

A Computation Model with Automatic Functions and Relations as Primitive Operations[☆]

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Abstract

Prior work of Hartmanis and Simon [36] and Floyd and Knuth [30] investigated what happens if a device uses primitive steps more natural than single updates of a Turing tape. One finding was that in the numerical setting, addition, subtraction and bit-wise Boolean operations of numbers preserve polynomial time while incorporating concatenation or multiplication allows to solve all PSPACE problems in polynomially many steps. Therefore we propose to use updates and comparisons with automatic functions as primitive operations and use constantly many registers; the resulting model covers all primitive operations of Hartmanis and Simon as well as Floyd and Knuth, but the model remains in polynomial time. The present work investigates in particular the deterministic complexity of various natural problems and also gives an overview on the nondeterministic complexity of this model.

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Dedication

This paper is dedicated to Cristian S. Calude who is a coauthor of some of the authors of this paper. Calude works in automata theory and recursion theory, namely on the subarea of these disciplines related to algorithmic randomness. Calude also bridges two countries (Romania and New Zealand) and migrated from the first to the second without cutting the links to his home-country. Calude was the founding director of the Centre for Discrete Mathematics and Theoretical Computer Science in Auckland and helped to recruit several outstanding mathematicians for it to his university.

Calude is an author of several books in this field [9, 10]. His work “Information and Randomness – An Algorithmic Perspective” [9] was the first textbook on this topic written by a New Zealand author – André Nies and Rodney G. Downey (jointly with Denis Hirschfeldt) are the other two New Zealanders who wrote textbooks on this topic. His upcoming book [10] will appear in May 2022 and has the title “To Halt, or Not to Halt. That is the question.” This book is dedicated to the study of the halting problem, an important example of a non-recursive but recursive enumerable set given by Alan Turing and studies the question on which computer programs e the machine is halting when

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input x is provided. This problem can also be asked for prefix-free universal machines from the field of algorithmic randomness and Calude jointly with Hertling, Khossainov, and Wang in his paper “Recursively enumerable reals and Chaitin Omega numbers” [19] showed that every Martin-Löf random real number between 0 and 1 equals the halting probability of some prefix-free universal Turing machine; this work was a ground breaking result and the team is one of two who obtained this result independently. A joint book of Cristian Calude and Gheorghe Păun “Computing with Cells and Atoms: an Introduction to Quantum, DNA and Membrane Computing” [25] gives an extensive overview of alternative computation models. A very well-cited paper (joint with Giuseppe Longo) is “The Deluge of Spurious Correlations in Big Data” [22].

Cristian Calude and his wife Elena Calude also jointly worked on various topics, as their research areas overlapped. They have 21 joint conference and journal papers; their first joint work was in collaboration with Bakhadyr Khossainov and investigated simulation, universality, and minimality of deterministic finite automata [15, 16]. This topic is in the heart of theoretical computer science since Myhill and Nerode gave an algorithm to minimize deterministic finite automata and characterized them in terms of number of derivatives. They jointly studied the complexity of mathematical problems [12, 13], quantum automata [11], and even a number-theoretical problem on taxicab numbers [14]. Their daughter Andreea also undertakes a scientific career and is now a senior lecturer at Waikato University and her research interests include the study of Māori loanwords in New Zealand English.

Before coming to Singapore, author Frank Stephan worked in Heidelberg and Australia and had a long-standing collaboration with Arun Sharma in Sydney which brought him annually into that region and he visited in the year 2000 the University of Auckland and met Calude and his colleagues. Since then they kept in contact and in 2008, Calude brought Stephan and Nies into his collaboration with Ludwig Staiger on universal recursively enumerable sets of strings [23, 24]. Besides algorithmic information theory and randomness, they also worked with Rūsiņš Freivalds and author Sanjay Jain on deterministic frequency pushdown automata [18]. One joint work in algorithmic game theory brought down the complexity of solving parity games from $n^{O(\sqrt{n})}$ to $O(n^{8+\log m})$ where m is the number of colors and n the number of nodes and $1 \leq m \leq n$ [20]. The recent joint work is recursion-theoretic about the search for least programs [21] and bi-immunity of sequences over non-binary alphabets [17].

The present work combines automata theory with aspects of computational complexity. The automata theoretic approach is studied both in Auckland and Singapore, and Calude’s work has various mentioning of finite automata. Thus the authors would like to dedicate the present work to their coauthor Cristian Calude at the occasion of his seventieth birthday.

1. Introduction

Sequential models of computation such as finite state machines, register machines and Turing machines typically carry out primitive operations at each computation step. For example, Turing machines have a finite number of rules for manipulating symbols on a tape and can only process symbols one at a time. Register machines, in particular random-access machines (RAM’s), can store numbers of arbitrary size in an unlimited number of registers and have basic arithmetic operators such as increment and decrement; a RAM may also be augmented with instructions for performing advanced operations such as multiplication, division, concatenation as well as number comparison [36, 61]. A fairly common theme in the study of these machines is the analysis of whether, and if so to what extent, computational power is boosted by allowing extra types of instructions. In the case of RAM’s, Hartmanis and Simon [36] showed that adding logical operations on bit vectors in parallel and either multiplication or concatenation allows polynomial time decidability of all languages in PSPACE, where the instructions use all counts as constant time operations. Schönhage [61] proved that incorporating the arithmetic operations of addition, multiplication and division as primitive steps allows polynomial time decidability of all languages in NP; if RAM’s only incorporate addition and multiplication as primitive steps, then the languages they decide in polynomial time are also decidable in polynomial time by probabilistic Turing machines.

A natural question is whether the computational power of register machines can be enhanced with primitive operators that generalise bit-wise operations, addition, subtraction and comparison, but do not lead to an “unreasonable” speedup in computational time. For the purpose of the current work, we define a “reasonable” primitive operator as one that can be simulated by an appropriate type of Turing machine or register machine with at most a polynomial increase in time complexity. As mentioned earlier, RAM’s that are equipped with bit operations as well as the multiplication or concatenation operator can decide languages in PSPACE in polynomial time; if one allows the addition,

multiplication and division operators, then RAM's can decide languages in NP in polynomial time. Consequently, under the assumption that neither PSPACE nor NP is equal to P, the present work does not regard multiplication or concatenation as a reasonable primitive operator. In this work, we propose the use of *automatic functions* or, more generally, *bounded automatic relations* as primitive steps in a register machine. We call such machines *Deterministic Automatic Register Machines* (DARM's) in the case that automatic functions are used, and *Non-deterministic Automatic Register Machines* (NARM's) in the case that bounded automatic relations are used. Automatic functions and relations belong to the wider class of *automatic structures*, which are relational structures whose domain and atomic relations can be realised by finite automata operating synchronously on their input. The idea of using automata to study structures was originally conceived by Büchi, who applied this notion to prove the decidability of a fragment of second-order arithmetic known as S1S [7, 8]. The definition of an automatic structure first appeared in Hodgson's doctoral thesis [41], parts of which were published later [39]. Khoussainov and Nerode [47] rediscovered the concept of an automatic structure and defined various notions of automatic presentations. They then studied the problem of characterising structures that have automatic presentations and analysed the complexity of problems formulated over automatic structures. A large body of work on automatic structures has since emerged, covering themes ranging from the model checking problem to the problem of characterising definable relations [33, 46, 55, 60]. Furthermore, there has been work done on automata-based models of computation for specific mathematical structures such as those of the reals and complex numbers in the framework of the Blum–Shub–Smale model of computation, where arithmetic operations on the reals and complex numbers have unit costs [4, 53, 54], as well as for algebraic structures [45] and for general structures [31]. There have also been research efforts to adapt the classical theory of discrete automata to analog domains or continuous time; the reader is referred to the survey by Bournez and Pouly [6] and the references therein for more information on automata-based analog models of computation.

The key link between automatic functions and one-tape deterministic Turing machines, which forms the basis for the DARM model, is that each computation of an automatic function can be simulated by a one-tape deterministic Turing machine in linear time [27]. Using a proof technique very similar to that for establishing the latter result, one can show that each computation of a bounded automatic relation can be simulated by a one-tape nondeterministic Turing machine in linear time; this means there is a one-tape nondeterministic Turing machine such that for any pair (x, y) in the automatic relation, there is a computational path that starts with the input x (with the tape head starting at the first position of the input) and eventually reaches an accept state with the final tape content y , where the first position of y matches that of the input. Thus a computation by an automatic function or bounded automatic relation fulfills our criterion for a reasonable primitive step. In particular, while automatic functions can perform bit-wise operations, addition, subtraction and comparison, they are not powerful enough to carry out multiplication or concatenation.

Automatic register machines (ARM's) may be viewed as a generalisation of *addition machines*, a class of register machines introduced by Floyd and Knuth [30]. Addition machines are limited to six types of operations, each carried out in constant time: reading input, writing output, adding, subtracting, copying the contents of one register to another register and comparing the contents of any two registers. It turns out that many natural operations like multiplication can be done by addition machines in linear time [30]. These nice results carry over to automatic function complexity, as the natural numbers with addition and comparison is an automatic structure. Moreover, as mentioned earlier, an ARM can also perform each bit-wise Boolean operation in a single step [36, 37]. Another class of devices closely related to ARM's is that of the *finite-state transducers*. It may seem reasonable to use the class of *rational relations* – relations recognised by finite-state transducers – as a primitive operator in register machines. However, as mentioned earlier, Hartmanis and Simon [36, Theorem 2] showed that bitwise operations together with addition, subtraction, comparisons and concatenation allow to decide in polynomially many steps sets from PSPACE; as all these primitive functions and relations can be implemented using transducers, it follows that transducers as primitive steps are too powerful, see Section 4.1 for more explanations and related results. For this reason, we have chosen not to implement rational relations as primitive operators.

One might also propose combining register machines with transducers that are limited by the constraint that for every update function which is computed by the transducer, there is a constant c such that for all inputs x the output y is at most c symbols longer than x ; such a transducer would therefore satisfy a size constraint property similar to that of the automatic functions. But such transducers might still be regarded as being too powerful, for they allow nonregular sets to be recognised in constantly many steps. One such example is the set of all words that have as many 0's as 1's: for each $a, b \in \{0, 1\}$, let $f_{a,b}(x)$ be the output of a transducer T which reads a word x and outputs a symbol

b whenever it reads the symbol a and which discards all other symbols without replacement; T , which evidently satisfies the required size constraint property, accepts an input x if and only if $f_{0,1}(x) = f_{1,1}(x)$. On the other hand, as we will show in this work, the class of languages recognised in constantly many steps by DARM's belongs to the class of regular languages, see Theorem 5.1 below. Thus, the computational power of register machines with primitive steps computed by transducers whose output has length bounded by the input length plus an additive constant already exceeds that of register machines combined with automatic functions.

An alternative model of transducers that can indeed be simulated by deterministic Turing machines in polynomial time is that recently proposed by Kutrib, Malcher, Mereghetti, and Palano [51], where transducers are restricted to be length-preserving. Although this transducer model yields interesting results, it might not be adequate for modelling primitive steps in a register machine since length-preserving transductions are a real restriction and allow recognition of only a subclass of P. Automatic functions are more robust in that they can output a string longer than the input by an additive constant, and cover the full class P in polynomially many steps. For all these reasons, we believe that automatic functions and bounded automatic relations are an appropriate choice of primitive operators in register machines.

We also mention a few other models of computation whose computational powers appear to be similar to that of ARM's. First, a d -dimensional array automaton consists of an 1-way input tape and a d -dimensional regular array of cells, each of which contains a finite-state machine, and each step of computation consists of an input head move and an array head move together with a state transformation for each finite-state machine in a cell. Kosaraju [48] showed that context-free languages can be recognised by 1-dimensional arrays in n^2 time; using a similar algorithm, we show the same time complexity can be achieved by a deterministic ARM (Theorem 5.4). Kutrib and Malcher [49, 50] also investigated one-dimensional arrays of interconnecting finite automata (known as *cellular automata* or *cells*), showing in particular that cellular automaton transducers enjoy better time complexities compared to iterative array transducers, where the input is processed in real time and the output is computed in linear time. We note that any computation of an automatic function can be simulated by one computation step of a cellular automaton; it would therefore appear that cellular automata are just as powerful as deterministic ARM's.

The contributions of this work are as follows. First, we lay out the general definition of an ARM, and then define two specific types of such machines, namely DARM's and NARM's, as well as their respective complexity classes for arbitrary functions. Next, we compare the computational power of ARM's to that of other models of computation including transducers, Generative systems (g-systems) [59] and Iterated Uniform Finite-state Transducers [51]. The main goal of these comparisons is to justify the use of automatic functions and bounded automatic relations in combination with register machines; we wish to show, in particular, that it yields a sufficiently expressive model that does not give an unrealistic speed-up. Furthermore, we show that an open question of Kutrib, Malcher, Mereghetti, and Palano [51] on whether their model of Iterated Uniform Finite-state Transducers covers the whole class of context-sensitive languages has a positive answer; this can be shown using a proof quite similar to that for establishing that context-sensitive languages are in nondeterministic linear space.

