Tutorial 8: Answer sketches.

A1: Suppose L_1 and L_2 are in NP as witnessed by nondeterministic polytime bounded machines M_1 and M_2 respectively. Then, to show that $L_1 \cup L_2 \in NP$, consider a machine M which on input x, runs $M_1(x)$ and $M_2(x)$ and accepts x iff at least one of M_1 and M_2 accepts x. It is easy to verify that M is a nondeterministic polytime bounded machine which accepts $L_1 \cup L_2$. It can be similarly shown that NP is closed under intersection.

A2 (sketch): To show that BPP is closed under intersection suppose L_1 and L_2 are two languages in BPP which are accepted by probabilistic machines M_1 and M_2, with error probability at most 1/4. Using boosting, one can reduce the error probability to 1/8 (by using machines M_1' and M_2' respectively). Note that the boosting can be done by running M_1 (respectively M_2) 101 times, and taking majority answer. Then consider M, which on input x simulates $M_1'(x)$ and then $M_2'(x)$. M accepts x iff M_1' and M_2' both accept x. The error probability of M is bounded by error probability of M_1' plus the error probability of M_2' on input x. Thus M makes error with probability at most 1/4.

One can similarly show that BPP is closed under union and complementation.

A3: Suppose NP is closed under complementation. Suppose L is in NP. Then $\overline{L}$ is also in NP. Thus, there exist two nondeterministic polytime bounded machines M_1 and M_2 which accept L and $\overline{L}$ respectively. Let M be a nondeterminstic machine which runs $M_1(x)$ and $M_2(x)$, and if $M_1(x)$ accepts then $M(x)$ accepts; if $M_2(x)$ accepts, then $M(x)$ rejects, and if neither accepts, then $M(x)$ outputs ? (neither accepts nor rejects). It is easy to verify that M above satisfies the requirements.

A4: To show that $ZPP \subseteq R$, convert ? outputs to 'no' (reject) outputs. As ZPP is closed under complementation, we get that $ZPP \subseteq coR$ also.

To show that $R \cap coR \subseteq ZPP$, suppose L and $\overline{L}$ are both in R as witnessed by M_1 and M_2 respectively. Now, $M(x)$ simulates $M_1(x)$ and $M_2(x)$. $M(x)$ accepts if $M_1(x)$ accepts; $M(x)$ rejects if $M_2(x)$ accepts, and outputs ? if both $M_1(x)$ and $M_2(x)$ reject. It is easy to verify that M witnesses that L is in ZPP.

Q5. Clearly, $PP \subseteq PP'$. Now suppose L is in PP' as witnessed by machine M'. Here we assume that M' uses two sided coin tosses, and runs in time $p(|x|)$, for some polynomial p (where $|x|$ denotes the length of input x).

Consider M'' which on input x, rejects with probability $1/2^{q(|x|)}$ and outputs the answer of $M'(x)$ with probability $1 - 2^{-q(|x|)}$. Then, if $x \notin L$, then $M''(x)$ accepts with probability $< 1/2$. If $x \in L$, then $M''(x)$ accepts with probability at least $(1 - 2^{-q(|x|)})(\frac{1}{2} + 2^{-p(|x|)})$, which is $> 1/2$ if we take $q(|x|)$ to be $p(|x|) + 2$.