $DTIME(T'(n)) \subseteq DTIME(T(n)) \subseteq NTIME(T(n)) \subseteq \bigcup_{c>0} DTIME(c^{T(n)}) \subseteq DTIME(n^{T(n)}) \subseteq$ $DTIME(T(n)^{T(n)}) = DTIME(T'(n))$.

Thus, $NTIME(T(n)) = DTIME(T(n)^{T(n)}$.

Also, $T(n) = T'(n)^2 \geq n^2$.

A2. Clearly, $NSPACE(n^a) \subseteq NSPACE(n^b)$.

Suppose by way of contradiction that $NSPACE(n^b) \subseteq NSPACE(n^a)$.

Let $s = b/a$.

By induction, we claim $NSPACE(n^{s^k b}) \subseteq NSPACE(n^a)$, for all k.

For $k = 0$, this is given by hypothesis.

Induction step:

Suppose by induction, $NSPACE(n^{s^k b}) \subseteq NSPACE(n^a)$.

By using $f(n) = n^s$ in translation lemma, we get

$NSPACE(n^{s^{k+1}b}) \subseteq NSPACE(n^{sa}) = NSPACE(n^b) \subseteq NSPACE(n^a)$.

Now, let k be such that $s^k > 2$. Then, from the claim we get,

$NSPACE(n^{2b}) \subseteq NSPACE(n^{s^k b}) \subseteq NSPACE(n^a)$.

But then, $DSPACE(n^{2b}) \subseteq NSPACE(n^{2b}) \subseteq NSPACE(n^a) \subseteq DSPACE(n^{2a})$, where the last inequality is by Savitch's theorem.

However, by space hierarchy theorem, $DSPACE(n^{2b})$ is a proper superset of $DSPACE(n^{2a})$, as $\lim_{n\to\infty} \frac{n^{2a}}{n^{2b}} = 0$.

A contradiction.

A3. The given problem is in NP, as one can guess a truth assignement to the variables, and verify if the assignment makes exactly one literal is true in every clause.

For NP-hardness, consider a reduction from 3-SAT.

Suppose (V, C) is an instance of 3-SAT. We reduce it to (V', C') an instance of Q3 as follows. Let $V' = V \cup \{W_{i,c} \mid i \leq 5 \text{ and } c \in C\}$, where $W_{i,c}$ are new variables. For each $c = (x\ y\ z) \in C$, C' contains the clauses

$(x\ W_{0,c}\ W_{1,c})$, $(y\ W_{0,c}\ W_{2,c})$, $(z\ W_{3,c})$, $(W_{0,c}\ W_{3,c}\ W_{4,c})$, and $(W_{1,c}\ W_{2,c}\ W_{5,c})$.

We claim that there exists a satisfying truth assignment for (V, C) iff there exists a truth assignment to variables in V' such that each clause in C' has one and only one true literal.

Suppose TA is a satisfying truth assignment for (V, C). Truth values assigned to variables in V' which appear in V are as in TA. For $c = (x\ y\ z)$, $W_{i,c}$ are assigned truth values as follows: $W_{3,c}$ is true iff z is false; $W_{0,c}$ is true iff both x and y are false; $W_{1,c}$ is true iff x is false and y is true; $W_{2,c}$ is true iff y is false and x is true. $W_{4,c}$ is true iff both $W_{0,c}$ and $W_{3,c}$ are false. $W_{5,c}$ is true iff both $W_{1,c}$ and $W_{2,c}$ are false.

It is easy to verify that each clause in C' has exactly one true literal.

Suppose TA is a truth assignment for variables in V' such that each clause in C' has exactly one true literal. Then, we claim that TA restricted to variables in V is a satisfying truth assignment for (V, C). To see this suppose by way of contradiction that, for some clause $c = (x\ y\ z)$ in C, none of the literals are true. Then consider the clauses $(x\ W_{0,c}\ W_{1,c})$, $(y\ W_{0,c}\ W_{2,c})$, $(z\ W_{3,c})$, $(W_{0,c}\ W_{3,c}\ W_{4,c})$, and $(W_{1,c}\ W_{2,c}\ W_{5,c})$ of C'. Note that $W_{3,c}$ must be true (due to $(z\ W_{3,c})$). Thus, $W_{0,c}$ must be false (due to $(W_{0,c}\ W_{3,c}\ W_{4,c})$). It follows that both $W_{1,c}$ and $W_{2,c}$ must be true (due to $(x\ W_{0,c}\ W_{1,c})$, and $(y\ W_{0,c}\ W_{2,c})$). But then, $(W_{1,c}\ W_{2,c}\ W_{5,c})$ has two true literals.