

Intrinsic complexity of learning geometrical concepts from positive data

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August 24, 2010

Abstract

Intrinsic complexity is used to measure the complexity of learning areas limited by broken-straight lines (called *open semi-hulls*) and intersections of such areas. Any strategy learning such geometrical concepts can be viewed as a sequence of *primitive basic* strategies. Thus, the length of such a sequence together with the complexities of the primitive strategies used can be regarded as the complexity of learning the concepts in question. We obtained the best possible lower and upper bounds on learning open semi-hulls, as well as matching upper and lower bounds on the complexity of learning intersections of such areas. Surprisingly, upper bounds in both cases turn out to be much lower than those provided by natural learning strategies. Another surprising result is that learning intersections of open semi-hulls turns out to be easier than learning open semi-hulls themselves.

1 Introduction

Learning geometrical concepts from examples is a popular topic in Computational Learning Theory (see for example, [BEHW89, CM94, CA99, GGDM94, Heg94, GG94, GGS96, GS99, DG95, Fis95, BGM98, BGGM98, GKS01]). The above listed papers mostly dealt with finite geometric concepts. The goal of this paper is to quantify the complexity of algorithmic learning of *infinite* geometrical concepts from growing finite segments. For this purpose, we will be using the learning in the limit model.

Consider, for example, an *open semi-hull* representing the space consisting of all points (x, y) with integer components x, y in the first quadrant of the plane bounded by the y -axis and the broken line passing through some points $(a_0, c_0), (a_1, c_1), \dots, (a_n, c_n), a_i, c_i \in \mathbb{N}, a_i < a_{i+1}, 0 \leq i < n$. The line is straight between any points $(a_i, c_i), (a_{i+1}, c_{i+1})$ and begins at $(a_0, c_0) = (0, 0)$; further we assume that the slope of the broken line is monotonically non-decreasing — that is $(c_{i+1} - c_i)/(a_{i+1} - a_i)$ is non-decreasing in i (we need this for learnability of the semi-hull). For technical ease we further assume that the first line segment $(0, 0), (a_1, c_1)$ is adjacent to the x -axis, that is, $c_1 = 0$. (See example semi-hull figure in Figure 1.) Note that each break point in the boundary of the semi-hull defines an angle in the semi-hull. Any such open semi-hull can be easily learned in the limit by the following strategy: given growing finite sets of points in the open semi-hull (potentially getting all points), learn the first break point (a_1, c_1) . Here, under our assumption, c_1 must be 0, therefore, we change our mind every time when we get in

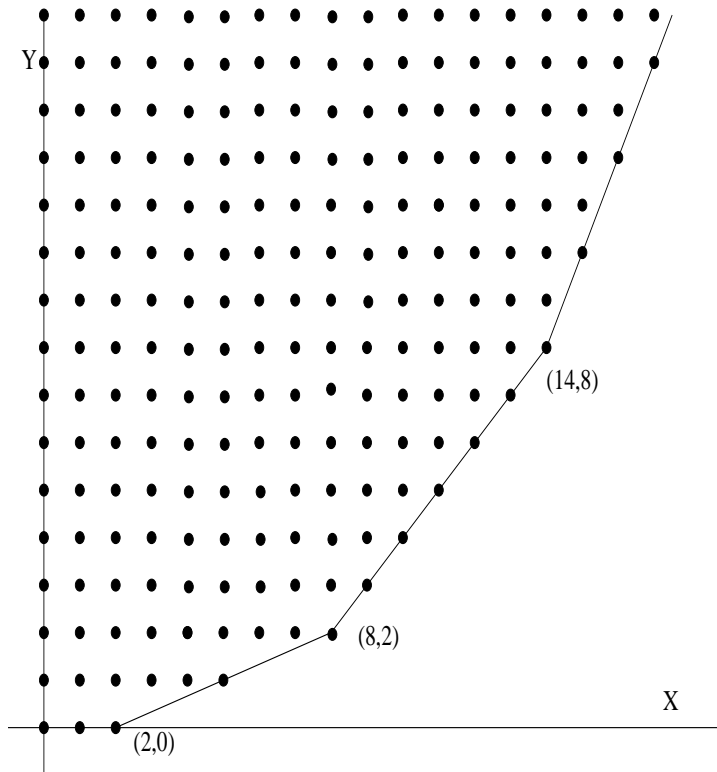


Figure 1: Open Semi Hull

the input a new point $(a, 0)$ with the value a greater than all values b in the points $(b, 0)$ seen so far. Now, once the first break point has been learned, we can try to learn the first slope $(c_2 - c_1)/(a_2 - a_1)$. The more points we get in the input, the more our hypothetical border-line may bend towards the x -axis. Since it can never cross the x -axis and since the points in the concept to be learned have integer components, we will eventually learn the slope. Now, moving along the border-line as more points on it become available from the input, we can learn the second break point (a_2, c_2) . Then we can learn the second slope $(c_3 - c_2)/(a_3 - a_2)$, etc. Is this strategy the best possible? Do there exist easier strategies of learning such geometrical concepts? How do we measure the complexity of learning such concepts? As we are interested in learning infinite objects from infinitely growing finite segments, we use inductive inference as our learning paradigm. This paradigm suggests several ways to quantify the complexity of learning. Among them are:

- a) counting the number of *mind changes* [BF72, CS83, LZ93] the learner makes before arriving at the correct hypothesis;
- b) measuring the amount of so-called *long-term* memory the learner uses [KS95];
- c) reductions between different learning problems (classes of languages) and respective degrees of the so-called *intrinsic* complexity [FKS95, JS96, JS97].

There have been other notions of the complexity of learning in the limit considered in literature (for example see [Gol67, DS86, Wie86]).

