# CS5230
# Tutorial 4

Below $\subseteq$ denotes subset, whereas $\subsetneq$ denotes proper subset.

Q1:

$DSPACE(n) \subseteq \bigcup_{c>0} DTIME(c^n) \subseteq DTIME(2^{n^{1.1}}) \subsetneq DTIME(2^{n^2})$, where

(i) $DSPACE(n) \subseteq \bigcup_{c>0} DTIME(c^n)$ by result done in class,

(ii) $\bigcup_{c>0} DTIME(c^n) \subseteq DTIME(2^{n^{1.1}})$, as $c^n \leq 2^{n^{1.1}}$, for any constant $c$ and large enough $n$,

(iii) $DTIME(2^{n^{1.1}}) \subsetneq DTIME(2^{n^2})$, by time hierarchy theorem as $\lim_{n\to\infty} \frac{2^{n^{1.1}} * n^{1.1}}{2^{n^2}} = 0$ and $2^{n^2}$ is fully time constructible.

Q2. Clearly, $DTIME(2^n)$ is a subset of $DTIME(2^n \lceil n^{2/3} \rceil)$.

We ignore floor and ceilings in the following for notation simiplicity. They don't matter due to linear speedup theorem.

Suppose by way of contradiction that

$DTIME(2^n n^{2/3}) \subseteq DTIME(2^n)$.

Then, in translation lemma using $f(n) = 2^n$ and $f(n) = 2^n + \lfloor 2n/3 \rfloor$ respectively we get

$DTIME(2^{2^n} 2^{2n/3}) \subseteq DTIME(2^{2^n})$ and

$DTIME(2^{2^n+2n/3}(2^n + 2n/3)^{2/3}) \subseteq DTIME(2^{2^n+2n/3})$,

and thus

$DTIME(2^{2^n+4n/3}) \subseteq DTIME(2^{2^n})$.

But, $\lim_{n\to\infty} \frac{2^{2^n} * \log(2^{2^n})}{2^{2^n+4n/3}} = 0$, and thus by time hierarchy theorrem

$DTIME(2^{2^n+n^{4/3}})$ is a proper superset of $DTIME(2^{2^n})$.

A contradiction. Thus, our assumption must have been false and

$DTIME(2^n n^{2/3}) \supset DTIME(2^n)$.

Q3. (Gap theorem for Space)

What we need to show is: For all recursive $h'$ (with $h'(n) \geq n$), there exists a $g'$ such that

$DSPACE(g'(n)) = DSPACE(h'(g'(n))$.

Let $h(m) = 2^{h'(m)*h'(m)}$.

Let $g$ be as given by Gap theorem (for time) for above $h$.

Then, we have:

$DSPACE(h'(g(n)) \subseteq DTIME(2^{(h'(g(n)))^2}) = DTIME(h(g(n))) = DTIME(g(n)) \subseteq DSPACE(g(n))$. Thus, $g' = g$ satisfies the requirement of gap theorem for space.

Gap theorem for NSPACE and NTIME can be proved similarly.

Q4. False.

Let $h(n) = 2^n$. Let $g$ be increasing function as in the gap theorem. Let $T_1(n) = h(g(n))$ and $T_2(n) = g(n) * g(n)$.

Then, $DTIME(T_1(n)) \subseteq DTIME(g(n))$ by gap theorem, and $DTIME(g(n)) \subseteq DTIME(T_2(n)) \subseteq DTIME(T_1(n))$ as $g(n) \leq T_2(n) \leq T_1(n)$ for all but finitely many $n$.

Thus, $DTIME(T_1(n)) \subseteq DTIME(T_2(n))$ even though $T_1(n)$ is not in $O((T_2(n)))$.

Here note that $g(n) \geq n$ and thus $T_2(n) = g(n) * g(n) \geq n^2$.

Q5. Suppose $\mathbf{M}$ accepts the language $\{wcw^R : w \in \{a,b\}^*\}$.

Without loss of generality assume that **M** accepts by moving to the right end of the input. Consider the behaviour of **M** on inputs of the form $wa^m ca^m w^R$, where $w \in \{a,b\}^m$ and $m$ is large enough.

Let $C_j^w$ denote the crossing sequence of **M** on input $wa^m ca^m w^R$ at the boundary left of $a^j c$. Suppose $s$ is the number of states of **M**. Let $d > 0$ be a constant such that $(2s+1)^{d*m} < 2^m/(m+1)$. Note that there exists such a constant $d$. Now consider the following cases.

Case 1: For some $w \in \{a,b\}^m$, all $C_j^w$ are of length at least $d*m$. In this case **M** takes time at least $dm^2$ on input $wa^m ca^m w^R$.

Case 2: For all $w \in \{a,b\}^m$, there exists a $j \leq m$ such that $C_j^w$ is of length at most $d*m$. Thus there exists $j \leq m$ such that for at least $2^m/(m+1)$ different $w$'s, $C_j^w$ is of length at most $d*m$. As the number of different $C_j^m$ of length at most $d*m$ is bounded by $(2s+1)^{d*m}$, we have that there are two different $w, w' \in \{a,b\}^m$ such that $C_j^w = C_j^{w'}$. But then **M** also accepts $wa^m ca^m (w')^R$, a contradiction.