Approximate Inference and Scientific Method [1]

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March 11, 2007

[1] Work supported in part by ONR/DARPA research contract number N00014-82-K-0193 and NSF grant CCR 8320136 to the University of Rochester.

Abstract

A new identification criterion, motivated by notions of successively improving approximations in the philosophy of science, is defined. It is shown that the class of recursive functions is identifiable under this criterion. This result is extended to permit somewhat more realistic types of data than usual. This criterion is then modified to consider restrictions on the quality of approximations, and the new criteria are compared to existing criteria.

1 Introduction

Research in inductive inference has historically been motivated by considerations of the philosophy of science, e.g. Case and Smith (1983). However, the criteria of success so far proposed seem unrealistic for science.

In the sequel we assume that scientific experiments and observations are encoded as natural numbers and that the process of scientific theory formation can be modeled by an algorithmic device which operates on the encoded experiments and observations.

Gold's (Gold (1967)) criterion demands that an inductive inference machine produce a final correct program (in the sense that it correctly computes the input function or set); others (e.g. Case and Smith (1983)) have liberalized that criterion to allow final programs that are correct except on finitely many inputs. Barzdin (1974) and Case and Smith (1983) also give criteria that permit infinite sequences of programs, nearly all of which are (perhaps only nearly) correct.

We hold with Peirce (1958) that science cannot be expected to produce a final theory of anything, nor even a cofinal sequence of nearly correct theories. Instead, the best we can hope for is that science produces an infinite sequence of improving approximations to reality.

This position has been assailed on the grounds that the notion of "approximation" and the way in which one approximation can improve on another have not been given a satisfactory definition (Belnap, *response to a Paper of Osherson and Weinstein,* given at the conference on Science and Discovery at the University of Pittsburgh in March, 1989). The most obvious definitions involve some sort of arbitrary choice: of an encoding of experiments, if limiting density is used, or of a measure on the set of experiments, or of a metric on experimental results. No such choice, it has been held, could be justified on *a priori* grounds. Furthermore, it has even been claimed that *no* reasonable "invariant" definition could be made of "approximation."

The main point of this paper is that a reasonable, "invariant," and non-trivial definition can be made of "approximation" and "improvement of approximation." We investigate the new definition (**Ap**-identification) and then investigate some non-invariant strengthenings of **Ap**. Some of the strengthened variations are interesting as representing a reasonable extension of Royer's notions of inferring approximations (Royer (1986), see also Smith and Velauthapillai (1986)).

2 Notation

N is the set of natural numbers. a, i, j, m, n, r, s, w, x, y, z, with or without decorations[1], range over natural numbers unless otherwise specified. A, B and S, with or without decorations, range over subsets of N. $\subseteq$ and $\subset$ denote subset and proper subset respectively. d, ϵ range over the real

[1] Decorations on variables refer to subscripts, superscripts, primes and the like.

interval $[0,1]$. f, g range over functions from N to N. $\emptyset$ denotes the null set. card(S) denotes the cardinality of the set S. $\max, \min$ denote the maximum and minimum of a set respectively. By convention $\max(\emptyset) = 0$ and $\min(\emptyset)$ is undefined. $\mu x.[Q(x)]$ is the least natural number x such that $Q(x)$ is true (if such exists). $\lfloor \frac{m}{n} \rfloor$ denotes $\max(\{x \in N \mid x \leq \frac{m}{n}\})$.

$\mathcal{R}$ denotes the class of recursive functions. $\mathcal{C}$ ranges over subsets of $\mathcal{R}$. φ denotes an acceptable numbering (Rogers (1958), Rogers (1967)). Φ denotes an arbitrary Blum complexity measure (Blum (1967)) for φ. We shall speak of programs and numerical names or codes for programs interchangeably; sometimes these numerical names are referred to as *indices*. In some contexts p ranges over natural numbers thought of, in those contexts, as programs. In other contexts p ranges over total functions, in which the range of p is thought of as a set of programs. $\langle \cdot, \cdot \rangle$ stands for an arbitrary computable one to one encoding of all pairs of natural numbers onto N (Rogers (1967)). $\langle \cdot, \cdot \rangle$ is extended to n-tuples in the usual way. domain(η) denotes the domain of (partial) function η. For any two partial functions η_1 and η_2, $\eta_1 =^n \eta_2$ means that card$(\{x \mid \eta_1(x) \neq \eta_2(x)\}) \leq n$ and $\eta_1 =^* \eta_2$ means that card$(\{x \mid \eta_1(x) \neq \eta_2(x)\})$ is finite. $S_1 \mathbf{\Delta} S_2$ denotes the symmetric difference of the sets S_1 and S_2. For any two sets S_1 and S_2, $S_1 =^n S_2$ means that card$(S_1 \mathbf{\Delta} S_2) \leq n$ and $S_1 =^* S_2$ means that card$(S_1 \mathbf{\Delta} S_2)$ is finite.