We then proceed to the main results concerning DARM's, first establishing that those languages recognised by such machines in constant time and polynomial time coincide with the regular languages and P respectively. It is shown that various problems such as recognising context-free languages, recognising Boolean languages and the multi-sources connectivity problem have a lower time complexity in the DARM model in comparison to plain RAM models. We obtain almost tight complexity bounds for some of these problems. Owing to the tight connection between automatic functions and deterministic one-tape Turing machines, traditional crossing sequence arguments used to establish lower bounds on the running time of one-tape Turing machines apply to DARM's as well.

The paper then moves on to NARM's, in particular showing that the class of languages recognised by NARM's in polynomial time coincides with NP. The time complexity of various problems in the NARM model, including that of recognising non-palindromes, 3SAT and context-free languages are then studied. The key observation for results concerning NARM's is similar to that in the case of DARM's, that is, one often gets a speedup in time complexity that is nonetheless not too unreasonable.

We conclude by examining more powerful variants of ARM's – first, by allowing unbounded automatic relations; second, by padding the input at the start of the computation with “working space” that is larger than the original input length. In the second variant, if the input is padded with a string that is at least polynomially longer than the original input length, then one obtains a Polynomial-Size Padded Automatic Register Machine (PARM), and if the padding string is at least exponentially longer than the original input length, then an Exponential-Size Padded Auto-

matic Register Machine (ExpARM) is obtained. The increase in computational power as a result of padding the input is illustrated by the fact that QSAT – the Quantified Boolean Formula Satisfaction problem – can be decided by an ExpARM in linear time (as we will show in Section 7), as well as the fact that the class of languages decidable in polynomial time by ExpARM’s is equal to PSPACE. Moreover, even a polynomial-size padding can boost computational power: one can show that a $(c \log(n))$ -variable version of 3SAT can be decided by a PARM in polylogarithmic time. Finally, we prove that under the Exponential Time Hypothesis (ETH), the class of languages decided by PARM’s in sublinear time is strictly greater than that decided by DARM’s in sublinear time. Many results in this paper build on prior work on automatic functions and relations [27, 52, 62].

2. Preliminaries

Let Σ denote a finite alphabet. We consider set operations including union (\cup), concatenation (\cdot), Kleene star ($*$), intersection (\cap), and complement (\neg). Let Σ^* denote the set of all strings over Σ . Let the empty string be denoted by ε . For a string $w \in \Sigma^*$, let $|w|$ denote the length of w and $w = a_1 a_2 \dots a_{|w|}$ where $a_i \in \Sigma$ denotes the i -th symbol of w . We also use $w_{i,j}$ to denote the substring $a_i a_{i+1} \dots a_j$. Fix a special padding symbol $\#$ not in Σ . Let $x, y \in \Sigma^*$ such that $x = x_1 x_2 \dots x_m$ and $y = y_1 y_2 \dots y_n$. Let $x' = x'_1 x'_2 \dots x'_r$ and $y' = y'_1 y'_2 \dots y'_r$ where $r = \max(m, n)$, $x'_i = x_i$ if $i \leq m$ else $\#$, and $y'_i = y_i$ if $i \leq n$ else $\#$. Then, the convolution of x and y is $\text{conv}(x, y) = \binom{x'_1}{y'_1} \binom{x'_2}{y'_2} \dots \binom{x'_r}{y'_r}$ [62]. In other words, a convolution of two (or more) strings is formed by making pairs symbol-by-symbol with appropriate padding at the end for shorter strings. A relation $R \subseteq X \times Y$ is automatic iff the set $\{\text{conv}(x, y) : (x, y) \in R\}$ is regular. Likewise, a function $f : X \rightarrow Y$ is automatic iff the relation $\{(x, y) : x \in \text{domain}(f) \wedge y = f(x)\}$ is automatic [63]. An automatic relation R is bounded iff \exists constant c such that $\forall (x, y) \in R, \text{abs}(|y| - |x|) \leq c$. On the other hand, an unbounded automatic relation has no such restriction. Automatic functions and relations have a particularly nice feature as shown in the following theorem.

Theorem 2.1 ([40, 47, 62]). *Every function or relation which is first-order definable from a finite number of automatic functions and relations is automatic, and the corresponding automaton can be effectively computed from the given automata.*

A grammar is a 4-tuple (N, Σ, S, P) where N and Σ are disjoint sets of non-terminal and terminal symbols. $S \in N$ is the starting symbol and P is the set of rules. Different restrictions on the rules in P define different family of languages. Moreover, $l \Rightarrow r$ denotes that r can be derived from l in one step while $l \Rightarrow^* r$ denotes that r can be derived from l in zero or more steps. A context-free grammar (N, Σ, S, P) is a grammar in which every rule in P is restricted to be in the form of $A \rightarrow \alpha$ where $A \in N$ and $\alpha \in (N \cup \Sigma)^*$. A context-free language is a language generated by a context-free grammar. For simplicity, we assume that the language does not contain the empty string. A context-free grammar (N, Σ, S, P) is in Chomsky Normal Form if every rule in P is restricted either in the form of $A \rightarrow u$ or $A \rightarrow BC$ where $A, B, C \in N$ and $u \in \Sigma$. A context-free grammar (N, Σ, S, P) is in Greibach Normal Form if every rule in P is restricted in the form of $A \rightarrow \alpha$ where $A \in N$ and $\alpha \in \Sigma N^*$. In informal definition by Okhotin [56], a Boolean grammar (N, Σ, S, P) extends context-free grammars by allowing two additional types of rules in P : (1) $A \rightarrow \alpha \ \& \ \beta$ where $A \in N$ and $\alpha, \beta \in (N \cup \Sigma)^*$: this type of rule states that for any $w \in \Sigma^*$, $A \Rightarrow^* w$ if $\alpha \Rightarrow^* w$ and $\beta \Rightarrow^* w$; and (2) $A \rightarrow \neg \alpha$ where $A \in N$ and $\alpha \in (N \cup \Sigma)^*$: this type of rule states that for any $w \in \Sigma^*$, $A \Rightarrow^* w$ if $\alpha \not\Rightarrow^* w$. A Boolean language is a language generated by a Boolean grammar. A Boolean grammar (N, Σ, S, P) is in Binary Normal Form if every rule in P except $S \rightarrow \varepsilon$ is restricted either in the form of $A \rightarrow u$ or $A \rightarrow B_1 C_1 \ \& \ \dots \ \& \ B_m C_m \ \& \ \neg D_1 E_1 \ \& \ \dots \ \& \ \neg D_n E_n \ \& \ \neg \varepsilon$ ($m \geq 1, n \geq 0$) where $A, B_i, C_i, D_i, E_i \in N$ and $u \in \Sigma$.

This paper uses a standard definition of Turing machines by Arora and Barak [2] and focuses on single-tape Turing machines. We consider a one-dimensional tape that is infinite to the right, consisting of a sequence of cells. Each cell can hold a symbol from a finite set Γ called the alphabet of the machine. Note that the alphabet of a language Σ will be a subset of Γ in this Turing machine model. The whole (non-empty) content of the tape at one point of time is defined as the content of the machine. The tape is equipped with a tape head pointing a cell to read or write the symbol on that cell in a single step. The tape head is initially pointing to the leftmost cell and can move left or right in a single step. A Turing machine M is described as a triple $\langle \Gamma, Q, \delta \rangle$ containing:

1. The alphabet of the machine Γ . Let $\square \in \Gamma$ be a symbol denoting an empty cell.

2. A set Q consisting of the possible states of the machine M . Let $q_{start}, q_{halt_accept}, q_{halt_reject} \in Q$ denote the start state, halt then accept state, and halt then reject state respectively.
3. A transition function $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$ describing the rules of M in performing each step. The rule describes: if the machine is in some state $\in Q$ and the tape head reads some symbol $\in \Gamma$ then the machine will change its state to some state $\in Q$, the tape head will write some symbol $\in \Gamma$ in the cell it is pointing to, and then it either moves left, right, or stay.

An input is initially written on the leftmost cells of the tape and other cells will be empty. The time complexity of a machine on input x is the minimum number of steps needed to reach q_{halt_accept} or q_{halt_reject} .

The exact concept of register machine varies from author to author. This paper uses a definition of register machine which only has constantly many registers — each capable of storing a non-negative integer — therefore the indirect access operation ($R_i \leftarrow R_{R_j}$) is not used. We use basic arithmetic operations and comparisons ($<, >, =$) as in Floyd and Knuth [30] in addition to allowing constant values assignment and bitwise operations (as found in various programming languages like C, C++, and Javascript) by Hartmanis and Simon [36]. Hartmanis and Simon observed that including either concatenation or multiplication makes the concept to cover $PSPACE$ in polynomially many steps therefore both are not used by us. Furthermore, inputs were read explicitly in the program and writing was only possible in programs which compute functions. The statements are numbered (with line numbers) and conditional and unconditional *goto* are allowed. For computing characteristic functions of set, the explicit *halt_accept* and *halt_reject* operations are used. Readers are advised to check Cook and Reckhow [29] for the relation between register machine (using a RAM model) and Turing machine.

3. Automatic Register Machines

3.1. Formal Definition

Formally, we denote an Automatic Register Machine (ARM) by M and represent it as a quintuple $\langle \Gamma, \Sigma, R, Op, P \rangle$ by the following definition. Semantically, our model tries to generalize the standard register machines where the operations are no longer limited and the registers are able to store string. The operations, however, are restricted depending on the type of M : in the most basic type, Deterministic Automatic Register Machine (DARM) only allows Op to be the set of automatic functions where the parameters (and the output) are convoluted. Note that in standard register machine, all operations are automatic, thus register machine is a subset of DARM. More types of M , which are also more general than DARM, will be discussed later in Subsection 3.3 and Section 7.

Definition 3.1 (Automatic Register Machine). Automatic Register Machine (ARM) is a machine $M = \langle \Gamma, \Sigma, R, Op, P \rangle$ where M is specified by:

1. a finite set of registers $R = \{r_1, r_2, \dots, r_n\}$, each capable of storing a string, possibly empty, over a fixed machine alphabet Γ ;
2. a finite set of possible operations $Op = \{f_1, f_2, \dots, f_m\}$, each are restricted to some rules depending on the type of M ;
3. a program P consisting of a finite list of instructions in form of $LINE_NO : INSTR$.

The $INSTR$ takes one of the following forms:

- *read*(R_i): assigning the input string over a fixed input alphabet $\Sigma \subseteq \Gamma$ to register $R_i \in R$,
- *write*(R_i): writing the output string of register $R_i \in R$,
- $R_i \leftarrow w$: assigning any string w over a fixed input alphabet $\Sigma \subseteq \Gamma$ to register $R_i \in R$,
- $R_i \leftarrow R_j$: assigning the value of register $R_j \in R$ to register $R_i \in R$,
- $R_i \leftarrow F(R_1, R_2, \dots, R_k)$: assigning the result of operation $F \in Op$ with parameters $R_1, R_2, \dots, R_k \in R$ to register $R_i \in R$,
- *goto*($LINE_NO$): go to the instruction in line $LINE_NO$ (i.e. unconditional jump),
- *if* $F(R_1, R_2, \dots, R_k)$ *then goto*($LINE_NO$): if the result of operation $F \in Op$ with parameters $R_1, \dots, R_k \in R$ is true, then go to the instruction in line $LINE_NO$ (i.e. conditional jump);

- *halt* (or *halt_accept* or *halt_reject*): halting the program (and accept or reject) the input.

Similar to register machines, the program runs sequentially from the first line number to the next one unless jumped either conditionally or unconditionally, while performing the instructions along the runtime. A single step is defined as performing one line of instruction. Initially, all registers store empty strings. After several steps, the program terminates after halted by *halt*. If one wants to recognize a language, both halt states (i.e. *halt_accept* or *halt_reject*) can be used. Alternatively, one can in deterministic model write a bit output of ‘1’/‘0’ then halt to denote the acceptance/rejection. If one wants to write a function value as an output, both halt states are also allowed especially for nondeterministic models. This is because the whole computation can be rejected when it turns out in guess-then-verify steps that some info had been guessed wrongly and thus leads to rejection of the computation. A partial constant-valued function can also be done in this way where the output might be skipped (by *halt_reject*) as only the output on members of some language L is considered.