The first two approaches, however, cannot capture the complexity of learning open semi-hulls with different numbers of angles: the number of mind changes cannot be bounded by any reasonable function (even for learning the very first break-point), and the long-term memory is maximum (linear) even for one angle. Thus, we have chosen reductions as the way to measure the (relative) complexity of geometrical concepts like open semi-hulls. An important issue

here is which classes of languages can be used as a *scale* for quantifying the complexity of open semi-hulls. One such scale, that turned out to be appropriate for our goal, had been suggested in [JKW00, JKW99]. This scale is a hierarchy of degrees of intrinsic complexity composed of simple natural ground degrees. Every such natural ground degree represents a natural type of learning strategy. For example, the degree *INIT* represents a strategy that tries to use a sequence of hypotheses equivalent to a sequence of monotonically growing finite sets. Another such strategy, *COINIT*, tries to use a sequence of hypotheses equivalent to a sequence of monotonically growing sets $N_a = \{x | x \in N, x \geq a\}$. Intuitively, capabilities of *INIT* and *COINIT*-strategies must be different (this has been formally established in [JS96]). For example, when a *COINIT*-strategy learns the language $\{5, 6, 7, \dots\}$, it actually tries to find the minimum number in this set (5 in our case). When the first number, say, 17 appears in the input, the strategy can immediately use it as the upper bound on its conjectures: no number greater than 17 can possibly be the desired minimum. Note also that the strategy is aware of the absolute lower bound 0 for all its conjectures. On the other hand, an *INIT*-strategy learning, say, the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ tries to find the maximum number 7. While, every next inputted number may increase the lower bound on its final conjecture, the strategy never is aware of any upper bound on it.

It has been demonstrated in [JS96] that many important simple learning problems (in particular, *pattern languages*) can be handled by strategies of these basic types. Now, the corresponding degrees of complexity *INIT* and *COINIT* can be used to form a complex hierarchy of degrees as follows. Imagine a three-dimensional language L . Suppose an *INIT*-type strategy M_1 can be used to learn its first dimension, L_1 . Once this dimension has been learned, a strategy of a different (or even same) type, say, *COINIT* can pick the grammar learned by M_1 and use this information to learn the second dimension L_2 . Consequently, the grammar learned by the *COINIT*-strategy M_2 can be used to learn the third dimension L_3 by a strategy M_3 of type, say, *INIT*. Thus, we get a strategy of the type (*INIT*, *COINIT*, *INIT*), where information learned by the learner M_i is relayed to the learner M_{i+1} making it possible to learn the next dimension. This idea can be naturally extended to any (finite) number of dimensions and to any sequences $Q = (q_1, q_1, \dots, q_k)$ of strategies $q_i \in \{\textit{INIT}, \textit{COINIT}\}$. It has been shown in [JKW00] that the degrees of complexity (classes of languages) corresponding to such Q -strategies form a rich hierarchy. For example, some classes learnable by (*INIT*, *COINIT*, *INIT*)-type strategies cannot be learned by any (*INIT*, *COINIT*)-strategy. In other words, such a class can be learned by a strategy that first works as an *INIT*-strategy, then it changes to a *COINIT*-strategy to learn another aspect of the concept, and then it changes to a *INIT*-strategy to learn the last component of the concept. On the other hand, changing strategies only one time (from *INIT* to *COINIT*) is not enough.

How can one apply the above hierarchy to quantify the complexity of learning semi-hulls? Let us take a closer look at the strategy learning semi-hulls (described in the beginning of the paper). The reader may have noticed that the strategy learning the first break point (as well as any other break point) is an *INIT*-type strategy: it tries to find the maximum number in a growing finite set. Now, once a break point has been found, what strategy can learn the slope? Let us assume that the slopes come from the set $N \cup \{1/n \mid n \in N\}$ (actually, sets of possible slopes in our model will be equivalent to this set). Suppose the actual slope to be learned is $1/7$. Suppose also that, based on some initial portion of the input, we conjectured the slope 5. That tells us that the actual slope cannot be greater than 5. Therefore, every following conjecture will be a number between 5 and $1/7$, getting closer and closer to $1/7$, but never being aware of any lower bound on its final guess $1/7$. This strategy seems to be similar to a *COINIT*-strategy discussed above. However, unlike a *COINIT*-strategy, it is never aware of any lower bound on possible conjectures. One can easily see that a very similar (equivalent) strategy can learn

sets $\{\dots - k, -(k - 1), \dots, 0, 1, 2, 3, \dots\}$. Based on this observation, we call such strategies *HALF*-strategies (learning “halfs” of the set of integers). We also show that these strategies, while being still quite primitive, are more powerful than *COINIT*-strategies (moreover, in certain sense, they can be regarded as Cartesian products of *INIT* and *COINIT*-strategies).

Thus, to learn the first break point we use an *INIT*-strategy, then we use a *HALF*-strategy to learn the first slope, then we change back to an *INIT*-strategy to learn the second break point, then to a *HALF*-strategy to learn the second slope, etc. If, for example, a semi-hull has two angles (the border line has two break points), then we use the sequence of strategies *INIT*, *HALF*, *INIT*, *HALF*. In other words, the corresponding learning problem belongs to the level (*INIT*, *HALF*, *INIT*, *HALF*) of the abovementioned hierarchy. Obviously, learning the semi-hulls with two angles by an (*INIT*, *HALF*, *INIT*)-strategy (if possible) can be viewed as more efficient. We will show, however, that such a strategy is not possible. Another question is if semi-hulls with two angles are learnable by an (*INIT*, *COINIT*, *INIT*, *COINIT*) - strategy. One must again agree that this strategy can be considered as a more efficient than our (*INIT*, *HALF*, *INIT*, *HALF*)-strategy, since, as we mentioned above, *COINIT*-strategies generally are more primitive (less capable) than *HALF*-strategies. We will show that such a strategy does exist for semi-hulls with two angles (Theorem 3; it is somewhat less intuitive than our original (*INIT*, *HALF*, *INIT*, *HALF*)- strategy). We will also show that for example, neither (*COINIT*, *INIT*, *COINIT*, *INIT*)- strategy nor any strategy with a number of components (from *INIT*, *COINIT*, *HALF*) smaller than 4 can learn the given class, as it easily follows from Corollary 6. In general, for every class of semi-hulls with fixed numbers of angles (two, or three, or four, etc.) we establish upper and lower bounds of this type.

We submit that this approach to measuring the complexity of learning is very reasonable for geometrical concepts of this and similar types. As we already mentioned, some more traditional measures of complexity (like the number of mind changes) are not applicable, since these measures cannot make distinction between learning one-angle semi-hulls and, say, three-angle ones.