Any unexplained notation is from Rogers (1967).

3 Preliminaries

In this section we briefly discuss notions from recursion theoretic machine learning literature. For detailed discussion see Osherson, Stob and Weinstein (1986), Case and Smith (1983), Gold (1967), Angluin and Smith (1983), Klette and Wiehagen (1980) and Blum and Blum (1975).

An *Inductive Inference Machine* (IIM) (Gold (1967)) is an algorithmic device which takes as its input a set of data given one element at a time, and which from time to time, as it is receiving its input, outputs programs. IIMs have been used in the study of machine identification of programs for computable functions as well as algorithmic learning of grammars for languages (Blum and Blum (1975), Case and Smith (1983), Chen (1981), Fulk (1985), Gold (1967), Osherson, Stob and Weinstein (1986) and Wiehagen (1978)).

$\mathbf{M}$, with or without decorations, ranges over the class of inductive inference machines. For inference of a computable function f by an IIM $\mathbf{M}$, the graph of f is fed to $\mathbf{M}$ in any order. Without loss of generality (Blum and Blum (1975) and Case and Smith (1983)), we will assume that $\mathbf{M}$ is fed the graph of f in the sequence $(0, f(0)), (1, f(1)), (2, f(2)), \ldots$. For all functions f, $f[n]$ denotes the finite initial segment $((0, f(0)), (1, f(1)), \ldots, (n-1, f(n-1)))$. Let INIT $= \{f[n] \mid f \in \mathcal{R} \wedge n \in N\}$. Variables σ and τ, with or without decorations, range over INIT. $|\tau|$ denotes the number of elements in τ. Thus $|f[n]| = n$. content(τ) denotes the set of pairs in the range of τ. Thus content$(f[n]) = \{(i, f(i)) \mid i < n\}$. $\mathbf{M}(\sigma)$ is the last output of $\mathbf{M}$ by the time it has received input σ. We will assume, without loss of generality, that $\mathbf{M}(\sigma)$ is always defined. We say that $\mathbf{M}(f)$ converges to i (written: $\mathbf{M}(f)\!\downarrow = i$) iff for all but finitely many n, $[\mathbf{M}(f[n]) = i]$; $\mathbf{M}(f)$ is undefined if no such i exists.

A criterion of success (called $\mathbf{Ex}$-identification) is for the machine to eventually output a last program, which computes (nearly computes) f. Formally,

Definition 1 (Gold (1967), Blum and Blum (1975) and Case and Smith (1983)) Let $a \in N \cup \{*\}$.
(a) $\mathbf{M}$ $\mathbf{Ex}^a$*-identifies* f (written $f \in \mathbf{Ex}^a(\mathbf{M})$) iff both $\mathbf{M}(f)\!\downarrow$ and $\varphi_{\mathbf{M}(f)} =^a f$.

(b) $\mathbf{Ex}^a = \{\mathcal{C} \subseteq \mathcal{R} \mid (\exists \mathbf{M})[\mathcal{C} \subseteq \mathbf{Ex}^a(\mathbf{M})]\}$.

In the above definition a stands for the number of anomalies allowed in the final program. $a = *$ means that unbounded but finite number of anomalies is allowed in the final program. Case and Smith (1983) introduced another infinite hierarchy of identification criterion which we describe below. "**Bc**" stands for *behaviorally correct*. Barzdin (1974) essentially introduced the notion $\mathbf{Bc}^0$.

Definition 2 (Case and Smith (1983)) Let $a \in N \cup \{*\}$.
(a) $\mathbf{M}$ $\mathbf{Bc}^a$-identifies f (written: $f \in \mathbf{Bc}^a(\mathbf{M})$) iff, for all but finitely many n, $\varphi_{\mathbf{M}(f[n])} =^a f$.
(b) $\mathbf{Bc}^a = \{\mathcal{C} \subseteq \mathcal{R} \mid (\exists \mathbf{M})[\mathcal{C} \subseteq \mathbf{Bc}^a(\mathbf{M})]\}$.