Remark. Note that reading in a register will erase the previous content of the register. Thus if you have a register machine which sums up all integer inputs until it reads 0, then this register machine needs at least two registers, as otherwise the reading of a new input would erase the old content. One register is only possible if one reads in the first statement a convolution of all inputs (or in other longer format like the concatenation of all inputs).

Example 3.2. We give an illustration of an Automatic Register Machine M recognizing $L_{two} = \{0^{2^i} \mid \text{for some non-negative integer } i\}$. Let $M = \langle \Gamma, \Sigma, R, Op, P \rangle$, $\Gamma = \{0, 1\}$, $\Sigma = \{0\}$, $R = \{r_1, r_2\}$, $Op = \{is_empty, is_one, is_odd, divide_by_two\}$, and

```
P = { 01 : read(r1)
      02 : if is_empty(r1) then goto(09)
      03 : if is_one(r1) then goto(08)
      04 : if is_odd(r1) then goto(09)
      05 : r2 ← divide_by_two(r1)
      06 : r1 ← r2
      07 : goto(03)
      08 : halt_accept
      09 : halt_reject }
```

with the following definition of the operations:

- *is_empty*(w) returns true iff w is an empty string,
- *is_one*(w) returns true iff w has exactly one occurrence of symbol ‘0’,
- *is_odd*(w) returns true iff w has odd occurrences of symbol ‘0’,
- *divide_by_two*(w) returns w' where w' is equal to w after modifying every second occurrence of ‘0’ to ‘1’ (e.g. if $w = 10010110$ then $w' = 10110111$.)

It is not hard to see that each operations can be constructed with finite automata thus they are automatic. Note that for *divide_by_two* in particular, the input w and the output w' are convoluted. As Op only consists of automatic functions, therefore M satisfies the requirement as a DARM. Let n be the occurrence number of ‘0’ in r_1 , which is initially equal to the length of the input string. The proof of correctness of the program comes from keep dividing n by 2 until n is either 1 — then accept because n is 2^i for some non-negative integer i — or other odd number — then reject. The program will halt and either accept or reject the input in no more than $O(\log n)$ steps.

3.2. Alternative Representation with Single Operation

The representation of an ARM $M = \langle \Gamma, \Sigma, R, Op, P \rangle$ could be simplified by rewriting each instructions in P to a form of $LINE_NO : F(R_1, R_2, \dots, R_k)$ for some operations F and $R_1, R_2, \dots, R_k \in R$. The operation F may be a member of Op or also a “keyword” operation (e.g. *read*, *goto*, *halt*). This new form resembles a neater operation format in register machine such as $INC(R_i)$, $ADD(R_i, R_j)$, and $IFEQUAL(R_i, R_j, LINE_NO)$; which are all automatic functions. We can then simplify the machine representation even more: the $LINE_NO$ could also be inserted as a parameter in F as well as all unused registers — which are eventually ignored when the operation is performed. Thus, the instructions in P could be fully rewritten as $F(LINE_NO, r_1, r_2, \dots, r_n)$ where F is either a member of Op or keyword operations.

Moreover, based on Theorem 2.1, the first-order definable function from automatic functions is also automatic. As members of Op and keyword operations are all automatic, thus we can combine them as a single operation $G(LINE_NO, r_1, r_2, \dots, r_n)$ where G will run the specific operations based on the $LINE_NO$. After one step, G will update the new $LINE_NO$ as well as some registers. The program P then has this single operation G alone which will be applied iteratively from $G(01, \epsilon, \epsilon, \dots)$, where 01 denotes the first line number, until it halts. Therefore, we can replace P by G and finally ARM can be re-represented as a triple $M = \langle \Gamma, \Sigma, G \rangle$. One single step is redefined as applying the operation G once.

Definition 3.3 (Alternative Representation of Automatic Register Machine). An Automatic Register Machine $M = \langle \Gamma, \Sigma, R, Op, P \rangle$ can also be re-represented as $M = \langle \Gamma, \Sigma, G \rangle$ where G is a single operation combining all instructions in P with a convolution of the line number and all registers in R as its parameter.

Example 3.4. We can rewrite the ARM in Example 3.2 recognizing L_{two} by letting $M = \langle \Gamma, \Sigma, G \rangle$, $\Gamma = \{0, 1\}$, $\Sigma = \{0\}$, and G with the following definition:

$$G(LINE_NO, r_1, r_2) = \begin{cases} G(02, input, r_2) & \text{if } LINE_NO = 01 \\ G(09, r_1, r_2) & \text{if } LINE_NO = 02 \text{ and } r_1 = \epsilon \\ G(03, r_1, r_2) & \text{if } LINE_NO = 02 \text{ and } r_1 \neq \epsilon \\ G(08, r_1, r_2) & \text{if } LINE_NO = 03 \text{ and '0' occurs 1 time in } r_1 \\ G(04, r_1, r_2) & \text{if } LINE_NO = 03 \text{ and '0' does not occur 1 time in } r_1 \\ G(09, r_1, r_2) & \text{if } LINE_NO = 04 \text{ and '0' occurs odd times in } r_1 \\ G(05, r_1, r_2) & \text{if } LINE_NO = 04 \text{ and '0' occurs even times in } r_1 \\ G(06, r_1, div2(r_1)) & \text{if } LINE_NO = 05 \\ G(07, r_2, r_2) & \text{if } LINE_NO = 06 \\ G(03, r_1, r_2) & \text{if } LINE_NO = 07 \\ halt_accept & \text{if } LINE_NO = 08 \\ halt_reject & \text{if } LINE_NO = 09 \end{cases}$$

where *input* is an input string and $div2(w)$ is the result of modifying w by changing every second occurrence of ‘0’ to ‘1’. Note that G is an automatic function. The machine will start with $G(01, \epsilon, \epsilon)$ and eventually will halt in the same number of steps as the original machine which is in $O(\log n)$ steps where n is the length of the input.

Remark. Note that it is also possible to represent all registers in a single (big) register which is simply a convolution of all registers. Moreover, while both representations of ARM are equivalent, the original representation is more natural and mainly used for the sake of readability. However, the alternative representation may also be used when proving some findings especially when they are related to Turing machine computation.

3.3. Basic Types of ARM and Relations to Turing machine

We now first define the two basic types of ARM which are our main focus throughout this paper. Other types will be discussed later in Section 7.

Definition 3.5 (Deterministic Automatic Register Machine). A Deterministic Automatic Register Machine (DARM) is an Automatic Register Machine $M = \langle \Gamma, \Sigma, R, Op, P \rangle$ where Op is restricted to automatic functions. Alternatively, a DARM is an Automatic Register Machine $M = \langle \Gamma, \Sigma, G \rangle$ where G is an automatic function.

Definition 3.6 (Nondeterministic Automatic Register Machine). A Nondeterministic Automatic Register Machine (NARM) is an Automatic Register Machine $M = \langle \Gamma, \Sigma, R, Op, P \rangle$ where Op is restricted to bounded automatic relations. Alternatively, a NARM is an Automatic Register Machine $M = \langle \Gamma, \Sigma, G \rangle$ where G is a bounded automatic relation.

In NARM, the bounded version of automatic relations is preferred to avoid the ability to guess a huge witness in one step by unboundedness, which may recognize a larger complexity class as discussed in Subsection 7.1. Furthermore, the two concepts differ when act as functions: a nondeterministic bounded automatic function can only have polynomial-sized output, while for the unbounded one it can have exponential-sized output e.g. a concatenation of all n -digits binary numbers separated by symbol ‘2’. We may now define the complexity class of the models.

Definition 3.7 (Deterministic Automatic Register Machine Complexity). Let $f(n)$ be any function in the input size n . $DAL[f(n)]$ is the class of languages accepted by DARM in $O(f(n))$ steps.

Definition 3.8 (Nondeterministic Automatic Register Machine Complexity). Let $f(n)$ be any function in the input size n . $NAL[f(n)]$ is the class of languages accepted by NARM in no more than $O(f(n))$ steps.

By this definition, L_{two} in Example 3.2 is in $DAL[\log n]$. Similar to nondeterministic Turing machine, a string in NARM is accepted iff at least one of the nondeterministic computation path halt with accept. Note that all accepting paths must be done in no more than $O(f(n))$ steps for the language being in $NAL[f(n)]$. Next, the following two theorems give the basic connection between automatic functions and bounded relations to Turing machine computations.

Theorem 3.9 ([26]). *Let f be an automatic function. Then there is a deterministic one-tape Turing machine which computes f in $O(n)$ steps and its head position starts and finishes in the original starting position i.e. the leftmost cell.*

Theorem 3.10 ([27](Based on Theorem 2.4)). *Any step by a bounded automatic relation R can be simulated by a one-tape nondeterministic Turing machine in $O(m)$ steps where m is the length of the longest current register content and m is bounded by $n + O(t)$ where t is the number of Automatic Register Machine steps done after the machine reads the input of length n .*

The core consequence of Theorem 3.9 is that we can translate a DARM to an equivalent deterministic Turing machine which runs linear-time longer. Note that the linear here is subjected to the length of the register. To show this, consider any DARM $M = \langle \Gamma, \Sigma, G \rangle$. The parameters of G are convoluted and this string could be pasted into a Turing tape. In a single step of M , the parameters are updated by applying an automatic function G on it. By the theorem, this can be simulated by the Turing machine in linear step of the length of the parameters, thus Q.E.D. Moreover, it is also possible to have the following converse: we can translate a deterministic Turing machine M which runs in $O(f(n))$ to an equivalent DARM which runs in $DAL[f(n)]$. The proof is quite similar where now the content of the Turing tape is pasted as a single register in DARM and show that each Turing machine step is an automatic function. Furthermore, the results can extended to NARM (with nondeterministic Turing Machine) as a consequence of Theorem 3.10 in similar manner. Formal proof of these results can be found later in Section 5 and 6.

4. ARM in Comparison to Other Models

This section is solely dedicated to justify that Automatic Register Machine is an adequate computation model which has primitive operations that are expressive and powerful yet not giving an unrealistic speed-up compared to other models.

4.1. Deterministic Polynomial Time Model with Transducers as Primitive Operations to Solve PSPACE-Complete

The computation model with transducers as primitive operations is similar to ARM with the only difference that for $M = \langle \Gamma, \Sigma, G \rangle$, G is a transducer. Informally, transducers allow one of the symbol to be ϵ when “convoluted” and such “convolution” can be recognized by an automaton (called Mealy machine), thus it is a superset of automatic functions and relations. This model is close to a language generating device called an Iterated Sequential Transducer by Bordihn, Fernau, Holzer, Manca and Martín-Vide [5] albeit the complexity is usually measured by its number of states (i.e. state complexity).

We show that the computation model with transducers as primitive operations gives an extreme speed-up by solving a PSPACE-complete problem in only polynomial number of steps. This can be deduced from our result in Theorem 7.9 later that a DARM with an access to exponential-length padding (i.e. allowing its working space to be exponential larger) can solve *Q SAT* in $O(n)$ steps, and one can simulate the exponential padding by a transducer which runs through a loop $p(n)$ times and each time doubles the length of y by mapping y to a string twice the length (to get a padding of length at least $2^{p(n)}$). Hartmanis and Simon [36] also show that $NPSPACE \subseteq$ polynomial steps on RAM with concatenation. They write later that they need only bitwise operations, concatenation, and division by 2; not even substring operation. All these operations can be realised by transducers, therefore their result also shows that polynomial number of transducer steps contains PSPACE. Though the exact construction is a bit technical, the result is not that important in the context of this paper therefore the working out of details and the verification is left to the reader. As a comparison, such blow-up will be unlikely to happen in DARM as based on Theorem 5.2 later, if 3SAT or *Q SAT* are in $DAL[poly(n)]$ then they are also in P .

4.2. ARM and G-systems

Generative systems (g-systems), introduced by Rovin [59] (stemming from Γ -systems by Wood [66]) as a natural model for language generating devices, depend on various types of transducers. We first define the g-systems.

