Tutorial 9: Answer sketches.

A1 (a) Similar to (b) below, except that we don't need to worry about error probability. The whole simulation is in PSPACE.

(b). Suppose $L \in BPP^{BPP}$. Suppose M_1 is polynomial time bounded probabilitic Turing Machine and $A \in BPP$ are such that L is accepted by M_1^A with the error probibility bounded by 2^{-n} on inputs of length n.

Let $A' = \{x\#^* : x \in A\}$. Note that A' is also in BPP. Furthermore, we can replace A by A' as an oracle without changing the fact that M_1^A accepts L with error probability bounded by 2^{-n} on inputs of length n.

Suppose M_2 is a polynomial time bounded probabilistic Turing Machine which accepts A' with error probability bounded by 2^{-n} on inputs of length n.

Note that such M_1 and M_2 exist using boosting technique done in class. Suppose M_1 is $p(n)$ time bounded.

Now let $M(x)$ simulate M_1 on input x and whenever questions are asked by M_1 for y, then simulate $M_2(y\#^{p(|x|)-|y|})$ and answer the question accordingly. As there are at most polynomially many queries, and for each of such queries, error is bounded by $2^{-p(|x|)}$, all questions are answered correctly with probability at least $1-p(|x|)*2^{-p(|x|)}$. Furthermore M_1 makes error with probability at most $2^{-|x|}$, if all its questions are answered correctly. Thus, M gives correct answer with probability at least $(1-2^{-|x|})*(1-p(|x|)*2^{-p(|x|)})$, which is more than $3/4$ for large enough x.

A2: Suppose M witnesses that $L \in PP$. Then consider the machine M' which rejects with probability $1/2$ and with probability $1/2$ just simulates M. Then, M' witnesses that L is in PP'''.

Now suppose $L \in PP''$ as witnessed by M. Without loss of generality assume that if $x \in L$ then M accepts with probability $> 1/4$ and if $x \notin L$, then M accepts with probability $< 1/4$ (see technique used in previous tutorial). To show that $L \in PP$, consider a machine M' which accepts with probability $1/3$, and with probability $2/3$ simulates $M(x)$ and accepts iff M does. Then, it is easy to verify that M' witnesses that L is in PP.

Note: The above uses 3-sided (along with 2-sided) coin tosses.

To simulate 3-sided coin toss using 2-sided coin toss, one can do the following. Suppose M is a machine which uses 3-sided coin toss, and runs in time $p(|x|)$. Then, one can simulate each 3-sided coin toss, by tossing a 2-sided coin $q(|x|)$ times, and dividing the $2^{q(|x|)}$ outcomes into three groups of nearly equal size (where the difference between any two groups is at most one). This simulation provides an "error" in simulation of each coin toss of probability at most $2^{-q(|x|)}$. Thus the total error introduced in the simulation is at most $3^{p(|x|)} * (2^{-q(|x|)})$, which is less than $7^{-p(|x|)}$, for $q(|x|) = |x| * p(|x|)$, and large enough $|x|$. This error is smaller than the "quantum" of each probability used by M (since acceptance and rejection probabilities of M are away from $1/2$ by at least $6^{-p(|x|)}$).