We usually write **Ex** for $\mathbf{Ex}^0$, and **Bc** for $\mathbf{Bc}^0$.

4 Approximation of Recursive Functions

Definition 3 $\mathbf{M}$ **Ap**-*identifies* f (written: $f \in \mathbf{Ap}(\mathbf{M})$) iff there is a sequence of sets $S_n^f \subset N$ such that the following four conditions are satisfied.
(i) For all n, for each $x \in S_n^f$, $\varphi_{\mathbf{M}(f[n])}(x) = f(x)$.
(ii) For all n, $S_n^f \subseteq S_{n+1}^f$.
(iii) For all x, there exists an n such that $x \in S_n^f$.
(iv) There exist infinitely many n such that $S_{n+1}^f - S_n^f$ is infinite.

If $\mathbf{M}$ **Ap**-identifies f, then we also say that $\mathbf{M}$ *approximates* f. Note that S_n^f may be a strict subset of $\{x \mid \varphi_{\mathbf{M}(f[n])}(x) = f(x)\}$. Also note that, if a machine $\mathbf{M}$ **Ex**-identifies f, then it also **Ap**-identifies f; although the choice of sets S_n^f in this case is a bit artificial.

Ap-identification is thus a notion of improving approximations that demands correctness on ever greater sets of experiments. A physicist's use of approximation often asks that a theory produce numbers that are close, in the usual metric on the real numbers, to a "correct answer." However, the physicist's notion of a correct answer is really a theoretical construct: the limit of the results of a sequence of experiments done to greater and greater degrees of precision, or the limit of the average result of a sequence of experiments. Approximation in this sense can be included in the **Ap** notion by noting that single experiments are always done to some finite precision; theories are better in the physicist's sense when they fit more experiments, namely the more precise ones.

Theorem 4 *There is an inductive inference machine* $\mathbf{M}$ *that approximates every recursive function.*

PROOF. The idea of the proof is to construct $\mathbf{M}$ which partitions N into infinitely many infinite subsets. $\mathbf{M}$ then carries out a separate induction process for each subset. In the inference process for a subset, $\mathbf{M}$ uses the number of the subset as a bound on the Gödel number of programs to be considered.

When $\mathbf{M}(f[n])$ is run on an input $x \geq n$ from the i-th element of the partition, it uses the program, with Gödel number less than i, that best fits $f[n]$, where n is used as a bound on computation time to find the "best fit" (see definition of err and best below). For inputs $x < n$, $\mathbf{M}(f[n])$ outputs $f(x)$.

Let patch, select, err, and best be recursive functions such that:

$$\varphi_{\text{patch}(i,\sigma)}(x) = \begin{cases} y, & \text{if } (\exists y)[(x,y) \in \text{content}(\sigma)]; \\ \varphi_i(x), & \text{otherwise.} \end{cases}$$

$$\varphi_{\text{select}(n,\langle a_0,\ldots,a_n\rangle)}(\langle i,x\rangle) = \begin{cases} \varphi_{a_i}(\langle i,x\rangle), & \text{if } i < n; \\ \varphi_{a_n}(\langle i,x\rangle), & \text{otherwise.} \end{cases}$$

$$\text{err}(j,\sigma) = \mu x.[x = |\sigma| \text{ or } \Phi_j(x) > |\sigma| \text{ or } \varphi_j(x) \neq \sigma(x)].$$

$$\text{best}(b,\sigma) = \mu i.[i \leq b \wedge (\forall j \leq b)[err(i,\sigma) \geq err(j,\sigma)]].$$

$$\mathbf{M}(\sigma) = \text{patch}(\text{select}(|\sigma|, \langle \text{best}(0,\sigma), ..., \text{best}(|\sigma|,\sigma)\rangle), \sigma).$$

patch patches σ into program i. select chooses one of $a_0, ..., a_n$ and runs it according to the left projection of the input. err finds the first apparent error committed by j relative to σ. best finds the best program, among programs with Gödel number $\leq b$, for σ, as measured by err. Note that best is a total function.