Definition 4.1 (Generative systems [59]). A generative system (g-system) is a 4-tuple $G = \langle N, \Sigma, M, S \rangle$ where N and Σ are disjoint set of non-terminal and terminal symbols, $S \in N$ is the starting symbol and M is a transducer with $M(w) = \emptyset$ for each $w \in \Sigma^+$. M is a 6-tuple $(K, N \cup \Sigma, N \cup \Sigma, H, q_0, F)$, where K is a finite set of states, q_0 is the initial state, $F \subseteq K$ is a set of accepting states and H is a finite subset of $K \times (N \cup \Sigma) \times (N \cup \Sigma)^+ \times K$. A computation of M is a word $h_1 \dots h_n \in H^+$ such that if, for each $i \in \{1, 2, 3, 4\}$, p_i is the homomorphism on H^* for which $p_i(x_1, x_2, x_3, x_4) = x_i$, then $p_1(h_1) = q_0$, $p_4(h_n) \in F$ and $p_1(h_{i+1}) = p_4(h_i)$ for $1 \leq i \leq n-1$. A computation of M of length n may be interpreted as a sequence of n configurations such that the end state of each of the first $n-1$ configurations is the start state of its succeeding configuration; furthermore, the first configuration starts with the initial state q_0 and the end state of the last configuration is an accepting state. Let \prod_M denote the set of all computations of M . For each language $L \subseteq (N \cup \Sigma)^*$, $M(L)$ is defined to be $p_3(p_2^{-1}(L) \cap \prod_M)$; in other words, for every $w \in L$, if $(x_1^1, x_2^1, x_3^1, x_4^1)(x_1^2, x_2^2, x_3^2, x_4^2) \dots (x_1^n, x_2^n, x_3^n, x_4^n)$ is a computation such that $x_2^1 x_2^2 \dots x_2^n = w$, then $x_3^1 x_3^2 \dots x_3^n \in M(L)$.

As is usually defined for grammars, the language generated by a g-system G consists of all $w \in \Sigma^*$ such that there is a finite sequence $S = v_1, v_2, \dots, v_k = w$ of words over $N \cup \Sigma$ with $v_{i+1} \in M(\{v_i\})$ for all $i \in \{1, \dots, k-1\}$. We note that the transducer M in Definition 4.1 maps subsets of $(N \cup \Sigma)^*$ to subsets of $(N \cup \Sigma)^*$ and is thus nondeterministic (otherwise, the language generated by G consists of at most one word).

As a generating language device, one step of derivation in the g-system is not using a set of rules P as in grammar but the transducer M . One step of $l \Rightarrow r$ is valid if and only if $r \in M(l)$. Various families of g-systems are differentiated by different types of their transducers. If the transducer does not allow deletion, then it is called ϵ -free g-system, else we can call it a g-system with deletion. If the transducer is some type of directed acyclic graph and either one goes forward in it or one stays on some state and copies and pastes the current symbols, then it is called a sequential g-system, else it is called a parallel g-system. It is not hard to see the similarity between g-system if used as a language accepting device with our model apart from the different types of operations. We list out the connection between them as follows.

1. If an ϵ -free g-system accepting (or generating) L , there exists a nondeterministic automatic function which also recognizes L such that if the g-system generates some word w in m steps then w can be accepted in $NAL[m]$ for some m . However, if NARM measures the time complexity in a weak-measure i.e.

$$\max\{\min\{\text{acceptance time for } x \text{ on a nondeterministic path}\} : |x| = n, x \in L\},$$

then such NARM might be faster as some languages are in $O(\log \log n)$ steps, as shown later in Theorem 6.4, while a g-system can in each step increase the length only by a constant factor and needs at least $\Omega(\log n)$ steps.

2. G-systems with deletion is more powerful than our model as they can recognize context-sensitive language in polynomial steps. The proof is based on the symbol duplication, which allows solving *PS PACE* in polynomial steps [36].

3. Sequential g-systems are weaker than the parallel one as they do not have the ability to duplicate a symbol. In fact, they are also weaker than our DAL model. The fact that sequential g-systems work symbol-by-symbol with a limitation of copy-pasting on some positions shows that it is actually a restrictive automatic function steps.

We conclude our connection with the following evaluation. The essence of this is that sequential g-systems do not allow too powerful steps but are slower than our deterministic model, while parallel g-systems with deletion allow speed-ups which incorporate larger complexity-classes into the class of NP in polynomial deterministic steps. Even the ϵ -free g-system is comparable to our nondeterministic model. So our model may somehow sit in the middle of the two by accommodating the adequate speed-up while avoiding unrealistic blow-up.

4.3. Iterated Uniform Finite-State Transducers and its Open Problem

Kutrib, Malcher, Mereghetti, and Palano [51] introduced Iterated Uniform Finite-State Transducers as a model of computation preserving the polynomial time. This model uses length-preserving transducers as primitive operations and might be a real restriction as allowing only a subclass of P for polynomial time recognition. As automatic functions can have output a constant longer than the input in each step, the overall model using automatic functions is not subjected to the length-limitation constraint and covers all polynomial-time decidable languages when recognizing languages in polynomially many steps, therefore more powerful. In a side note, the authors left it as an open problem whether their model (in arbitrary time) can cover the whole of context-sensitive languages. We prove the answer is positive by the following proof.

Roughly speaking, an Iterated Uniform Finite-State Transducer is a finite-state transducer that processes the input in multiple sweeps; in the first sweep, it reads the input followed by an endmarker and outputs a word; in subsequent sweeps, it reads the output word of the previous sweep and outputs a new word. Formally, a nondeterministic iterated uniform finite-state transducer is a system $T = \langle Q, \Sigma, \Delta, q_0, \triangleleft, \delta, F_+, F_- \rangle$, where Q is the set of internal states, Σ is the set of input symbols, Δ is the set of output symbols, $q_0 \in Q$ is the initial state, $\triangleleft \in \Delta \setminus \Sigma$ is the endmarker, $F_+ \subseteq Q$ is the set of accepting states, $F_- \subseteq Q \setminus F_+$ is the set of rejecting states and $\delta : Q \times (\Sigma \cup \Delta) \mapsto 2^{Q \times \Delta}$ is the transition function (which is total on $(Q \setminus (F_+ \cup F_-)) \times (\Sigma \cup \Delta)$); the endmarker \triangleleft is output only if it is read. (One observes from this definition that any word emitted by an Iterated Uniform Finite-State Transducer has the same length as the input word.) T halts whenever the transition function is undefined or T enters an accept or reject state at the end of a sweep. A computation of the nondeterministic iterated uniform finite-state transducer T on input $w \in \Sigma^*$ is a sequence of words $w_1, \dots, w_i, w_{i+1}, \dots$ such that $w_1 \in T(w\triangleleft)$ and $w_{i+1} \in T(w_i)$; the computation halts if there is an $r \geq 1$ such that T halts on w_r . The set of possible words output by T in a complete sweep on input $w \in (\Sigma \cup \Delta)^*$ is denoted by $T(w)$. The input word w is accepted by T iff all computations on w halt and at least one computation halts in an accepting state; w is rejected by T iff all computations on w halt and none of the computations halt in an accepted state.

The proof is quite similar to the proof of showing that context-sensitive languages are in nondeterministic linear space. The essential idea is to construct a transducer that reads and outputs a convolution of three components: first, the input word (whose membership in the given context-sensitive language is to be tested); second, a “work-tape” for simulating context-sensitive grammar derivations; third, a counter for measuring the length of a derivation. At the beginning, let us fix a constant-sized ordered set of digits (in the alphabet) for counting e.g. $\{0, 1, \dots, 9\}$, such that this set has size equal to the total number of terminals and non-terminals plus one (this is to ensure that whenever there is a repetition-free derivation of the word, the process does not run out of the steps to be counted, but can complete the derivation). Here, 9 acts as the largest digit but it does not mean that there are exactly 10 digits. The input/output alphabet of the transducer consists of all convolutions of the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a (resp. b) either represents a letter in the alphabet of the input grammar or is a blank symbol, and c is either a digit in the counting alphabet or a blank symbol. The first run of the transducer translates the input to a same-length-word in the tape alphabet which is a convolution of the input, the counter “00...0” and the start symbol S (using blank symbols as appropriate to ensure that all three components are of equal length). Now for any given n , one may determine from the input grammar the maximum length of a repetition-free derivation of a word of length n . The counter starts from “0000...” and goes increasingly to “9999...” in some sufficiently large alphabet with a word as long as the input and forces a reject, if the input has not been derived in the corresponding maximum possible number of steps in a repetition-free derivation. If a derivation of a symbol A becomes, under context, a word w of length at least 1, the transducer will nondeterministically guess the distribution of its symbols over the position of A so that they go into the target position. If there is no non-terminal

in the word, we just simply increment the counter. The machine will accept if the generated word and the given word are equal (independently of the counter number). The machine will reject if the last counter (“9999...”) is reached, thus if there is any incorrect guess then the derivation will be stuck and eventually be rejected.

Finally, we remark that every automatic function which maps inputs to outputs of the same length can be computed by a transducer which is length-preserving. The reason is that an automatic function is verified by a synchronous nfa which reads one symbol of each the input and the output per cycle; now turning the “reading of the symbol” into “writing of a symbol” where the successor state depends in the same way as before on the two symbols processed (the input and the output symbol), turns the nfa into a transducer which in each cycle reads one input symbol and writes one output symbol. At the input-end-symbol, this nfa indicates that the sweep was successful iff the nfa on the sequence of all the inputs processed and outputs generated would say that the (input,output)-pair is in the automatic relation. This way one can translate the nfa recognising the automatic relation into a transducer given as a Mealy machine which computes a multivalued function of the possible next steps in the derivation from the given step (with the corresponding maintenance of the counter and comparison with the input-word).

5. Key Results on DARM

As we can translate a DARM to an equivalent deterministic Turing machine which runs linear-time longer, theoretically it may be — but not always — possible for some languages to be recognized by DARM in up to linear-time fewer steps than an established algorithm for such language in standard Turing machine steps. This section highlights several key results of those interesting languages in DAL class.

5.1. Deterministic Polynomial ARM and Lower Bound Technique

We start with the following theorem which is a direct consequence of authors’ result in another work [32].

Theorem 5.1. $DAL[1] = Regular$.

Proof. Let P_k be the class of languages decided by a k -step DARM program. We will show that languages in P_k are regular over constant k by induction. For $k = 1$, any language $L \in P_1$ must be regular as the automatic function step is a Deterministic Finite Automaton (DFA) recognizing L . Now assume that all languages in P_k are regular. For any language $L \in P_{k+1}$ and its DARM $M = \langle \Gamma, \Sigma, G \rangle$, let g_k be the content (parameters) of G after k -th step. One can construct a Nondeterministic Finite Automaton (NFA) guessing g_k then verify with the automatic function step (i.e. a DFA) on the $(k + 1)$ -th step whether to accept or reject the string. This whole process can be replaced by another DFA of exponential-larger size by processing the convolution of the content of G generated from the first step to the $(k - 1)$ -th step. Note that the program now consists of k steps and the last step is done by a DFA therefore $L \in P_{k+1}$ is regular. \square

Mentioned briefly in Subsection 3.3 and 4.1, we can also prove the following equality.

Theorem 5.2. $DAL[poly(n)] = P$.

Proof. To show $DAL[poly(n)] \subseteq P$, note that in a single step of an automatic function the space used grows by a constant, thus each step update of the registers can be done in polynomial time in the length of the register string. For $P \subseteq DAL[poly(n)]$, we refer to the result from Hartmanis and Simon [36] where P can be covered by polynomial many steps of basic operations which are in fact automatic functions without any use of indirect addressing. \square

We also give the definition of a crossing sequence by Hennie [38] as a tool to compute a lower bound on the computational complexity of a language and then connect it in the context of the execution of automatic relations in DARM.

Definition 5.3 (Crossing Sequence [38]). Given a deterministic one-tape Turing Machine and an input, a crossing sequence of the i -th cell is a sequence of states whenever the head of Turing Machine crosses between the i -th and $i + 1$ -th cells along the computation. Informally, a crossing sequence of the i -th cell describes how the Turing Machine carries the information between the left and right cells separated by the border of i -th and $i + 1$ -th cell. The total computation time is the sum of the lengths of all crossing sequences.

The connection of crossing sequences to the execution of automatic relations in a DARM $M = \langle \Gamma, \Sigma, G \rangle$ is the following. Each step in the execution of G can be simulated by a double-pass of a deterministic one-tape Turing machine. The machine passes first forward and then backward over the content of the register to replace the convolution (of r and r' , where r' is the content of r after the execution of G) by their updated versions in all cells. As the state may change at the end of each pass, hence the crossing sequence becomes only at most two symbols (states) longer in one step update. In general, a program line number update in DARM can also be simulated similarly. Thus, the length of a $\text{DAL}[f(n)]$ -computation gives a crossing sequence of length $O(f(n))$. So a lower bound on the crossing sequence length gives a lower bound on the DAL-computation too.

5.2. Context-Free Languages and Boolean Languages in DARM

We first mention the CYK algorithm, discovered independently by Cocke, Kasami, and Younger [1, 44, 67]: a parsing algorithm for context-free languages based on a bottom-up approach of dynamic programming. Given a context-free language L in its grammar $G = (N, \Sigma, S, P)$ in Chomsky Normal Form, the algorithm tries to parse an input string $s = s_1 \cdots s_n$ with length n using CYK triangular matrix. Here is an example of the CYK triangular matrix for $|s| = 4$.

