Tutorial 11:

Q1. Fill in the details to prove that $AM[k] \subseteq AM$.

Q2. Suppose SAT $\in PCP(r(n), 1)$ for some $r(n) = o(\log n)$. Then show that $P = NP$.

Q3: Suppose L can be expressed as an intersection of a language in NP, and a language in coNP. That is, there exists $L_1 \in NP$ and $L_2 \in coNP$, such that $L = L_1 \cap L_2$. Then show that $L \in PP$.

For the following questions, let $A_M(x)$ and $R_M(x)$ denote the number of accepting paths and number of rejecting paths on input x by nondeterministic Turing Machine M. Let $D_M(x) = A_M(x) - R_M(x)$.

Q4. Suppose that M is a polynomial time bounded nondeterministic Turing machine and $[x \in L \text{ iff } D_M(x) > 0]$. Then show that $L \in PP$.

Q5. Suppose we are given two non-deterministic Turing machines M_1 and M_2.

(a) Show that one can construct a non-deterministic Turing machine M such that $D_M(x) = D_{M_1}(x) - D_{M_2}(x)$.

(b) Show that one can construct a non-deterministic Turing machine M such that $D_M(x) = D_{M_1}(x) * D_{M_2}(x)$.

Make sure that your M is polynomial time bounded, if M_1 and M_2 are polynomial time bounded.

Q6. Suppose x and y are integers.

Let $P_n(x) = (x-1)\Pi_{i=1}^{n}(x-2^i)^2$.

Let $S_n(x) = \frac{P_n(-x)-P_n(x)}{P_n(-x)+P_n(x)}$.

Show that:

(a) If $1 \leq x \leq 2^n$, then $0 \leq 4P_n(x) < -P_n(-x)$.

(b) If $1 \leq x \leq 2^n$, then $1 \leq S_n(x) < 5/3$.

(c) If $-2^n \leq x \leq -1$, then $-5/3 < S_n(x) \leq -1$.

Below, $1 \leq |x|, |y| \leq 2^n$.

Let $A_n(x,y) = S_n(x) + S_n(y) - 1$. Show that

(d) if $1 \leq x \leq 2^n$ and $1 \leq y \leq 2^n$, then $A_n(x,y) > 0$.

(e) if $-2^n \leq x \leq -1$ or $-2^n \leq y \leq -1$, then $A_n(x,y) < 0$.

Use techniques of the above questions to show that PP is closed under intersection.

Q7. QuadEQ is the following problem:

INPUT: Given a set of n variables, $u_1, u_2, \ldots, u_n$ and a set of r quadratic equations (mod 2) over the variables (that is, each term in the equation could be having degree two (either as u_i^2 or as $u_i u_j$)).

QUESTION: Is there a boolean vector $u_1, u_2, \ldots, u_n$ such that the equations hold?

Show that QuadEQ is NP-complete.