Note that for any sufficiently large i and for sufficiently large n (depending on i) best$(i, f[n])$ will be a program for f. Let

$$S_n^f = \{0,\ldots,n-1\} \bigcup \{\langle k,x\rangle \mid x \in N \wedge k < n \wedge (\forall n' \geq n)[\varphi_{\text{best}(k,f[n'])} = f]\}$$

Verification of the properties (i)-(iv) in Definition 3 is immediate. □

5 Density Restrictions in the Ap-criterion

The **Ap** criterion is unsatisfactory in the following respect: the sets S_n^f can be very sparse. In fact Theorem 13 below shows that, if **M** **Ap**-identifies $\mathcal{R}$, then, for some recursive f, the sets S_n^f must be exceedingly sparse. We seem to believe that we can predict the outcomes of many new experiments quite reliably; experience seems to indicate that this belief is justifiable. The results of the previous section do not contribute to any understanding we might have of this phenomenon.

As a first try at understanding such things, we investigate strengthenings of **Ap** that depend on notions of limiting density (Rogers (1967)). Unfortunately, doing so destroys the "invariance" of the **Ap** notion. These results, however, also have a separate interest. Royer (1986) and Smith and Velauthapillai (1986) have investigated notions related to $\mathbf{Ex}^a$, in which the anomalies of the final program are of bounded limiting density. The extensions of their results to **Bc**-style criteria are all trivial because of the basic result in Case and Smith (1983) that $\mathbf{Bc}^* = \mathcal{R}$. Our results represent a more interesting extension of the work of Royer, Smith, and Velauthapillai to **Bc**-like criteria.

It is easy to choose a pairing function such that, for each recursive function f, $\lim_{n\to\infty} \mathbf{d}(S_n^f) > 0$, where S_n^f is as defined in the proof of Theorem 4. In this section we study the effects of requiring the limiting density of the sets $S_0^f, S_1^f, \ldots$ as in Definition 3 to be above a certain prespecified value. First we formally define what we mean by density of a set. See Smith and Velauthapillai (1986) for similar definitions.

Definition 5 (S. Tennenbaum: see page 156 in Rogers (1967), Royer (1986)) The *density* of a set $A \subseteq N$ in a finite and nonempty set B (denoted: $\mathbf{d}(A;B)$) is card$(A \cap B)$/card(B).

Intuitively, $\mathbf{d}(A;B)$ can be thought of as the probability of selecting an element of A when choosing an arbitrary element from B.

Definition 6 (S. Tennenbaum: see page 156 in Rogers (1967), Royer (1986)) The *density* of a set $A \subseteq N$ (denoted: $\mathbf{d}(A)$) is $\liminf_{n\to\infty}\{\mathbf{d}(A; \{z \mid z \leq x\}) \mid x \geq n\}$.

Note that the above definitions are only a special case of Tennenbaum's definition.

Definition 7 Let $d \in [0,1]$. An IIM $\mathbf{M}$ $\mathbf{DAp}^d$*-identifies* a function f (written: $f \in \mathbf{DAp}^d(\mathbf{M})$) iff there exists a sequence of sets $S_n^f \subset N$ such that the following five conditions are satisfied.

(i) For all n, for each $x \in S_n^f$, $\varphi_{\mathbf{M}(f[n])}(x) = f(x)$.

(ii) For all n, $S_n^f \subseteq S_{n+1}^f$.

(iii) For all x, there exists an n such that $x \in S_n^f$.

(iv) There exist infinitely many n such that $S_{n+1}^f - S_n^f$ is infinite.

(v) $\lim_{n\to\infty} \mathbf{d}(S_n^f) \geq d$.

Definition 8 Let $d \in [0,1]$. $\mathbf{DAp}^d = \{\mathcal{C} \subseteq \mathcal{R} \mid (\exists \mathbf{M})[\mathcal{C} \subseteq \mathbf{DAp}^d(\mathbf{M})]\}$.

Even though the limiting density of a set may be 1, there may be arbitrarily large gaps. We thus introduce another form of identification which prohibits such large gaps.

Definition 9 (Royer (1986)) The *uniform density* of a set A in intervals of length $\geq n$ (denoted: $\mathbf{ud}_n(A)$) is $\inf(\{\mathbf{d}(A; \{z \mid x \leq z \leq y\}) \mid x, y \in N$ and $y - x \geq n\})$. *Uniform density* of A (denoted: $\mathbf{ud}(A)$) is $\lim_{n\to\infty} \mathbf{ud}_n(A)$.