			$\alpha_{1,4}$		
		$\alpha_{1,3}$	$\alpha_{2,4}$		
	$\alpha_{1,2}$	$\alpha_{2,3}$	$\alpha_{3,4}$		
	$\alpha_{1,1}$	$\alpha_{2,2}$	$\alpha_{3,3}$	$\alpha_{4,4}$	
$s =$	s_1	s_2	s_3	s_4	

The variables $\alpha_{i,j}$ will be the set of all non-terminals generating substring $s_i \cdots s_j$ i.e. $\alpha_{i,j} = \{A \mid A \in N, A \Rightarrow^* s_{i,j}\}$. The algorithm starts with the bottom row computing $\alpha_{i,i} = \{A \mid A \rightarrow s_{i,i} \in P\}$. Iteratively, $\alpha_{i,j}$ can be efficiently computed using its (lower-)left diagonal: $\alpha_{i,k}$ where $k < j$; and (lower-)right diagonal: $\alpha_{k,j}$ where $k > i$. This is captured by the matrix such that $\alpha_{i,j} = \bigcup_{i \leq k < j} \alpha_{i,k} \times \alpha_{k+1,j}$ where $X \times Y = \{A \mid B \in X, C \in Y, A \rightarrow BC \in P\}$. Finally, $s \in L$ iff $S \in \alpha_{1,n}$. The algorithm clearly runs in $O(n^3)$ in modern RAM model and translates to an $O(n^4)$ -algorithm on a one-tape deterministic Turing machine. We show that the problem needs less number of steps in DARM.

Theorem 5.4. *Given a context-free language L in its grammar $G = (N, \Sigma, S, P)$ in Chomsky Normal Form, recognizing an input with length n in L can be done in $\text{DAL}[n^2]$.*

Proof Sketch. The idea is similar to using 1-dimensional arrays in solving context-free languages in quadratic steps by Kosaraju [48]. We simulate CYK algorithm but for each variables $\alpha_{i,j}$ we only need constant steps to compute $\bigcup_{i \leq k < j} \alpha_{i,k} \times \alpha_{k+1,j}$ which is done in parallel. We present the DARM algorithm as follows.

For $0 \leq k < n$, let the k -th layer of CYK triangular matrix be the $(k+1)$ -th row from the bottom i.e. the set of $\alpha_{i,i+k}$ for $1 \leq i \leq n-k$. Assign two registers R_l and R_r in DARM to be responsible of left and right diagonals respectively. The DARM algorithm will run in n iterations and process layer-by-layer from 0-th to $(n-1)$ -th. When it finished processing the k -th layer, the algorithm guarantees the following properties:

1. Started with some padding, R_l will be a concatenation of all left diagonals up to the k -th layer (i.e. left-diagonals of $\alpha_{i,j}$ for $i \leq j \leq \min\{n, i+k\}$) written from left-to right and separated by some separator.
2. R_r will be a concatenation of all right diagonals up to the k -th layer (i.e. right-diagonals of $\alpha_{i,i}$ for $1 \leq i \leq k$ and $\alpha_{i,j}$ for $i \leq j \leq \min\{n, i+k\}$) written from left-to right and separated by some separator, followed by some padding.
3. The separators in R_l and R_r are lined up in such a way that $\alpha_{1,1}$ in R_l is matched with $\alpha_{2,2+k}$ in R_r . In other words, full “blocks” of diagonals are aligned to compute $\bigcup_{i \leq k < j} \alpha_{i,k} \times \alpha_{k+1,j}$ in the next iteration.

Below is the illustration of above properties when processing a string of length 4 for the first three iterations. A dashed line denote a separator symbol. Note that each of possible subsets of N of G can be denoted by a single symbol in Γ of DARM M thus each $\alpha_{i,j}$ is represented as a single symbol.

$$0: \begin{array}{l} R_l \parallel \# \mid \alpha_{1,1} \mid \alpha_{2,2} \mid \alpha_{3,3} \mid \alpha_{4,4} \\ R_r \parallel \alpha_{1,1} \mid \alpha_{2,2} \mid \alpha_{3,3} \mid \alpha_{4,4} \mid \# \end{array}$$

$$\begin{array}{l}
1 : \quad \begin{array}{c} R_l \\ R_r \end{array} \left\| \begin{array}{cccc|cccc|cccc} \# & \# & \# & \alpha_{1,1} & \alpha_{1,2} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{3,3} & \alpha_{3,4} & \alpha_{4,4} \\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{3,3} & \alpha_{3,4} & \alpha_{4,4} & \# & \# & \# \end{array} \\
2 : \quad \begin{array}{c} R_l \\ R_r \end{array} \left\| \begin{array}{cccc|cccc|cccc|cccc} \# & \# & \# & \# & \# & \# & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & \alpha_{3,3} & \alpha_{3,4} & \alpha_{4,4} \\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{2,2} & \alpha_{1,3} & \alpha_{2,3} & \alpha_{3,3} & \alpha_{2,4} & \alpha_{3,4} & \alpha_{4,4} & \# & \# & \# & \# & \# & \# \end{array}
\end{array}$$

By the construction, the final iteration will eventually compute $\alpha_{1,n}$ thus the membership of s is tested by checking whether $S \in \alpha_{1,n}$. To process 0-th layer in particular, it can be done in linear steps by first transforming s_i to $\alpha_{i,i}$, adding one padding symbol in the beginning of R_l and at the end of R_r , then adding n many times the separators.

To process k -th layer from $(k-1)$ -th layer, one can in a single automatic step compute all variables in the k -th layer by looking at the diagonals formed by the pairs in R_l and R_r — ignoring incomplete blocks of diagonals. Note that $\bigcup_{i \leq k < j} \alpha_{i,k} \times \alpha_{k+1,j}$ could be computed in that single step as there is an automaton recognizing the union of subsets of N given that each $\alpha_{i,k}$ and $\alpha_{k+1,j}$ are already convoluted therefore the automaton can also compute a subset $\alpha_{i,k} \times \alpha_{k+1,j}$. After reading the separator, the variable $\alpha_{i,i+k}$ can be formed and convoluted to the separator (or stored in the third register R_t) and it continues to form the next variables. After all variables in the k -th layer is formed, the algorithm needs to put it on the correct positions: to the right of its left diagonal and to the left of its right diagonal. As its diagonals are directly before the separator, inserting it to the correct position can also be done in constant steps. Lastly, some paddings are added in the beginning of R_l and at the end of R_r such that $\alpha_{1,1}$ in R_l is matched with $\alpha_{2,2+k}$ in R_r . Adding one padding needs one step, in addition, some separators might be added as well.

Finally, to compute the total complexity time, one can observe that all operations are constant and could be reduced to how many times it inserts a symbol into R_l and R_r . As the length of both will be equal to the size of CYK triangular matrix, therefore the algorithm runs in $DAL[n^2]$. \square

We can further extend the proof in Theorem 5.4 for Boolean languages.

Theorem 5.5. *Given a Boolean language L in its grammar $G = (N, \Sigma, S, P)$ in Binary Normal Form, recognizing an input with length n in L can be done in $DAL[n^2]$.*

Proof. The CYK algorithm can be extended to parse Boolean grammars by Okhotin's work [56]. The algorithm is exactly the same with some modifications on computing $\alpha_{i,j}$ as now it allows conjunctions and negations. The algorithm also runs in $O(n^3)$ in modern RAM model. In DARM, we can only modify the automaton when computing $\alpha_{i,j}$. Note that set operations such as intersection and complement can be done by introducing an additional Boolean flag and that will only multiply the number of states in the automaton by a factor of constant. Therefore the complexity will still be $O(n^2)$. \square

Remark. Note that the runtime complexity in Theorem 5.4 and 5.5 still translates to an $O(n^4)$ -algorithms on a one-tape Turing machine due to the length of the register becoming $O(n^2)$.

5.3. Multi-Sources Connectivity Problem in DARM

We define multi-source connectivity problem as the following.

Definition 5.6 (Multi-Source Connectivity Problem). Given a directed graph $G = (V, E)$ with n vertices $\{1, 2, \dots, n\}$, and some vertices as sources $S \subseteq V$, Multi-Source Connectivity Problem is to find all vertices reachable from at least one of the vertices in S .

We then define a string format of a directed graph $G = (V, E)$ with n vertices $\{1, 2, \dots, n\}$ as the following. Note that '[' is a separator symbol.

$$| v_1 | e_{1,1} \quad e_{1,2} \quad \cdots \quad e_{1,n} | v_2 | e_{2,1} \quad e_{2,2} \quad \cdots \quad e_{2,n} | \cdots | v_n | e_{n,1} \quad \cdots \quad e_{n,n}$$

This format represents an adjacency matrix of G which is written row-by-row. For each row, it starts with the vertex number $v_i = i$ followed by its neighbours: $e_{i,j} = j$ if $(i, j) \in E$, otherwise $e_{i,j} = 0$. Note that v_i and $e_{i,j}$ are written in binary form thus each have length $\log n$. In this way, the string format of G with n vertices has total length of $O(n^2 \log n)$.

As an input of multi-source connectivity problem, in addition to above string of G , we can also have a separate input for S : a string of length n denoting the membership of each vertices in S where the i -th symbol is 1 iff $i \in S$ (else, i -th symbol will be 0). Alternatively, if we just want the input to be a single string, all symbols of v_i in the string of G will be convoluted by a tick symbol \checkmark (i.e. $\lfloor \left(\frac{v_i}{\checkmark} \right) \rfloor$) iff $i \in S$. Note that both input representations are equivalent and interchangeable with an additional $O(n)$ simple automatic function steps to construct one from the other. To solve the multi-source connectivity problem, the algorithm must tick all reachable vertices by the similar convolution (or alternatively construct the membership string of length n of reachable vertices). We then have the following results.

Theorem 5.7. *Multi-source connectivity problem of a directed graph with n vertices can be solved in $DAL[n \log n]$.*

Proof. We use the input format with a single string (i.e. the vertices in S is convoluted with tick symbol) and try to tick all reachable vertices. We give the following algorithm:

1. Find any ticked vertex u having at least one out-going edge. If no such vertex exists, then terminate the algorithm.
2. Find any v such that $e_{u,v} = v$.
3. Tick vertex v (even if v is already ticked).
4. Remove all edges going to v and repeat the algorithm.

The algorithm uses the concept of flood-fill where it relays the connectivity from any visited (i.e. ticked) vertex to another vertex, shown by points 1, 2, and 3. By removing all in-going edges to vertex v in point 4, v will be guaranteed to be traversed at most once thus the algorithm will repeat at most n times.

Now we will calculate the time complexity. Point 1 can trivially be done in a constant number of steps. Point 2 is done similarly as the continuation. But in addition, the vertex v is copied to another register in $O(\log n)$ steps. The copy of v then compared to each v_i in parallel and the matched vertex number will be ticked in point 3. By “comparing in parallel” we mean here that the first character of v is matched to the first character of every vertex, then the second character of v is matched to the second character of every vertex, and so. As each matching takes one round and v is represented by $\log(n)$ bits, these comparisons need only $O(\log n)$ steps. Point 4 is done in the same fashion by comparing v for each $e_{i,j}$ then set the matched ones to 0, again in $O(\log n)$ steps. As there are at most n iterations, therefore the overall time complexity is $DAL[n \log n]$. \square

Corollary 5.8. *Any graph problem with n vertices whose solution relies on Breadth-First-Search (BFS) such as Shortest Path in Unweighted Graph, Cycle Detection, Bipartite Checking, and Eulerian Checking; can be solved in $DAL[n \log n]$.*

Proof. BFS can be done with some modifications of the algorithm of Theorem 5.7. Instead of convoluting v_i with a tick, v_i will be convoluted with its distance, starting from 0 on the source. Assign a register as a counter cnt , initialized with 0. The modified algorithm is then the following:

1. Find any vertex u with distance = cnt and having at least one out-going edge. If no such vertex exists and $cnt < n$, then increment cnt . If no such vertex exists and $cnt = n$, then terminate the algorithm.
2. Find any v such that $e_{u,v} = v$.
3. If the distance of vertex v is not set yet, set it to $cnt + 1$.
4. Remove all edges going to v and repeat the algorithm.

The proof of correctness is still the same as the previous algorithm but now “the ticks” are done in increasing order of the distance, instead of in an arbitrary order. The time complexity remains the same. \square

We also show that, if the graph is acyclic then a faster solution exists given that the graph is in a nice format.

Definition 5.9 (Sorted-Directed-Acyclic-Graph). A Sorted-Directed-Acyclic-Graph (SDAG) is defined as a directed graph with no cycles and each edge $(i, j) \in E$ satisfies $i < j$. Note that every directed acyclic graph can be converted into its respective SDAG e.g. by performing a topological sort to relabel the vertex numbers.