We use also a similar approach to examine the power of learning semi-hulls (and other figures) from a slightly different perspective. Namely, suppose we are given some strategy to learn two-angle semi-hulls. How can knowledge of such a strategy help us to learn problems like (*INIT*, *COINIT*, *INIT*) (viewed this time as families of languages rather than strategies)? We can show that, being armed with such a strategy, one can learn, for example, the class of languages (*HALF*, *INIT*) (Theorem 5). On the other hand, being able to learn two-angle semi-hulls does not help to learn the class (*INIT*, *INIT*, *INIT*) (Corollary 7) or (*COINIT*, *INIT*, *INIT*) (Corollary 8). It also does not help to learn any similar class with *COINIT* as the second component (Corollaries 9 and 10). In this respect, (*HALF*, *INIT*) can be regarded as a lower bound on the power of learning two-angle semi-hulls.

The paper has the following structure. Section 2 introduces notations and preliminaries. In Section 3 we define the reductions and the degrees of complexity. In Section 4 we give formal definition of the *Q*-classes and degrees. This definition extends the definition of the *Q*-classes in [JKW00]: in addition to the classes *INIT* and *COINIT*, we use in vectors *Q* a new class of strategies/languages, *HALF*, that turns out to be different from *INIT* and *COINIT* and is useful for classifying geometrical concepts. In Section 5 we show that the *Q*-hierarchy can be appropriately extended to the class of learning strategies/languages involving *HALF*. In Section 6 we define the classes of the type *SEMI_HULL* that formalize intuitive geometrical concepts described above. In this section we also prove some useful technical propositions. In Sections 7 and 8 we establish upper and lower bounds for the *SEMI_HULL* degrees in terms of the *Q*-hierarchy. In particular, we establish that semi-hulls with *n* angles can be learned by a (*INIT*, *COINIT*, ..., *INIT*, *COINIT*)- strategy with $2n$ components (Theorem 3). On the other

hand, no strategy, with alternating *INIT*s and *COINIT*s and a smaller number of components, can learn such concepts (Corollary 6). We also examine how learning semi-hulls can help to learn the Q -classes (Theorem 5). Corollaries 7, 8, 9 and 10 give examples of the Q -classes that cannot be learned by strategies armed with a strategy capable of learning semi-hulls. These examples show that the Q -classes found in Theorem 5 are the most complex Q -classes that can be learned using a strategy for learning semi-hulls. Therefore, they can be regarded as the best possible lower bounds (based on the Q -classes considered here) on the power of learning semi-hulls.

In Section 9 we introduce the classes *coSEMI_HULL* that consist of complements of languages in *SEMI_HULL*. Sections 10 and 11 establish lower and upper bounds for *coSEMI_HULL*s in terms of the Q -hierarchy. In particular, we show that the most efficient strategy for learning complements of semi-hulls with n angles is $(INIT, COINIT, \dots, COINIT)$ with n occurrences of *COINIT* (Theorem 10). We also give examples of the Q -classes that can and cannot be learned by strategies having access to strategies learning *coSEMI_HULL*s, providing the best possible lower bounds for the learnability power of complements of semi-hulls.

Upper and lower bounds for *SEMI_HULL*s and *coSEMI_HULL*s come close, but do not match (though, upper bounds are much lower than the ones suggested by intuitive strategies learning the classes in question). In Section 12 we define the classes of *open hulls* formed by intersections of languages in *SEMI_HULL*s adjacent to x and y -axis; Figure 2 shows an example of open hull. For the complexity of learning these classes, we have established matching upper and lower bounds in terms of the Q -hierarchy. Namely, we show that the most efficient strategy for learning open hulls with at most n angles on each side is $(INIT, INIT, \dots, INIT)$ with n components *INIT* (Theorem 14). It turns out also (Theorem 13) that a strategy armed with a strategy for learning open hulls can learn the class $(INIT, INIT, \dots, INIT)$ with n components. (It seems a bit counterintuitive that open hulls can be learned by strategies simpler than semi-hulls. We will give some intuition behind the corresponding strategy in Theorem 14).

In Section 13 we define the classes of languages formed by complements of open hulls and establish matching upper and lower bounds for the corresponding degrees of intrinsic complexity. All the abovementioned upper bounds are much lower than the ones suggested by intuitive learning strategies.

2 Notation and Preliminaries

Any unexplained recursion theoretic notation is from [Rog67]. The symbol N denotes the set of natural numbers, $\{0, 1, 2, 3, \dots\}$. Z denotes the set of integers. Z^- denotes the set of negative integers. $i \dot{-} j$ is defined as follows:

$$i \dot{-} j = \begin{cases} i - j, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Symbols \emptyset , \subseteq , \subset , \supseteq , and \supset denote empty set, subset, proper subset, superset, and proper superset, respectively. D_0, D_1, \dots , denotes a canonical recursive indexing of all the finite sets [Rog67]. We assume that if $D_i \subseteq D_j$ then $i \leq j$ (the canonical indexing defined in [Rog67] satisfies this property). Cardinality of a set S is denoted by $\text{card}(S)$. The maximum and minimum of a set are denoted by $\max(\cdot)$, $\min(\cdot)$, respectively, where $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$.

We let $\langle \cdot, \cdot \rangle$ stand for an arbitrary, computable, bijective mapping from $N \times N$ onto N [Rog67]. We assume without loss of generality that $\langle \cdot, \cdot \rangle$ is monotonically increasing in both its arguments. We define $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$. $\langle \cdot, \cdot \rangle$ can be extended to n -tuples in a natural way (including $n = 1$, where $\langle x \rangle$ may be taken to be x). Projection functions π_1, \dots, π_n