Definition 10 Let $d \in [0,1]$. An IIM $\mathbf{M}$ $\mathbf{UDAp}^d$*-identifies* a function f (written: $f \in \mathbf{UDAp}^d(\mathbf{M})$) iff there exists a sequence of sets $S_n^f \subset N$ such that the following five conditions are satisfied.

(i) For all n, for each $x \in S_n^f$, $\varphi_{\mathbf{M}(f[n])}(x) = f(x)$.

(ii) For all n, $S_n^f \subseteq S_{n+1}^f$.

(iii) For all x, there exists an n such that $x \in S_n^f$.

(iv) There exist infinitely many n such that $S_{n+1}^f - S_n^f$ is infinite.

(v) $\lim_{n\to\infty} \mathbf{ud}(S_n^f) \geq d$.

Definition 11 Let $d \in [0,1]$. $\mathbf{UDAp}^d = \{\mathcal{C} \subseteq \mathcal{R} \mid (\exists \mathbf{M})[\mathcal{C} \subseteq \mathbf{UDAp}^d(\mathbf{M})]\}$.

Harrington (Case and Smith (1983)) showed that $\mathcal{R} \in \mathbf{Bc}^*$. Chen (1981) in his thesis proved several results showing that any machine which $\mathbf{Bc}^*$-identifies $\mathcal{R}$ must necessarily perform quite badly on some functions. We consider below one of Chen's results.

Theorem 12 (Chen (1981), Theorem 5.2) *Suppose* $\mathbf{M}$ $\mathbf{Bc}^*$*-identifies* $\mathcal{R}$*. Then for all recursive functions* g*, there exists a recursive function* f *such that, for infinitely many* i*,*

$$\mathrm{card}(\{x \mid \varphi_{\mathbf{M}(f[i])}(x) \neq f(x)\}) > g(i).$$

Thus the number of errors committed by programs output by a machine $\mathbf{Bc}^*$-identifying $\mathcal{R}$ cannot be recursively bounded. We now prove the following theorem (Theorem 13) which is in some sense analogous to Chen's theorem[2]. As a consequence of the following theorem we have that for any machine which $\mathbf{Ap}$-identifies all the recursive functions, there must exist a recursive f, for which the corresponding S_n^f's (as in Definition 3) must be exceedingly sparse. Even though the particular notion of density employed depends on a particular *ad hoc* coding of experiments, this sparseness property holds for all possible encoding of experiments.

[2]This analogy was pointed out to us by an anonymous referee.

Theorem 13 $(\forall d, 0 < d \leq 1)[\mathcal{R} \notin \mathbf{DAp}^d]$.

Corollary 14 $(\forall d, 0 < d \leq 1)[\mathcal{R} \notin \mathbf{UDAp}^d]$.

Proof. We prove the theorem for $d > 1/3$. The proof can be easily generalized to the case when $1/q < d \leq 1/(q-1), q > 3$. Suppose by way of contradiction that IIM $\mathbf{M}$ $\mathbf{DAp}^{1/3+\epsilon}$-identifies $\mathcal{R}$, $\epsilon > 0$. We will now exhibit a recursive function f, such that $\mathbf{M}$ fails to $\mathbf{DAp}^{1/3+\epsilon}$-identify f. We will use operator recursion theorem (Case (1974)) to construct such an f.

By implicit use of operator recursion theorem (Case (1974)) there exists a recursive, one-to-one function p, such that the (partial) functions $\varphi_{p(\cdot)}$ may be described in stages as follows.

For each i, let $\varphi_{p(i)}^s$ denote the part of $\varphi_{p(i)}$ defined before stage s. Let $\varphi_{p(0)}(0) = 0$. Also let x_s denote the least x not in domain($\varphi_{p(0)}^s$). Go to stage 1.

Stage s

Dovetail steps 1 and 2 until step 1 succeeds. If and when step 1 succeeds go to step 3.

1. Search for a τ, extending $\varphi_{p(0)}[x_s]$, and for m, k, w_1, w_2, $\ldots$, w_k such that the following three conditions are satisfied.
   (1a) $k > m \cdot (2/3 + \epsilon/2)$ and $|\tau| < m \cdot \epsilon/200$.
   (1b) $|\tau| < w_1 < w_2 < \cdots < w_k < m$.
   (1c) $\varphi_{\mathbf{M}(\tau)}(w_i)\downarrow$, for $1 \leq i \leq k$.
2. For $x < x_s$, let $\varphi_{p(s)}(x) = \varphi_{p(0)}(x)$.