Theorem 5.10. *Multi-source connectivity problem of a SDAG with n vertices can be solved in $DAL[n]$.*

Proof. We give the following algorithm in high-level. Assign two new registers denoted as R_v and R_e which has the same length as the input string. Initially, R_v will have exactly a single marker symbol (denoted as \uparrow) located at the same position as the first symbol of v_1 . R_e instead will have exactly n marker symbols located at the same positions as all the first symbol of $e_{i,1}$ for $1 \leq i \leq n$. All other symbols in R_v and R_e are blank.

Now, for the next n iterations, we maintain the markings in R_v and R_e such that on the j -th iteration: R_v will have exactly one single marker at the first symbol of v_j and R_e will have exactly n markers at the first symbol of each $e_{i,j}$ for $1 \leq i \leq n$. Here is the illustration of the movement of the markers. Note that, as we only mark the first symbol, the movement can be done in constant steps by moving the markers to their next spot in parallel.

Iteration	G	v_1	$e_{1,1}$	$e_{1,2}$	\dots	$e_{1,n}$	v_2	$e_{2,1}$	$e_{2,2}$	\dots	$e_{2,n}$	\dots	v_n	$e_{n,1}$	$e_{n,2}$	\dots	$e_{n,n}$
1	R_v R_e	\uparrow			\dots					\dots							
			\uparrow		\dots			\uparrow		\dots				\uparrow			
2	R_v R_e				\dots		\uparrow			\dots							
				\uparrow	\dots				\uparrow	\dots					\uparrow		
\dots																	
n	R_v R_e				\dots					\dots			\uparrow				\uparrow
					\dots	\uparrow				\dots	\uparrow						\uparrow

After the marking is maintained on the j -th iteration, our next job is to check whether j is a reachable vertex. This can be done by checking if there exists any ticked (with \checkmark) vertex number v_i that has an out-going edge to j (i.e. $v_{i,j} = j$), if yes then tick vertex number v_j . Note that we only need constant steps as all relevant variables are already marked. After n iterations, the program terminates and we show that all reachable vertex have been ticked.

The proof of correctness is by an induction of this statement: after j -th iteration, all reachable vertices $i \leq j$ are all ticked. The base case of $j = 1$ is trivial because 1 is reachable if and only if $1 \in S$ (which holds if and only if vertex 1 is already ticked at the beginning). For the inductive step, suppose the statement is already correct for $j = k$. The $k + 1$ -th iteration checks whether vertex $k + 1$ is reachable and that is true iff there exists a reachable vertex i going to $k + 1$ i.e. $v_{i,k+1} = k + 1$. As the graph is SDAG, i must be less than $k + 1$. As we already correctly ticked the reachable vertices up to vertex number k , therefore the reachability of $k + 1$ can be fully checked. \square

The upper bound of the multi-source connectivity problem is almost tight. In fact, it is match with the lower bound if the graph is a SDAG by the following theorem.

Theorem 5.11. *The lower bound of multi-source connectivity of a directed graph with n vertices is $DAL[\Omega(n)]$.*

Proof. Assume that n is even so we can split the vertices into two equal groups: $\{1, 2, \dots, \frac{n}{2}\}$ and $\{\frac{n}{2} + 1, \dots, n\}$. Let us construct an input graph such that $(i, j) \in E$ iff $i + \frac{n}{2} = j$ and $S \subseteq \{1, 2, \dots, \frac{n}{2}\}$ i.e. the sources are the subset of the first group. By this construction, we reduce the problem to a duplication problem: the algorithm must tick the vertices in the second group such that they are equal to the ticks of the first group. It is also equivalent to transforming w to ww where $|w| = \frac{n}{2}$. Between both w , at least $\frac{n}{2}$ bits has to be transmitted as it carries the copy of w . Thus by its crossing sequence, the lower bound will be $\Omega(n)$ -DAL-computation. \square

5.4. Sorting in DARM

We also have a nice result in sorting where the solution is optimal.

Theorem 5.12. *Given n many m -digits numbers as an input, sorting those n numbers can be solved in $DAL[nm]$. Moreover, it is also matched with the lower bound.*

Proof. We can directly implement parallel neighbour-sort by Habermann [35]. The algorithm runs in n rounds. In odd-numbered round, $\lfloor n/2 \rfloor$ pairs are set: $(1, 2), (3, 4), \dots$; while in even-numbered round, $\lfloor (n-1)/2 \rfloor$ pairs are set: $(2, 3), (4, 5), \dots$. These pairs denote the two index numbers to be compared and those two numbers will be swapped if the former is larger. In a single round, all comparisons and swappings can be done in parallel and need $O(m)$ steps since each number has m digits. Thus, the total complexity time is $DAL[nm]$.

For lower bound, an input can be generated by putting $n/2$ large numbers in the first half and smaller numbers in the rest last half of the input. To sort this requires transferring all numbers in the first half to the last half and vice

versa. Therefore, at least $nm/2$ bits of information have to be transmitted between the middle point. As crossing sequences are only determined up to a constant factor, thus the length of the crossing sequence between both halves will be in length $\Omega(nm)$, hence the lower bound is attained. \square

6. Key Results on NARM

In the same spirit of DARM, this section presents the relationship between NARM and standard Turing machine complexity classes followed by membership of some interesting languages in NAL class.

6.1. Nondeterministic Turing Machine, Regularity, and NONPALINDROME

We first start our result with the following theorem.

Theorem 6.1. $NAL[f(n)] \subseteq NSPACE[f(n)] \cap NTIME[(n + f(n))f(n)]$.

Proof. Consider the space-time diagram of the register content in NARM $M = \langle \Gamma, \Sigma, G \rangle$. Let l be the maximum length of a string in the computation of G automata until it halts. The input is padded with $\#$'s such that the length from first to last computation is always l . Let r_i be the i -th symbol of the register.

step	r_1	r_2	\dots	r_l
1	$r_1^{(1)}$	$r_2^{(1)}$	\dots	$r_l^{(1)}$
2	$r_1^{(2)}$	$r_2^{(2)}$	\dots	$r_l^{(2)}$
\dots	\dots	\dots	\dots	\dots
$cf(n)$	$x_1^{(cf(n))}$	$r_2^{(cf(n))}$	\dots	$r_l^{(cf(n))}$

Note that it suffices to keep track of the state of a deterministic finite automaton accepting the automatic relation step updates in order to simulate it. Hence, the Turing machine simply maintains a column of states. These states correspond to the states of the automata recognizing the automatic step updates. The Turing machine then guesses the contents column by column and simulates the automata to process these contents. At the end of all l columns, it then verifies that all step updates are valid. The total amount of space used is simply $O(f(n))$. Hence, we have $NAL[f(n)] \subseteq NSPACE[f(n)]$.

Secondly, in each bounded automatic relation step, the length of the string is allowed to increase by at most a constant c . Hence, the length of the string after $O(f(n))$ steps is bounded by $O(n + f(n))$. By Theorem 3.10, the execution of automatic relations can be simulated by a nondeterministic Turing Machine in $O((n + f(n))f(n))$ steps. Hence, $NAL[f(n)] \subseteq NTIME[(n + f(n))f(n)]$. \square

Corollary 6.2. $NAL[poly(n)] = NP$.

Corollary 6.3. For $f(n) = \Omega(n)$, under the assumption that $NTIME[(n + f(n))f(n)] \not\subseteq NSPACE[f(n)]$, $NAL[f(n)] \subset NSPACE[f(n)]$.

Next, we show several results on a NARM with a smaller number of computations. Note that an automatic relation step is an NFA which can be constructed as a DFA but with exponential-larger size. Therefore, we can directly use the proof in Theorem 5.1 to show that $NAL[1] = DAL[1] = Regular$. However, a better bound can be achieved.

Theorem 6.4. $NAL[o(\log n)] = Regular$, moreover if NARM measures the time complexity in a weak-measure i.e. $\max\{\min\{\text{acceptance time for } x \text{ on a nondeterministic path}\} : |x| = n, x \in L\}$ then $NAL[o(\log \log n)] = Regular$.

Proof. In Subsection 5.1 we show that the length of a $DAL[f(n)]$ -computation gives a crossing sequence of length $O(f(n))$. This result can be extended to NARM in a similar manner as a bounded automatic relation can be simulated by a double-pass of a nondeterministic one-tape Turing machine. Thus an $f(n)$ -steps NARM produces an $f(n)$ crossing sequence length accepting computation on the best possible run. Szepietowski [65] and Pighizzini [57] show that if a nondeterministic one-tape Turing machine uses a crossing sequences of length at most $o(\log n)$ then it accepts regular language. This result allows to translate that NARM in $o(\log n)$ steps accept regular language too.

Moreover, Pighizzini [57] also shows that in a weak-measure where the acceptance of a string is only measured to its shortest computation path, then the previous bound goes down to $o(\log \log n)$. Their proof is based on accepting a unary string of length n where the smallest integer non dividing n is not a power of 2, by guessing s and t such that $2^s < t < 2^{s+1}$, $n \bmod 2^s = 0$, and $n \bmod t \neq 0$. This translates well for NARM in a weak-measure by guessing a string where every t -th symbol is a 2 and in between are symbols 1, for some non-negative integer t — allowing NARM to check the divisibility of t . \square

The next lemma nicely follows the hierarchy that in $O(\log n)$ steps, NARM may recognize a non-regular language.

Definition 6.5 (PALINDROME and NONPALINDROME). $PALINDROME = \{x \in \{0, 1\}^* : \forall i, x_i = x_{|x|-i+1}\}$ while $NONPALINDROME = \{0, 1\}^* \setminus PALINDROME$.

Theorem 6.6 ([3, 38, 58]). *PALINDROME cannot be accepted on a nondeterministic one-tape Turing Machine in $o(n^2)$ steps.*

The subsequent corollary follows from Theorems 6.1 and 6.6.

Corollary 6.7. $PALINDROME \notin NAL[o(n)]$.

Lemma 6.8. $NONPALINDROME \in NAL[\log n]$.

Proof. The idea is the following. After r_1 reads an input w where $|w| = n$, r_2 guesses (i.e. nondeterministically chooses) a string $0^i \cdot 3 \cdot 1^j \cdot 3 \cdot 2^i$ where $2i + j + 2 = n$ such that the symbols of w at positions $i + 1$ and $n - i$ are different, thus w is nonpalindrome. As r_2 can only guess in form of $0^*31^*32^*$ with an automatic relation, our task is then to verify that the length is exactly n and the number of symbol ‘0’ and ‘2’ are equal. Verifying that the length is exactly n is trivial by comparing r_1 and r_2 (reject if one string is longer/shorter). To verify that the occurrences of ‘0’s and ‘2’s are the same, while there exists symbol ‘0’ or ‘2’, repeat the following rounds:

1. If the parities of ‘0’s and ‘2’s are different, then reject.
2. Else, modify the 1, 3, 5, \dots -th occurrences of ‘0’ to ‘1’, and modify the 1, 3, 5, \dots -th occurrences of ‘2’ to ‘1’.

If the iteration terminates without reject, both digits are disappearing at the same time and proven to be always equal. The final step is then to verify that the symbols on r_1 in the same positions of both ‘3’s on r_2 are different, thus a nonpalindrome. Note that both the number of ‘0’s and ‘2’s are halved in each round and therefore the runtime of the verification is logarithmic in the length of the input. \square

Finally, we wrap up this subsection with the following result.

Theorem 6.9. $NAL[\log n] \subset NLOGSPACE \subseteq P$.

Proof. Combining Corollary 6.7 and Lemma 6.8, $NAL[\log n]$ is not closed under complement, whereas $NLOGSPACE$ is closed under complement [42][64]. From Theorem 6.1, we also have $NAL[\log n] \subseteq NLOGSPACE$. Hence, $NAL[\log n] \subset NLOGSPACE$. \square

6.2. 3SAT and Context-free Languages in NARM

We employ the following binary encoding of 3SAT as an input.

Definition 6.10 (Binary Encoding of 3SAT). Consider a binary encoding of 3SAT where a variable x_i is represented by a binary number i . We use $s_i \in \{+, -\}$ to represent whether the i -th literal is positive or negative such that a clause $(x_i \vee \neg x_j \vee \neg x_k)$ is represented by $i + |j - |k-$. Different clauses are separated by symbol “&”.

Let n be the length of the formula (and not the number of variables). We can show a nice result that in unary encoding $3SAT \in NAL[n]$. However, using a similar technique, we can actually show that $3SAT \in NAL[\frac{n}{\log n}]$ if it is in binary encoding. The catch is that an additional small constraint has to be imposed: if there are k variables then the first k binary numbers (i.e. 1 to k) must be used as their namings and are in the same length $O(\log k)$. The following is the proof.