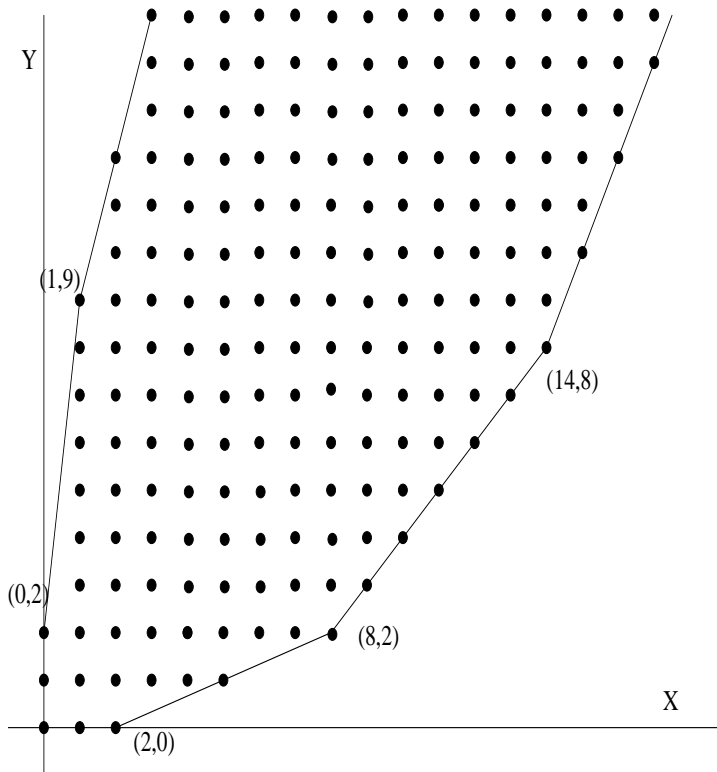


Figure 2: Open Hull

corresponding to n -tuples can be defined similarly (where the tuple size would be clear from context). Due to above isomorphism between N^n and N , we often identify (x_1, \dots, x_n) with $\langle x_1, \dots, x_n \rangle$. Thus we can say $L_1 \times L_2 = \{\langle x, y \rangle \mid x \in L_1, y \in L_2\}$.

By φ we denote a fixed *acceptable* programming system for the partial computable functions: $N \rightarrow N$ [Rog67, MY78]. By φ_i we denote the partial computable function computed by the program with number i in the φ -system. Symbol \mathcal{R} denotes the set of all total computable functions. By Φ we denote an arbitrary fixed Blum complexity measure [Blu67, HU79] for the φ -system. By W_i we denote $\text{domain}(\varphi_i)$. W_i is, then, the r.e. set/language ($\subseteq N$) accepted (or equivalently, generated) by the φ -program i . We also say that i is a grammar for W_i . Symbol \mathcal{E} will denote the set of all r.e. languages. Symbol L , with or without decorations, ranges over \mathcal{E} . By \bar{L} , we denote the complement of L , that is $N - L$. Symbol \mathcal{L} , with or without decorations, ranges over subsets of \mathcal{E} . We denote by $W_{i,s}$ the set $\{x < s \mid \Phi_i(x) < s\}$.

\downarrow denotes defined or converges. \uparrow denotes undefined or diverges.

A partial function F from N to N is said to be partial limit recursive, iff there exists a recursive function f from $N \times N$ to N such that for all x , $F(x) = \lim_{y \rightarrow \infty} f(x, y)$. Here if $F(x)$ is not defined then $\lim_{y \rightarrow \infty} f(x, y)$, must also be undefined. A partial limit recursive function F is called (total) limit recursive function, if F is total.

We now present concepts from language learning theory. The next definition introduces the notion of a *sequence* of data.

Definition 1 (a) A *finite sequence* σ is a mapping from an initial segment of N into $(N \cup \{\#\})$. The empty sequence is denoted by Λ .

(b) The *content* of a finite sequence σ , denoted $\text{content}(\sigma)$, is the set of natural numbers in the range of σ .

(c) The *length* of σ , denoted by $|\sigma|$, is the number of elements in σ . So, $|\Lambda| = 0$.

(d) For $n \leq |\sigma|$, the initial sequence of σ of length n is denoted by $\sigma[n]$. So, $\sigma[0]$ is Λ .

Intuitively, $\#$'s represent pauses in the presentation of data. We let σ , τ , and γ , with or without decorations, range over finite sequences. We denote the sequence formed by the concatenation of τ at the end of σ by $\sigma \diamond \tau$. Sometimes we abuse the notation and use $\sigma \diamond x$ to denote the concatenation of sequence σ and the sequence of length 1 which contains the element x . SEQ denotes the set of all finite sequences.

Definition 2 [Gol67] (a) A *text* T for a language L is a mapping from N into $(N \cup \{\#\})$ such that L is the set of natural numbers in the range of T .

(b) The *content* of a text T , denoted $\text{content}(T)$, is the set of natural numbers in the range of T .

(c) $T[n]$ denotes the finite initial sequence of T with length n .

We let T , with or without decorations, range over texts. We let \mathcal{T} range over sets of texts.

Definition 3 [Gol67] A *language learning machine* is an algorithmic device which computes a mapping from SEQ into N .

We let \mathbf{M} , with or without decorations, range over learning machines. $\mathbf{M}(T[n])$ is interpreted as the grammar (index for an accepting program) conjectured by the learning machine \mathbf{M} on the initial sequence $T[n]$. We say that \mathbf{M} converges on T to i , (written $\mathbf{M}(T)\downarrow = i$) iff $(\forall^\infty n)[\mathbf{M}(T[n]) = i]$.

There are several criteria for a learning machine to be successful on a language. Below we define identification in the limit introduced by Gold [Gol67].

Definition 4 [Gol67, CS83] Suppose $a \in N \cup \{*\}$.

(a) \mathbf{M} **TxtEx**-identifies a text T if and only if $(\exists i \mid W_i = \text{content}(T)) (\forall^\infty n)[\mathbf{M}(T[n]) = i]$.

(b) \mathbf{M} **TxtEx**-identifies an r.e. language L (written: $L \in \mathbf{TxtEx}(\mathbf{M})$) if and only if \mathbf{M} **TxtEx**-identifies each text for L .

(c) \mathbf{M} **TxtEx**-identifies a class \mathcal{L} of r.e. languages (written: $\mathcal{L} \subseteq \mathbf{TxtEx}(\mathbf{M})$) iff \mathbf{M} **TxtEx**-identifies each $L \in \mathcal{L}$.

(d) $\mathbf{TxtEx} = \{\mathcal{L} \subseteq \mathcal{E} \mid (\exists \mathbf{M})[\mathcal{L} \subseteq \mathbf{TxtEx}(\mathbf{M})]\}$.

Other criteria of success are finite identification [Gol67], behaviorally correct identification [Fel72, OW82, CL82], and vacillatory identification [OW82, Cas99]. In the present paper, we only discuss results about **TxtEx**-identification.