   Let $\varphi_{p(s)}^{s,s'}$ denote the part of $\varphi_{p(s)}$ defined before substage s'. Let $x_{s,s'}$ denote the least x not in domain($\varphi_{p(s)}^{s,s'}(x)$).

   Go to substage 0.

   Substage s'

   2.1. Search for a τ', extending $\varphi_{p(s)}[x_{s,s'}]$, and for m', k', w'_1, w'_2, $\ldots$, $w'_{k'}$ such that the following three conditions are satisfied.
   (2.1a) $k' > m' \cdot (1/3 + \epsilon/2)$ and $|\tau'| < m' \cdot \epsilon/200$.
   (2.1b) $|\tau'| < w'_1 < w'_2 < \cdots < w'_{k'} < m'$.
   (2.1c) $\varphi_{\mathbf{M}(\tau')}(w_i)\downarrow$, for $1 \leq i \leq k'$.

   2.2. If and when step 2.1 succeeds, let τ', m', k', $w'_1, \cdots$ be as found in step 2.1.
   For x such that $x_{s,s'} \leq x < |\tau'|$, let $\varphi_{p(s)}(x) = y$, where $(x, y) \in \text{content}(\tau')$.
   For $x \in \{w'_i \mid 1 \leq i \leq k'\}$, let $\varphi_{p(s)}(x) = \varphi_{\mathbf{M}(\tau')}(x) + 1$.
   For x such that $|\tau'| \leq x \leq m'$, $x \notin \{w'_i \mid 1 \leq i \leq k'\}$, let $\varphi_{p(s)}(x) = 0$.
   Go to substage $s' + 1$.

   End substage s'.
3. If and when step 1 succeeds, let τ, m, k, w_1, $\cdots$ be as found in step 1.
   For $x_s \leq x < |\tau|$, let $\varphi_{p(0)}(x) = y$, where $(x, y) \in \text{content}(\tau)$.
   For $x \in \{w_i \mid 1 \leq i \leq k\}$, let $\varphi_{p(0)}(x) = \varphi_{\mathbf{M}(\tau)}(x) + 1$.
   For $|\tau| \leq x \leq m$, $x \notin \{w_i \mid 1 \leq i \leq k\}$, let $\varphi_{p(s)}(x) = 0$.

   Go to stage $s + 1$.

End stage s.

Now consider the following cases:
Case 1: There are infinitely many stages.

In this case, let $f = \varphi_{p(0)}$. Clearly f is recursive. We claim that no $S^f_0, S^f_1, \ldots$ can exist, satisfying (i)-(v) in Definition 7, for $d = 1/3 + \epsilon$. Suppose otherwise. Then there exists n_1 such that $\mathbf{d}(S^f_{n_1}) > 1/3 + 60\epsilon/100$. Also, there exists n_2 such that, for all $x \geq n_2$, $\mathbf{d}(S^f_{n_1}; \{0, \ldots, x\}) > 1/3 + 55\epsilon/100$. Since, for all i, $S^f_i \subseteq S^f_{i+1}$, we have that, for all $n, r > \max(\{n_1, n_2\})$, $\mathbf{d}(S^f_n; \{0, \ldots, r\}) \geq 1/3 + 55\epsilon/100$. But, then, in all stages greater than $\max(\{n_1, n_2\})$, due to step 3 and the way $\tau, m, w_1, \cdots$ were chosen, we have that, there exists an error point for $\varphi_{\mathbf{M}(\tau)}$ in $S^f_{|\tau|}$ (since $|\tau| > \max(\{n_1, n_2\})$ and the fraction of points upto m on which $\varphi_{\mathbf{M}(\tau)}$ commits error is at least $2/3 + \epsilon/2$). This contradicts (i) in Definition 7. Thus $\mathbf{M}$ does not $\mathbf{DAp}^{1/3+\epsilon}$-identify f.

Case 2: Stage s starts but never halts.

In this case, for every extension τ of $\varphi_{p(0)}[x_s]$, for all $r \geq 200 \cdot |\tau|/\epsilon$, $\mathbf{d}(\text{domain}(\varphi_{\mathbf{M}(\tau)}); \{0, \ldots, r\}) < 2/3 + 51\epsilon/100$.