Theorem 6.11. *In a nice format, $3SAT \in NAL[\frac{n}{\log n}]$ where n is the length of the 3SAT formula.*

Proof. After reading the input of formula f , assign a register r_c as a counter. Register r_c will copy the input f but every variables in it is replaced by 0. Then the following iteration is performed:

1. Increment every variables in r_c .
2. Nondeterministically choose a truth value T or F .
3. For each variable in f which is equal to the one in r_c , assign above truth value to it.
4. If every variables in f has been assigned, terminate the iteration; else repeat.

The idea is that in the i -th iteration, the algorithm will assign the i -th variable a same truth value. Each operations in the iterations such as increment and assigning a truth value can be done in constant steps as it is possible update all variables both in r_c and f in parallel. After the iteration terminates, every variables has been assigned and a single step can be performed to check the satisfiability of the f .

Note that the algorithm runs in $O(k)$ steps where k is the number of iterations. The number of iterations is equal to the number of variables if the first k binary numbers are used as the naming. As each variable is written in the same length of $O(\log k)$ therefore there must be at most $O(\frac{n}{\log n})$ many variables, thus the runtime of the algorithm is $NAL[\frac{n}{\log n}]$. \square

Beside 3SAT, we also have a result in context-free languages.

Theorem 6.12. *Context-free languages and their closure under union, intersection, concatenation, and Kleene star are in $NAL[n]$; moreover, it is also optimal.*

Proof. The upper bound comes from a direct translation from grammars in Greibach normal form [34]. Meanwhile, the lower bound follows from Corollary 6.7. As PALINDROME is a context-free language, there is no $NAL[o(n)]$ algorithm to recognize the membership for context-free language. \square

7. Advanced Types of ARM

In this last section, we further extend the types of ARM by allowing unboundedness of automatic relations, a booster operation to generate padding, and also a concept similar to Alternating Turing machines.

7.1. Unbounded Nondeterministic Automatic Register Machine

This type of machine is similar to NARM but it uses unbounded automatic relations instead.

Definition 7.1 (Unbounded Nondeterministic Automatic Register Machine). An Unbounded Nondeterministic Automatic Register Machine (UNARM) is an Automatic Register Machine $M = \langle \Gamma, \Sigma, R, Op, P \rangle$ where Op is restricted to unbounded automatic relations. Alternatively, an UNARM is an Automatic Register Machine $M = \langle \Gamma, \Sigma, G \rangle$ where G is an unbounded automatic relation.

Definition 7.2 (Unbounded Nondeterministic Automatic Register Machine Complexity). Let $f(n)$ be any function in the input size n . $UNAL[f(n)]$ is the class of languages accepted by UNARM in no more than $O(f(n))$ steps.

As unbounded automatic relation has the ability to guess a huge witness in one step, it may recognize a larger complexity class compared to NARM as shown by the following result.

Theorem 7.3. *For $f(n) = \Omega(n)$, $NSPACE[f(n)] \subseteq UNAL[f(n)]$.*

Proof. Consider a one-tape nondeterministic Turing Machine with $cf(n)$ cells for some constant c . Let C_i denote the string on the tape of Turing Machine along with the head location (marked on the string) and state of the machine at the i -th step, padded with ‘#’ such that $|C_i| = cf(n)$. Let $|$ be a new symbol not in the alphabet of the Turing Machine. Now UNARM nondeterministically guesses

$$r = \begin{array}{c|c|c|c|c|c} C_1 & C_2 & \dots & C_i & \dots & C_{t-1} \\ C_2 & C_3 & \dots & C_{i+1} & \dots & C_t \end{array}$$

and verifies in one step that C_1 is the input (with starting state and the head location marked the leftmost symbol), C_i is in accepting state, and C_i to C_{i+1} is a valid transition in each column separated by |. In the next $cf(n)$ steps, it verifies that $\forall i, C_i$ in the first row is the same as C_i in the second row. Let $m = cf(n)$, expanding C_i into $C_{i,1}C_{i,2} \dots C_{i,m}$:

$$r = \begin{array}{c|c|c|c} C_{1,1}C_{1,2}C_{1,3} \dots C_{1,m} & C_{2,1}C_{2,2}C_{2,3} \dots C_{2,m} & C_{3,1}C_{3,2}C_{3,3} \dots C_{3,m} & \dots \\ C_{2,1}C_{2,2}C_{2,3} \dots C_{2,m} & C_{3,1}C_{3,2}C_{3,3} \dots C_{3,m} & C_{4,1}C_{4,2}C_{4,3} \dots C_{4,m} & \dots \end{array}$$

In the j -th step, more specifically, it verifies that $\forall i, C_{i,j}$ from the second row of the $(i-1)$ -th block is the same as $C_{i,j}$ from the first row of the i -th block. It also marks off $C_{i,j}$ with the marker symbol @ after checking. The @ will help the automatic relation to identify which is the j -th symbol: the j -th symbol is exactly the first non @ symbol in each block.

After one step:

$$r = \begin{array}{c|c|c|c} @ C_{1,2}C_{1,3} \dots C_{1,m} & @ C_{2,2}C_{2,3} \dots C_{2,m} & @ C_{3,2}C_{3,3} \dots C_{3,m} & \dots \\ @ C_{2,2}C_{2,3} \dots C_{2,m} & @ C_{3,2}C_{3,3} \dots C_{3,m} & @ C_{4,2}C_{4,3} \dots C_{4,m} & \dots \end{array}$$

After two steps:

$$r = \begin{array}{c|c|c|c} @ @ C_{1,3} \dots C_{1,m} & @ @ C_{2,3} \dots C_{2,m} & @ @ C_{3,3} \dots C_{3,m} & \dots \\ @ @ C_{2,3} \dots C_{2,m} & @ @ C_{3,3} \dots C_{3,m} & @ @ C_{4,3} \dots C_{4,m} & \dots \end{array}$$

Lastly, it also verifies all C_i are having the same length, i.e. $|C_i| = m$, and it can be done with the help of a counter (for example, in another register). Hence, we have $\text{NSPACE}[f(n)] \subseteq \text{UNAL}[f(n)]$ for $f = \Omega(n)$. Note that the condition $f(n) = \Omega(n)$ is necessary as the simulation needs to verify that the correct input are used in the computation which takes $\Omega(n)$ steps. \square

Corollary 7.4. $\text{UNAL}[\text{poly}(n)] = \text{PSPACE}$.

Remark. While it is clear that $\text{NP} \subseteq \text{PSPACE}$, the equality/inequality part is still not known thus it is also unclear whether polynomial time models with bounded and unbounded automatic relations as operations are actually the same or different as a decision problem. As mentioned before, if both act as functions instead then the two concepts differ as a nondeterministic bounded automatic function can only have polynomial-sized output while the unbounded one can have exponential-sized output.

7.2. Polynomial-Size and Exponential-Size Padded Automatic Register Machines

Automatic functions and bounded automatic relations have a nice property that the length of the string only increases by a constant. As unboundedness introduces more power by its ability to guess a huge witness, we are curious whether the length of the string is what matters. What happens if we are still using automatic functions and bounded automatic relations yet we initially give the "working space" big enough? Here we define such models with a booster operation.

Definition 7.5 (Polynomial-Size Padded Automatic Register Machine). A Polynomial-Size Padded Automatic Register Machine (PARM) is an ARM $M = \langle \Gamma, \Sigma, R, Op, P \rangle$ where instead of operation $\text{read}(R_i)$, there is an operation $\text{read_and_boost}(R_i, R_j, p)$ where $R_i, R_j \in R$ and p is a polynomial. This operation will:

1. Assign the input string to register R_i .
2. Let n be the input length, then a padding string 0^* is generated in an adversarial way such that its length is at least $p(n)$ and then is assigned to R_j .

Definition 7.6 (Exponential-Size Padded Automatic Register Machine). An Exponential-Size Padded Automatic Register Machine (ExpARM) is an ARM $M = \langle \Gamma, \Sigma, R, Op, P \rangle$ where instead of operation $\text{read}(R_i)$, there is an operation $\text{read_and_boost}(R_i, R_j, p, q)$ where $R_i, R_j \in R$ and p, q are polynomials. This operation will:

1. Assign the input string to register R_i .
2. Let n be the input length, then a padding string 0^* is generated in an adversarial way such that its length is at least $2^{p(n)} \cdot q(n)$ and then is assigned to R_j .

With PARM model, a working space with polynomial-size larger than the original length of the input is generated and it can only be done once when reading the input. Any polynomial p can be chosen by the designer of the algorithm but, due to the adversary nature of the booster step, it does not guarantee that the padding string has exactly $p(n)$ symbols — it only guarantees that there are at least $p(n)$ symbols. The intention is that in $O(1)$ steps only automatic functions and relations can be computed, i.e. it is not possible to have a function $h : x \rightarrow 0^{|x|}$, thus a booster step is solely used to enlarge the working space. Note that one can however in $O(\log n)$ steps trim the booster string to get the exact length of $p(n)$. This concept also applies for ExpARM model where the padding length will be at least $2^{p(n)} \cdot q(n)$ and all other properties can be chosen adversially.

Remark. As any upper bound of PARM and ExpARM are also welcome, one can modify the boosting operation to *read_and_boost*(R_i, R_j, c) then just choose a constant c such that the padding string has to have at least length n^c in PARM and 2^{n^c+c} in ExpARM.

We then have the following definitions for the complexity.

Definition 7.7 (Polynomial-Size Padded Deterministic Automatic Register Machine Complexity). Let $f(n)$ be any function in the input size n . PDAL[$f(n)$] is the class of languages accepted by deterministic PARM in $O(f(n))$. Here the PARM has to finish in $O(f(n))$ steps no matter what length of the padding string a booster step generate.

Definition 7.8 (Exponential-Size Padded Deterministic Automatic Register Machine Complexity). Let $f(n)$ be any function in the input size n . ExpDAL[$f(n)$] is the class of languages accepted by deterministic ExpARM in $O(f(n))$. Here the ExpARM has to finish in $O(f(n))$ steps no matter what length of the padding string a booster step generate.

It turns out allowing a large working space gives the machine more power as shown by the following result.

Theorem 7.9. *The problem QSAT is in ExpDAL[n].*

Proof. Given an input x which is a QSAT-formula of length n , let m be the number of variables in it. One assumes the following coding and only in that case the number returned by the algorithm needs to be correct:

1. There is a prefix of the variables saying whether they are universal or existential quantified and the variables are ordered as numbers $m-1, m-2, \dots, 2, 1, 0$ in this prefix;
2. Each variable name has k binary digits and $m \leq 2^k < 2m$;
3. The quantified part is in square brackets and within this, the variables are either prefixed by + or by - to indicate that they are non-negated or negated in the clause, respectively, and they have furthermore an underscore _ after the variable to record the value and they are separated by commas inside the clause and clauses are separated by semicolons;
4. Every number $0, 1, \dots, m-1$ (in binary) occurs as a variable name somewhere in the inner part of the formula;
5. The instance is redundancy-free, that is, no clause α logically implies a clause β .