3 Reductions

We first present some technical machinery.

We write $\sigma \subseteq \tau$ if σ is an initial segment of τ , and $\sigma \subset \tau$ if σ is a proper initial segment of τ . Likewise, we write $\sigma \subset T$ if σ is an initial finite sequence of text T . Let finite sequences $\sigma^0, \sigma^1, \sigma^2, \dots$ be given such that $\sigma^0 \subseteq \sigma^1 \subseteq \sigma^2 \subseteq \dots$ and $\lim_{i \rightarrow \infty} |\sigma^i| = \infty$. Then there is a unique text T such that for all $n \in N$, $\sigma^n = T[\sigma^n]$. This text is denoted by $\bigcup_n \sigma^n$. Let \mathbf{T} denote the set of all texts, that is, the set of all infinite sequences over $N \cup \{\#\}$.

We define an *enumeration operator* (or just operator), Θ , to be an algorithmic mapping from SEQ into SEQ such that for all $\sigma, \tau \in \text{SEQ}$, if $\sigma \subseteq \tau$, then $\Theta(\sigma) \subseteq \Theta(\tau)$. We further assume that for all texts T , $\lim_{n \rightarrow \infty} |\Theta(T[n])| = \infty$. By extension, we think of Θ as also defining a mapping from \mathbf{T} into \mathbf{T} such that $\Theta(T) = \bigcup_n \Theta(T[n])$.

A final notation about the operator Θ . If for a language L , there exists an L' such that for each text T for L , $\Theta(T)$ is a text for L' , then we write $\Theta(L) = L'$, else we say that $\Theta(L)$ is undefined. The reader should note the overloading of this notation because the type of the argument to Θ could be a sequence, a text, or a language; it will be clear from the context which usage is intended.

We let $\Theta(\mathcal{T}) = \{\Theta(T) \mid T \in \mathcal{T}\}$, and $\Theta(\mathcal{L}) = \{\Theta(L) \mid L \in \mathcal{L}\}$.

We also need the notion of an infinite sequence of grammars. We let α , with or without decorations, range over infinite sequences of grammars. From the discussion in the previous section it is clear that infinite sequences of grammars are essentially infinite sequences over N . Hence, we adopt the machinery defined for sequences and texts over to finite sequences of grammars and infinite sequences of grammars. So, if $\alpha = i_0, i_1, i_2, i_3, \dots$, then $\alpha[3]$ denotes the sequence i_0, i_1, i_2 , and $\alpha(3)$ is i_3 . Furthermore, we say that α converges to i if there exists an n such that, for all $n' \geq n$, $i_{n'} = i$.

Let \mathbf{I} be any criterion for language identification from texts, for example $\mathbf{I} = \mathbf{TextEx}$. We say that an infinite sequence α of grammars is \mathbf{I} -admissible for text T just in case α witnesses \mathbf{I} -identification of text T . So, if $\alpha = i_0, i_1, i_2, \dots$ is a \mathbf{TextEx} -admissible sequence for T , then α converges to some i such that $W_i = \text{content}(T)$; that is, the limit i of the sequence α is a grammar for the language $\text{content}(T)$.

Intuitively, a learning problem \mathcal{L}_1 is reducible to a learning problem \mathcal{L}_2 if, using a strategy learning \mathcal{L}_1 one can learn \mathcal{L}_2 .

We now formally introduce our reductions. Although in this paper we will only be concerned with \mathbf{TextEx} -identification, we present the general case of the definition.

Definition 5 [JS96] Let $\mathcal{L}_1 \subseteq \mathcal{E}$ and $\mathcal{L}_2 \subseteq \mathcal{E}$ be given. Let identification criterion \mathbf{I} be given. Let $\mathcal{T}_1 = \{T \mid T \text{ is a text for } L \in \mathcal{L}_1\}$. Let $\mathcal{T}_2 = \{T \mid T \text{ is a text for } L \in \mathcal{L}_2\}$. We say that $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ if and only if there exist operators Θ and Ψ such that, for all $T, T' \in \mathcal{T}_1$ and for all infinite sequences α of grammars, the following hold:

- (a) $\Theta(T) \in \mathcal{T}_2$ and
- (b) if α is an \mathbf{I} -admissible sequence for $\Theta(T)$, then $\Psi(\alpha)$ is an \mathbf{I} -admissible sequence for T .
- (c) if $\text{content}(T) = \text{content}(T')$ then $\text{content}(\Theta(T)) = \text{content}(\Theta(T'))$.

We say that $\mathcal{L}_1 \equiv^{\mathbf{I}} \mathcal{L}_2$ iff $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ and $\mathcal{L}_2 \leq^{\mathbf{I}} \mathcal{L}_1$.

The reduction defined above was called strong-reduction in [JS96]. The above reduction without clause (c) was called weak-reduction. Since in this paper we will be only concerned with strong-reductions, we just refer to them as reductions. (Weak reductions are not sharp enough to provide real distinction between most of the classes considered in this paper).

Intuitively, $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ just in case there exists an operator Θ that transforms texts for languages in \mathcal{L}_1 into texts for languages in \mathcal{L}_2 and there exists another operator Ψ that behaves as follows: if Θ transforms text T (for a language in \mathcal{L}_1) to text T' (for a language in \mathcal{L}_2), then Ψ transforms \mathbf{I} -admissible sequences for T' into \mathbf{I} -admissible sequences for T . Thus, informally, the operator Ψ has to work only on \mathbf{I} -admissible sequences for such texts T' . In other words, if α is a sequence of grammars which is *not* \mathbf{I} -admissible for any text T' in $\{\Theta(T) \mid \text{content}(T) \in \mathcal{L}_1\}$, then $\Psi(\alpha)$ can be defined *arbitrarily*. This property will be used implicitly at all places below where we have to define operators Ψ witnessing (together with operators Θ) some reducibility. Note that this approach both simplifies the corresponding definitions and preserves the computability of the so defined operators.

Additionally different texts for some language $L \in \mathcal{L}_1$, are transformed into (possibly different) texts for same language $L' \in \mathcal{L}_2$.

Now, a *degree* of learnability under our reduction is, naturally, a set of families \mathcal{L} reducible to each other (i.e. the equivalence class under the reduction considered).