Case 2.1: In stage s, there are infinitely many substages.

In this case, let $f = \varphi_{p(s)}$. Clearly f is recursive. Now, since for all but finitely many n, for all $r \geq 200 \cdot n/\epsilon$, $\mathbf{d}(\text{domain}(\varphi_{\mathbf{M}(f[n])}); \{0, \ldots, r\}) < 2/3 + 51\epsilon/100$, arguing as in case 1, we have that $\mathbf{M}$ does not $\mathbf{DAp}^{1/3+\epsilon}$-identify f.

Case 2.2: In stage s, substage s' starts but never halts.

In this case, we have that on every extension of $\varphi_{p(s)}[x_{s,s'}]$, $\mathbf{M}$ fails to output a program which has a domain of limiting density more than $1/3 + 51\epsilon/100$. Let f be such that

$$f(x) = \begin{cases} \varphi_{p(s)}(x), & \text{if } x < x_{s,s'}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbf{M}$ does not $\mathbf{DAp}^{1/3+\epsilon}$-identify f.

From the above cases we have that $\mathbf{M}$ does not $\mathbf{DAp}^{1/3+\epsilon}$-identify $\mathcal{R}$.

When $1/q < d \leq 1/(q-1)$ the above proof can be generalized by taking $q-1$ levels of iteration instead of 2 as done in the above procedure. We leave the details to the reader. □

Theorem 15 $(\forall d_1, d_2 \mid 0 \leq d_1 < d_2 \leq 1)[\mathbf{UDAp}^{d_1} - \mathbf{DAp}^{d_2} \neq \emptyset]$.

Proof. Without loss of generality, suppose $d_1 = m/n$ and $d_2 = (m+1)/n$, where $m + 1 \leq n$. Let $\mathcal{C} = \{f \in \mathcal{R} \mid (x \bmod n) < m \Rightarrow f(x) = 0\}$. An easy modification of the procedure to $\mathbf{Ap}$-identify all the recursive functions given in the previous section, gives us a procedure to $\mathbf{Ap}$-identify $\mathcal{C}$ with $S^f_0 = \{x \mid (x \bmod n) < m\}$. Thus $\mathcal{C} \in \mathbf{UDAp}^{d_1}$.

We now prove that $\mathcal{C} \notin \mathbf{DAp}^{d_2}$. Suppose by way of contradiction that IIM $\mathbf{M}$ $\mathbf{DAp}^{d_2}$-identifies $\mathcal{C}$. Let r be a recursive function such that

$$r(x) = n * \lfloor \frac{x}{n-m} \rfloor + m + x - (n-m) * \lfloor \frac{x}{n-m} \rfloor$$

r is an one to one, onto, increasing mapping from N to $\{x \mid x \ \mathbf{mod}\ n \geq m\}$. For each recursive function f, define f' as follows.

$$f'(x) = \begin{cases} f(r^{-1}(x)), & \text{if } x \ \mathbf{mod}\ n \geq m; \\ 0, & \text{otherwise.} \end{cases}$$

Let p be a recursive function such that, for all i and x,

$$\varphi_{p(i)}(x) = \varphi_i(r(x))$$

Let $\mathbf{M}'$ be such that, for all f, n, $\mathbf{M}'(f[n]) = p(\mathbf{M}(f'[n]))$. It is easy to verify that, if $\mathbf{M}$ $\mathbf{DAp}^{d_2}$-identifies $\mathcal{C}$, then $\mathbf{M}'$ $\mathbf{DAp}^{1/(n-m)}$-identifies $\mathcal{R}$. But this is not possible (Theorem 13). Thus no such $\mathbf{M}$, $\mathbf{DAp}^{d_2}$-identifying $\mathcal{C}$, can exist. □

Theorem 16 $(\forall d, 0 < d \leq 1)[\mathbf{DAp}^1 - \mathbf{UDAp}^d \neq \emptyset]$.