For $m = 3$, an example of a correct coding is the following:

$$\exists 10 \forall 01 \exists 00 [+00_ , +01_ , +10_ ; -01_ , -10_]$$

and it says that variable number 2 is existentially quantified followed by variable 1 being universally quantified followed by variable number 0 being existentially quantified such that one of the variables is positive and one of the variables number 1 and number 2 is negative. The variable order is in each quantification from top to bottom and that exactly the variables are quantified which occur inside the formula. If this is not the case then the algorithm can return any value. Now the algorithm is as follows:

1. Read input x and let n denote its length;
2. One guesses (i.e. uses booster step) a string y of length at least $2^{n+2} \cdot n$ and in the next step one makes all symbols in y to 3;
3. While the position of the second 3 in y is not at least two symbols after the end of x do Begin Change in y the second, fourth, sixth, \dots occurrence of 3 to 5 End;
4. Make the symbol after the last occurrence of 3 to 4 and delete everything afterwards (this can either increase the length of y by 1 or leave it as it is or make y shorter);

5. For all symbols a between the two square brackets in x (which might be read out by using a marker in the loop) and then for $a = “.”$ Do Begin For each occurrence of a 3 in y go to the first 5 after it and replace it by a End;
6. Make a copy z of y ; Replace all 1 in z by 0;
7. If there are still some variable names which do not have an s or u after them and either there is no position where the variable names in y and z coincide and or there is a position where the variable name in z consists only of 1 then return the value 0;
8. If there is no variable name in y which is not followed by either s or u then go to line 13;
9. Go over y and z in parallel and with a bit b having value u at the beginning such that whenever one passes a 3 in y then one swaps b from u to s or from s to u and whenever one passes a variable name which coincides in y and z then one replaces in y the $_$ after it by the value of b ;
10. Go over y and change the second, fourth, sixth, \dots 3 to 2;
11. Go over z and increment all binary variable names by 1 (an automatic function can do this);
12. Go to line 7;
13. In y , make the symbol after the second occurrence of a 3 to 4 (this 3 sits after the first instance where all variables are evaluated to s) and discard all symbols after this 4;
14. Now replace in y all 2 by 3;
15. For each instance in y (where an “instance” in y refers to the segment between two consecutive occurrences of 3), if every clause contains either a substring $\{0, 1\}^*s$ or a substring $\{0, 1\}^*u$ then all clauses are satisfied and the first 5 after the dot at the end of the instance is made an s else the instance is not satisfied and the first 5 after the dot at the end of the instance is made a u ;
16. Go backward through the quantifier prefix of x (using some marker) until again y has only two 3s Do Begin
17. For each subsequent two occurrences of s, u after a dot do Begin
 - if the quantifier is universal and both are s then replace the first by 5 and the second one by s else
 - if the quantifier is universal and at least one is u then replace the first by 5 and the second by u else
 - if the quantifier is existential and both are u then replace the first by 5 and the second by u else
 - if the quantifier is existential and at least one is s then replace the first by 5 and the second by s End End;
18. Now there is only one s or u left after a dot and this is the result of the formula (s is satisfied and u is unsatisfied).

Here are some explanations. First note that functions which are computed by one-way Turing machines going either over one input alone or over several inputs synchronously and doing changes on the spot are automatic and all automatic functions in this algorithm are of this type; for line 11 the Turing machine has to go backward for easily seen to be deterministic. Actually nondeterministic Turing machines going forward can also do the job. The characterisation [26] says that the Turing machine processing the variable y (or y and z in parallel) for an automatic function can change the direction constantly many times, but here the easier one-directional pass is sufficient for all automatic functions; when comparing the variable names in y and z , the Turing machine goes over the word by accessing in both variables the same position in each step.

- In line 2, $4n$ is a safe upper bound of the length of copy of the inner part of the quantified variable plus some additional space and n is a safe upper bound on the number of variables, thus the length reserved in y is sufficient (one cannot fix the length exactly but only get it above the bound for avoiding of coding).
- Line 3 thins out symbols 3 in y by always converting every second to 5 (what an automatic function can do) until the gap between the first 3 (at the beginning of the word) and the second 3 is longer than the length of x , this are $O(\log n)$ many runs through the body of the loop which is implemented by one automatic function.
- Line 4 rectifies the end of the word and removes trailing 5s and adds a 4 as an end marker.
- Line 5 copies in $O(n)$ rounds the inner part of the formula into y starting after each 3 (there is enough space for this) by always for each symbol a copied replacing the first 5 after each 3 by a . Each copying of a symbol a into all these places can be done by invoking one automatic function to update y and thus in overall $O(n)$ steps the copying can be done.
- Line 6 copies y into a new variable z and makes in z all variable names to 00 \dots 0 by making all 1 to 0.

- Lines 7–12 are one big loop in order to set in y all values of the variables in a consistent manner with the additional property that all 2^m ways of putting the variable values are covered in an instance.
- Line 7 causes early termination of the algorithm in the case that the names of the used variables are not contiguous.
- Line 8 leaves the loop in the case that all variables are assigned s (for satisfied or true) or u (for unsatisfied or false).
- Line 9 and line 10 make the values of the ℓ -th variable to change every 2^ℓ instances, initially the zeroth variables changes from every instance to the next between u and s and subsequently each second, fourth, sixth, \dots 3 becomes a 2 to increase the number of instances with the same value from 2^ℓ to $2^{\ell+1}$ before processing the ℓ -first variable.
- Before going into the next iteration of the loop, all variable names in z are increased from ℓ to $\ell + 1$.
- Line 13 is just to make sure that exactly 2^m instances will survive and these have the variable values (u, \dots, u, u) , (u, \dots, u, s) , (u, \dots, s, u) , (u, \dots, s, s) , \dots , (s, \dots, s, s) and a one-way Turing machine going over the word y can detect the position of the instance where all variables are s and change the symbol after the 3 after that instance to 4 and discard the symbols after that; thus this modification can be computed by an automatic function.
- Line 14 is only a step needed to make the separation by two neighbouring instances always by a 3 and not by a 3 or a 2. This eases the writing of the algorithm, but has no further significance.
- For each instance in y , a one-way Turing machine can check whether all literals with value s, u of the variable are set such that the instance is satisfied, if so then the instance gets after its final dot the value s else the instance gets after its final dot the value u . This is done in one scan for all instances in y .
- The loop in Lines 16 and 17 goes backwards over the quantifiers in x and for each quantifier, it identifies the still active values of blocks of instances in the form $.s$ and $.u$ after the instance and it evaluates the first and the second with the result in the second, the third and the fourth with the result in the fourth, the fifth and the sixth with the result in the sixth and so on in order to process the currently quantified variable; in the ℓ -th round (ℓ goes from 0 to $m - 1$) this loop treats the quantification of the ℓ -th variable and takes the result of two neighbouring blocks of 2^ℓ instances with the ℓ -th variable being u and s respectively from before and puts the result into the entry for the second of these blocks while the entry for the first of these blocks get erased. After doing this for all m variables, one entry for the block of 2^m instances remains and this is the result of this loop and value of the QSAT formula.
- Line 18 returns this value. Note that the number of rounds in each of the loops is $O(\log n)$ in the loop of Line 3, $O(n)$ for the loop copying the formula in all applicable positions in Lines 4–5, number m of variables for the big loop in lines 7–12, number m of variables for the loop in lines 16–17. Thus the overall amount of operations in the evaluation of the QSAT formula is $O(n)$; note that $m \leq n$.

This completes the algorithm and its explanations for this proposition. \square

We also show the connection between ExpARM and UNARM as the following.

Theorem 7.10. $PSPACE = ExpDAL[poly(n)] = UNAL[poly(n)]$.

Proof. We first prove $PSPACE \subseteq ExpDAL[poly(n)]$. Let a problem R in $PSPACE$ be given which is many-one reduced by a function f in polynomial time to QSAT where f increases the size of an instance n' to at most $p(n')$ for some polynomial p . Let an input x' of R be given. One generates in padding an input y of length $2^{p(n') + 4} \cdot p(n')$ and then one translates the instance x' into $x = f(x')$ of length n with $n \leq p(n')$ in $poly(n')$ steps. Afterwards one runs above QSAT algorithm in $O(n) = O(p(n'))$ steps to decide whether x is in QSAT, without loss of generality f produces a formula which satisfies all the constraints of the above algorithm so that the algorithm is correct. Thus the algorithm is correct.

Next, we prove that $\text{ExpDAL}[\text{poly}(n)] \subseteq \text{UNAL}[\text{poly}(n)]$: One just replaces the step of producing the y by exponential padding having at least 2^{n^c+c} symbols (where c is a sufficiently large constant) by an unrestricted guessing of the y and then one makes all bits in y to 1 and then one verifies that $n^c + c$ many steps of the algorithm which converts in each step half of the 1s to 0s does not produce a 0 only word. Then the word is long enough and one can proceed with the previous $\text{ExpDAL}[\text{poly}(n)]$ algorithm. By Corollary 7.4, we know that $\text{UNAL}[\text{poly}(n)] = \text{PSPACE}$ therefore the equality is established. \square

We move on to the result in polynomial-size padding.

Theorem 7.11. *Let c be a constant. Solving $(c \log n)$ -variable 3SAT is in $\text{PDAL}[\text{polylog}(n)]$.*

Proof. Without loss of generality the input is given as a pair (x, y') where y' has length n^{c+2} and x is a 3SAT formula (all variables existentially quantified) written in the conventions of Theorem 7.9. In particular the variables are written in binary and the instance x is redundancy-free. Thus the instance x has at most length $10(c \log n \log \log n)^3 + c'$ for some constant c' and the factor 10 is larger than the expected factor 8 to absorb constant add-ons per literal and the quantifier prefix. If x violates this length bound then one can return the value 0, as the input is not in adequate form.

Note that n can be computed from y' in $\log(n)$ rounds. The computation and checking of the length-bound can be done in polylogarithmic many steps. Furthermore, one can do a padding using that the y guessed must have $2^{c \log(n)}$ instances of length $|x|$ (which is polylogarithmic and thus shorter than n) so that n^{c+2} is a save lower bound for the length of the padding parameter. The algorithm of Theorem 7.9 can now run with x and y using $|y| \geq 2^m \cdot |x| \cdot 4$ and it needs $O(|x|)$ steps where $|x| \leq 10(\log n \cdot c)^3 + c'$. Thus the overall amount of steps done is polylogarithmic in n . \square

Lastly, we prove that $\text{PDAL}[o(n)] \neq \text{DAL}[o(n)]$ if the following hypothesis is true.

Hypothesis 7.12 (Exponential Time Hypothesis [43]). *Exponential Time Hypothesis (ETH) states that solving an h -variable 3SAT instance needs time $\Omega(2^{\delta h})$ for some $\delta > 0$.*

Theorem 7.13. *If ETH is true then $\text{PDAL}[o(n)] \neq \text{DAL}[o(n)]$.*

Proof. Let $h = c \log n$ such that $c\delta > 2$. It follows from ETH that solving the h -variable 3SAT needs $\Omega(2^{c\delta \log n})$. As we set $c\delta > 2$ thus the lower bound will be $\Omega(n^2)$. Suppose there is an algorithm in DARM solving it in $\text{DAL}[o(n)]$. By Theorem 3.9, such algorithm can be translated to a Turing machine algorithm which runs in $o(n^2)$ thus we get a contradiction. It is clear that the problem can be solved in $\text{PDAL}[o(n)]$ by Theorem 7.11 therefore $\text{PDAL}[o(n)] \neq \text{DAL}[o(n)]$. \square

7.3. Alternating Automatic Register Machine

Alternating Turing machine is formulated by Chandra, Kozen and Stockmeyer [28] to generalize the concept of non-determinism. We bring such notion of alternation to ARM called Alternating Automatic Register Machine (AARM) and have defined and investigated the model in another work [32]. We denote AARM complexity by AAL and its polynomial-size padded version by PAAL. The important findings of AARM are the following:

1. $\bigcup_k \text{AAL}[n^k] = \text{PSPACE}$.
2. $\text{AAL}[1] = \text{Regular}$, while there is already an NP-complete problem in $\text{AAL}[\log^* n]$.
3. $\text{PH} \subseteq \text{PAAL}[\log^* n] \subseteq \text{PSPACE}$.

8. Conclusion

This paper introduced and studied several variants of register machines that employ automatic functions as primitive steps. The first, known as a *Deterministic Automatic Register Machine* (DARM), is provably more computationally powerful than conventional register machines. For example, it was shown that any graph problem with n vertices whose solution relies on Breadth-First-Search (BFS) can be solved by a DARM in $O(n \log n)$ time; moreover, the multi-source connectivity problem can be solved by a DARM in $O(n)$ time. Allowing non-determinism gives rise to a more powerful type of register machine known as a *Non-Deterministic Automatic Register Machine* (NARM). It was shown that the class of context-free languages can be recognised by a NARM in linear time, an improvement on the

quadratic time complexity of this class for DARMs; furthermore, 3SAT can be recognised by a NARM in $O\left(\frac{n}{\log n}\right)$ time when the input formula is given in a particular format, where n is the length of the input formula. Finally, we introduced Automatic Register Machines that are given more “working space” by means of a padding of the input with a string of 0’s; a *Polynomial-Size Padded Automatic Register Machine* (PARM) is provided with a polynomial-size padding of the input while an *Exponential-Size Padded Automatic Register Machine* (ExpARM) is provided with an exponential-size padding. It turned out that an ExpARM is just as powerful as an unbounded NARM (unbounded in the sense that during each computation of an automatic relation, the output can be arbitrarily longer than the input), and both types of machines can recognise all languages in PSPACE in polynomial time.

It may be natural to draw parallels between the models of computation studied in the present work and other automata-based machines. In this regard, we mention one such class of machines: cellular automata. One can show that for any function f , $\text{NAL}[f(n)]$ is characterised by a certain class of non-deterministic cellular-automata with $O(n + f(n))$ cells. In addition, both $\text{DAL}[f(n)]$ and $\text{NAL}[f(n)]$ can be characterised by tiling systems, though these characterisations are somewhat unnatural. As such machine comparisons are not the main focus of this work, we omit the details.

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