Intuitively, for many identification criteria \mathbf{I} such as **TxtEx**, if $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ then the problem of identifying \mathcal{L}_2 in the sense of \mathbf{I} is at least as hard as the problem of identifying \mathcal{L}_1 in the sense of \mathbf{I} , since the solvability of the former problem implies the solvability of the latter one. That is given any machine \mathbf{M}_2 which \mathbf{I} -identifies \mathcal{L}_2 , one can construct a machine \mathbf{M}_1 which \mathbf{I} -identifies \mathcal{L}_1 . To see this, for $\mathbf{I} = \mathbf{TxtEx}$, suppose Θ and Ψ witness $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$. $\mathbf{M}_1(T)$, for a text T is defined as follows. Let $p_n = \mathbf{M}_2(\Theta(T)[n])$, and $\alpha = p_0, p_1, \dots$. Let $\alpha' = \Psi(\alpha) = p'_0, p'_1, \dots$. Then let $\mathbf{M}_1(T) = \lim_{n \rightarrow \infty} p'_n$. Consequently, \mathcal{L}_2 may be considered as a hardest problem for \mathbf{I} -identification if for *all* classes $\mathcal{L}_1 \in \mathbf{I}$, $\mathcal{L}_1 \leq^{\mathbf{I}} \mathcal{L}_2$ holds. If \mathcal{L}_2 itself belongs to \mathbf{I} , then \mathcal{L}_2 is said to be *complete*. We now formally define these notions of hardness and completeness for the above reduction.

Definition 6 [JS96] Let \mathbf{I} be an identification criterion. Let $\mathcal{L} \subseteq \mathcal{E}$ be given.

- (a) If for all $\mathcal{L}' \in \mathbf{I}$, $\mathcal{L}' \leq^{\mathbf{I}} \mathcal{L}$, then \mathcal{L} is $\leq^{\mathbf{I}}$ -hard.
- (b) If \mathcal{L} is $\leq^{\mathbf{I}}$ -hard and $\mathcal{L} \in \mathbf{I}$, then \mathcal{L} is $\leq^{\mathbf{I}}$ -complete.

Proposition 1 ([JS96]) $\leq^{\mathbf{TxtEx}}$ is reflexive and transitive.

The above proposition holds for most natural learning criteria.

Proposition 2 (based on [JS97]) Suppose $\mathcal{L} \leq^{\mathbf{I}} \mathcal{L}'$, via Θ and Ψ . Then, for all $L, L' \in \mathcal{L}$, $L \subseteq L' \Rightarrow \Theta(L) \subseteq \Theta(L')$.

We will be using Proposition 2 implicitly when we are dealing with reductions. Since, for $\mathcal{L} \leq^{\mathbf{I}} \mathcal{L}'$ via Θ and Ψ , for all $L \in \mathcal{L}$, $\Theta(L)$ is defined (= some $L' \in \mathcal{L}'$), when considering reductions, we often consider Θ as mapping finite sets to (possibly infinite) sets instead of mapping sequences to sequences.¹ This is clearly without loss of generality, as one can easily convert such Θ to Θ as in Definition 5 of reduction.

4 Q-classes

In this section we introduce the classes of languages and corresponding degrees of intrinsic complexity that form the scale being used for estimating the complexity of learning open semi-hulls and open hulls. First we define ground natural classes that are being used as bricks to build our hierarchy of degrees.

Definition 7 $INIT = \{L \subseteq N \mid (\exists i \in N)[L = \{x \in N \mid x \leq i\}]\}$.

$COINIT = \{L \subseteq N \mid (\exists i \in N)[L = \{x \in N \mid x \geq i\}]\}$.

$HALF = \{L \subseteq Z \mid (\exists i \in Z)[L = \{x \in Z \mid x \geq i\}]\}$.

Note that officially our definition for languages and r.e. sets as in the Section 2, only allows subsets of N . Since, one can easily code Z onto N , by slight abuse of convention, we can consider subsets of Z also as languages. We thus assume an implicit coding of Z onto N whenever we deal with languages and language classes involving *HALF*, without explicitly stating so.

In the sequel, we will use the above notation in two different contexts. Namely, we will use *INIT* to denote the class of languages as defined above, and to denote the degree of *all* classes of languages equivalent to the class *INIT* under our reductions. Similarly, we will use in two different contexts *COINIT*, *HALF* and all their combinations defined below in this section. In every use of this notation, the reader will be able to easily determine an appropriate context.

¹For infinite X , by definition, $\Theta(X)$ would be $\bigcup_{X' \subseteq X: \text{card}(X') < \infty} \Theta(X')$.

While both the classes *INIT* and *COINIT* are monotonically learnable, the types of conjectures being used to learn *INIT* and, respectively, *COINIT* are obviously different. For *INIT* the maximum element in the input gives a code for the language, whereas in *COINIT* the minimum element gives the code. Note that the maximum element used in *INIT* strategy is unbounded, whereas the minimum element for *COINIT* is bounded by 0. So, not surprisingly, the degrees of *INIT* and *COINIT* were proven in [JS96] to be different.

Classes *HALF* and *COINIT* are learnable by similar strategies, however, the minimum element in *HALF* is unbounded. We will formally prove below that degree of *HALF* is different from degrees of both *INIT* and *COINIT*. Furthermore, we will show that the degree of *HALF* can be viewed as a cross product of the degrees of *INIT* and *COINIT*.

There are several other natural classes considered in the literature such as *FINITE* (degree of which is equivalent to *INIT*), *SINGLE*, *COSINGLE*, etc. but we will not be concerned with them here since they will not be relevant to our results.

Now we define the cross product of arbitrary classes \mathcal{L}_1 and \mathcal{L}_2 .

Definition 8 Let $\mathcal{L}_1, \mathcal{L}_2$ be two classes of languages. Then $\mathcal{L}_1 \times \mathcal{L}_2 = \{L_1 \times L_2 \mid L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$.

This definition can be naturally extended to any finite number of dimensions. For example, one can naturally define $\mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$, etc.

Theorem 1 $HALF \equiv^{\text{TxtEx}} INIT \times COINIT$.

Proof of the above Theorem is given in the Appendix.