Proof. Let $\mathcal{C} = \{f \in \mathcal{R} \mid (\forall n)(\forall x \mid 2^n < x < 2^{n+1} - n)[f(x) = 0]\}$. Again an easy modification of the procedure to $\mathbf{Ap}$-identify all the recursive functions given in the previous section, gives us a procedure to $\mathbf{Ap}$-identify $\mathcal{C}$ with $S_0^f = \{x \mid (\exists n)[2^n < x < 2^{n+1} - n]\}$. Thus $\mathcal{C} \in \mathbf{DAp}^1$. Suppose by way of contradiction that IIM $\mathbf{M}$ $\mathbf{UDAp}^d$-identifies $\mathcal{C}$. Then $\mathbf{M}$ can be easily modified to obtain $\mathbf{M}'$ (in a way similar to that used in the proof of Theorem 15) which $\mathbf{UDAp}^d$-identifies all the recursive functions. But this is not possible (Corollary 14). Thus no such $\mathbf{M}$, $\mathbf{UDAp}^d$-identifying $\mathcal{C}$ can exist. □

Now we show that even though $\mathcal{R}$ cannot be $\mathbf{DAp}^d$-identified (for $d > 0$), there are large classes of functions which can be $\mathbf{DAp}^1$-identified.

Theorem 17 $(\forall i \in N)[\mathbf{Bc}^i \subseteq \mathbf{DAp}^1]$.

Proof. Given IIM $\mathbf{M}$ and i, we construct another machine $\mathbf{M}'$, which $\mathbf{DAp}^1$-identifies all functions $\mathbf{Bc}^i$-identified by $\mathbf{M}$. Let $fl(n) = \max(\{2^k \mid 2^k \leq n\})$. Let $\mathbf{M}'(f[n]) = \text{patch}(\mathbf{M}(f[fl(n)]), f[n])$, where patch is as in the proof of Theorem 4.

Now suppose $f \in \mathbf{Bc}^i(\mathbf{M})$. Let n_0 be such that $fl(n_0) = n_0$ and, for all $n \geq n_0$, $\varphi_{\mathbf{M}(f[n])} =^i f$. Let $X_f = N - \{x \mid (\exists n \geq n_0)[\varphi_{\mathbf{M}(f[n])}(x) \neq f(x)]\}$. It is easy to verify that $\mathbf{d}(X_f) = 1$. Thus, $\mathbf{M}$ $\mathbf{DAp}^1$-identifies f. □

6 Further Considerations and Open Problems

Consider a situation in which an IIM, trying to learn a function f, may initially receive incorrect values for some data points. However, for each of the data points, the IIM is eventually told about the correct value of f. Note that in the above situation $\mathbf{M}$, as in the proof of Theorem 4 (with minor modifications), can still approximate all the recursive functions.

Scientific experiments are not always deterministic; even if quantum mechanical indeterminacy is not important, one can never be certain that all of the significant variables have been controlled. On the other hand it is arguable that the set of possible outcomes of an experiment is always finite.

These considerations lead to a formulation of inductive inference in which the function to be learned carries experimental descriptions to finite sets of outcomes, and the data to the inductive inference machine consists of experiments paired with one outcome at a time. It should be clear that, even under these circumstances, the result of Theorem 4 holds.

In some respects, these results are not very satisfactory. One would like to be able to give some account of the confidence we have in the outcomes of certain experiments. Also, for the inference machine given in the proof of Theorem 4, the lower numbered partitions of the data are never completely predicted. Science does partition experiments into classes, and treat each class to some extent separately; but the classes are not simply the arbitrary choices made by a pairing function but reflect, to some extent, the results of the experiments.

It remains open whether or not one can approximate a set from positive data only; we conjecture not.

Note that the IIM **M** given in the proof of Theorem 4 does not $\mathbf{Bc}^*$-identify $\mathcal{R}$. Also, it is easy to see that the IIM $\mathbf{M}_H$ constructed by Harrington (Case and Smith (1983)), to $\mathbf{Bc}^*$-identify $\mathcal{R}$, does not **Ap**-identify $\mathcal{R}$. We do not know if there exists an IIM which identifies all the recursive functions, simultaneously in both **Ap** and $\mathbf{Bc}^*$ senses[3]. We conjecture that no IIM can identify all the recursive functions, simultaneously in both **Ap** and $\mathbf{Bc}^*$ senses.

7 Acknowledgements

We thank an anonymous referee for several helpful comments. We would like to thank John Case, David Haussler, Arun Sharma, Rajeev Raman and Lata Narayanan for helpful discussions. Part of this work was in the dissertation of the first author at SUNY/Buffalo.

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