Now, following [JKW00], we are going to combine the classes *INIT*, *COINIT*, and *HALF* to form classes of multidimensional languages, where, to learn the dimension L_{k+1} of a language L , the learner must first learn the parameters i_1, \dots, i_k of the dimensions L_1, \dots, L_k ; then L_{k+1} is the projection $\{x_{k+1} \mid \langle i_1, \dots, i_k, x_{k+1}, x_{k+2}, \dots, x_n \rangle \in L\}$ with a simple sublanguage whose description is specified yet by i_1, \dots, i_k . Once it has been determined which projection must be learned, the learner can use a predefined *INIT*, *COINIT*, or *HALF*-type strategy to learn the projection in question. For example, one can consider a class of two-dimensional languages (*INIT*, *COINIT*), where the first dimension $L_1 = \{x \mid \langle x, y \rangle \in L\}$ of any language L belongs to *INIT*, and if i is the parameter describing L_1 (that can be learned by an *INIT*-type strategy) then the projection $\{y \mid \langle i, y \rangle \in L\}$ is in *COINIT*.

Below for any tuples X and Y , let $X \cdot Y$ denote the concatenation of X and Y . That is if $X = \langle x_1, x_2, \dots, x_n \rangle$ and $Y = \langle y_1, y_2, \dots, y_m \rangle$ then $X \cdot Y = \langle x_1 \dots, x_n, y_1, \dots, y_m \rangle$. Let $BASIC = \{INIT, COINIT, HALF\}$.

In part (c) of the following definition and in later situations in languages involving *HALF* we sometimes abuse notation slightly and allow elements of Z as components of the pairing function $\langle \dots \rangle$. This is for ease of notation, and one could easily replace these by using some coding of Z onto N .

Definition 9 [JKW00] Suppose $k \geq 1$. Let $Q \in BASIC^k$. Let $I \in N^k$. Then inductively on k , we define the languages L_I^Q and $T(L_I^Q)$ and $P(L_I^Q)$ as follows.

If $k = 1$, then

- (a) if $Q = (INIT)$ and $I = (i)$, $i \in N$, then
 $T(L_I^Q) = \{\langle x \rangle \mid x \in N, x < i\}$, $P(L_I^Q) = \{\langle i \rangle\}$, and $L_I^Q = T(L_I^Q) \cup P(L_I^Q)$.
- (b) if $Q = (COINIT)$ and $I = (i)$, $i \in N$, then
 $T(L_I^Q) = \{\langle x \rangle \mid x \in N, x > i\}$, $P(L_I^Q) = \{\langle i \rangle\}$, and $L_I^Q = T(L_I^Q) \cup P(L_I^Q)$.
- (c) if $Q = (HALF)$ and $I = (i)$, $i \in Z$, then

$T(L_I^Q) = \{\langle x \rangle \mid x \in Z, x > i\}$, $P(L_I^Q) = \{\langle i \rangle\}$, and $L_I^Q = T(L_I^Q) \cup P(L_I^Q)$.

Now suppose we have already defined L_I^Q for $k \leq n$. We then define L_I^Q for $k = n + 1$ as follows. Suppose $Q = (q_1, \dots, q_{n+1})$ and $I = (i_1, \dots, i_{n+1})$. Let $Q_1 = (q_1)$ and $Q_2 = (q_2, \dots, q_{n+1})$. Let $I_1 = (i_1)$ and $I_2 = (i_2, \dots, i_{n+1})$. Then,

$T(L_I^Q) = \{X \cdot Y \mid X \in T(L_{I_1}^{Q_1}), \text{ or } [X \in P(L_{I_1}^{Q_1}) \text{ and } Y \in T(L_{I_2}^{Q_2})]\}$,

$P(L_I^Q) = \{X \cdot Y \mid X \in P(L_{I_1}^{Q_1}) \text{ and } Y \in P(L_{I_2}^{Q_2})\}$, and

$L_I^Q = T(L_I^Q) \cup P(L_I^Q)$.

Intuitively, in the above definition $T(L_I^Q)$ denotes the terminating part of the language that is specified yet by i_1, \dots, i_n, i_{n+1} , and $P(L_I^Q)$ denotes the propagating part of the language L_I^Q that could be used for adding a language in dimension $n + 2$. (See [JKW00] for more details and motivation on the terminology of terminating and propagating.)

For ease of notation we often write $L_{(i_1, i_2, \dots, i_k)}^Q$ as $L_{i_1, i_2, \dots, i_k}^Q$.

Definition 10 Let $k \geq 1$. Let $Q = (q_1, \dots, q_k) \in \text{BASIC}^k$ and $R = R_1 \times R_2 \times \dots \times R_k$, where for $1 \leq i \leq k$, $R_i \subseteq N$ if $q_i \in \{\text{INIT}, \text{COINIT}\}$, and $R_i \subseteq Z$, if $q_i = \text{HALF}$. Then the class $\mathcal{L}^{Q,R}$ is defined as

$$\mathcal{L}^{Q,R} = \{L_I^Q \mid I \in R\}.$$

For technical convenience, for $Q = ()$, $I = ()$, $R = \{I\}$, we also define $T(L_I^Q) = \emptyset$, $P(L_I^Q) = \{\langle \rangle\}$, and $L_I^Q = T(L_I^Q) \cup P(L_I^Q)$, and $\mathcal{L}^{Q,R} = \{L_I^Q\}$.

Note that we have used a slightly different notation for defining the classes $\mathcal{L}^{Q,R}$ (for example instead of INIT , we now use $\mathcal{L}^{(\text{INIT}, N)}$). This is for clarity of notation.

Also, our main interest is for R_i 's being N or Z (based on whether $q_i \in \{\text{INIT}, \text{COINIT}\}$ or $q_i = \text{HALF}$), though (as the following proposition shows) it doesn't matter as long as R_i is (or contains) an infinite recursive subset of N , if $q_i \in \{\text{INIT}, \text{COINIT}\}$, and R_i is (or contains) an infinite recursive subset of Z with infinite intersection with both N and Z^- , if $q_i = \text{HALF}$. The usage of general R is more for ease of proving some of our theorems.

Proposition 3 Suppose $k \geq 1$. Let $Q \in \text{BASIC}^k$. Let $R = R_1 \times R_2 \times \dots \times R_k$, where each R_i is an infinite recursive subset of N , if $q_i \in \{\text{INIT}, \text{COINIT}\}$ and R_i is a recursive subset of Z , with infinite intersection with both N and Z^- , if $q_i = \text{HALF}$. Let $R' = R'_1 \times R'_2 \times \dots \times R'_k$, where R'_i is N , if $q_i \in \{\text{INIT}, \text{COINIT}\}$ and R'_i is Z , if $q_i = \text{HALF}$. Then, $\mathcal{L}^{Q,R} \equiv^{\text{TxtEx}} \mathcal{L}^{Q,R'}$.

For ease of notation, if $Q = (q_1, q_2, \dots, q_n)$ and $R = R_1 \times R_2 \times \dots \times R_n$, where $R_i = N$ if $q_i \in \{\text{INIT}, \text{COINIT}\}$ and $R_i = Z$ if $q_i = \text{HALF}$, then we drop R from $\mathcal{L}^{Q,R}$, using just \mathcal{L}^Q .

The immediate question is which classes Q represent different degrees.

Proposition 4 Suppose $Q = (q_1, \dots, q_{k-1}, q_k, q_{k+1}, \dots, q_l)$, and $Q' = (q_1, \dots, q_{k-1}, q', q_{k+1}, \dots, q_l)$, where $q_k \in \{\text{INIT}, \text{COINIT}\}$ and $q' = \text{HALF}$, and other $q_i \in \text{BASIC}$.

Then, $\mathcal{L}^Q \leq^{\text{TxtEx}} \mathcal{L}^{Q'}$.

PROOF. Above can be easily shown using $\text{INIT} \leq^{\text{TxtEx}} \text{HALF}$. ■

Since $\text{INIT} \times \text{COINIT} \leq^{\text{TxtEx}} \mathcal{L}^Q$, for $Q = (\text{INIT}, \text{COINIT})$, or $Q = (\text{COINIT}, \text{INIT})$ [JKW00], we have the following proposition.

Proposition 5 Suppose

$Q = (q_1, \dots, q_{k-1}, \text{HALF}, q_{k+1}, \dots, q_n)$, and $Q' = (q_1, \dots, q_{k-1}, \text{INIT}, \text{COINIT}, q_{k+1}, \dots, q_n)$ or $Q' = (q_1, \dots, q_{k-1}, \text{COINIT}, \text{INIT}, q_{k+1}, \dots, q_n)$. Then $\mathcal{L}^Q \leq^{\text{TxtEx}} \mathcal{L}^{Q'}$.

PROOF. Above can be easily shown using $HALF \leq^{\text{TxtEx}} (INIT, COINIT)$ and $HALF \leq^{\text{TxtEx}} (COINIT, INIT)$. ■

Proposition 6 (Based on [JS96]) Suppose $Q = (INIT)$, $R = R_1$, $Q' = (COINIT)$, and $R' = R'_1$, where R_1 and R'_1 are infinite subsets of N . Then $\mathcal{L}^{Q,R} \not\leq^{\text{TxtEx}} \mathcal{L}^{Q',R'}$, and $\mathcal{L}^{Q',R'} \not\leq^{\text{TxtEx}} \mathcal{L}^{Q,R}$.

The following technical definition introduces an ordering on all k -tuples of parameters i_1, \dots, i_k of languages in an arbitrary class \mathcal{L}^Q . This ordering will be helpful for the result in the next section.

Definition 11 Suppose $Q = (q_1, \dots, q_k)$, where each $q_i \in \{HALF, INIT, COINIT\}$, for $1 \leq i \leq k$. Let $Q' = (q'_1, \dots, q'_k)$. We say that $\langle i_1, \dots, i_k \rangle \leq_Q \langle j_1, \dots, j_k \rangle$ iff

- (a) if $q_1 = INIT$, then $[i_1 < j_1]$ or $[i_1 = j_1 \text{ and } \langle i_2, \dots, i_k \rangle \leq_{Q'} \langle j_2, \dots, j_k \rangle]$;
- (b) if $q_1 = COINIT$ or $q_1 = HALF$, then $[i_1 > j_1]$ or $[i_1 = j_1 \text{ and } \langle i_2, \dots, i_k \rangle \leq_{Q'} \langle j_2, \dots, j_k \rangle]$.

Here, for $Q = ()$, we assume that $\langle \rangle \leq_Q \langle \rangle$.

5 Q-hierarchy Involving HALF

In this section we establish a hierarchy among the Q -classes that will serve as a scale for the complexity of our geometrical concepts.

Definition 12 Q is said to be a pseudo-subsequence of Q' , iff there exists a subsequence Q'' of Q' , such that Q'' is obtainable from Q by

- (i) replacing some $INIT$ with $HALF$,
- (ii) replacing some $COINIT$ with $HALF$,
- (iii) replacing some $HALF$ with $(COINIT, INIT)$, or
- (iv) replacing some $HALF$ with $(INIT, COINIT)$.

For example, $Q = (INIT, HALF, COINIT)$ is a pseudo-subsequence of $Q' = (COINIT, HALF, COINIT, INIT, INIT, COINIT)$: dropping the first $COINIT$ and the second $INIT$ from Q' , we get the subsequence $Q'' = (HALF, COINIT, INIT, COINIT)$; now Q can be promoted to Q'' by replacing $INIT$ with a more powerful $HALF$ and replacing $HALF$ (in its middle) with a more powerful combination $COINIT, INIT$.

In other words, a pseudosubsequence of a vector Q is a subsequence where some $HALF$ s are replaced with more primitive $INIT$ s or $COINIT$ s and some combinations $INIT, COINIT$ or $COINIT, INIT$ are replaced with more primitive $HALF$ s.

Proposition 7 Suppose $Q, Q', Q'' \in \text{BASIC}^*$. Then if Q is a pseudo-subsequence of Q' and Q' is a pseudo-subsequence of Q'' , then Q is a pseudo-subsequence of Q'' .

PROOF. Follows from definition of pseudo-subsequence. ■

Proposition 8 Suppose $Q, Q' \in \text{BASIC}^*$ and Q is a pseudo-subsequence of Q' . Then $\mathcal{L}^Q \leq^{\text{TxtEx}} \mathcal{L}^{Q'}$.

PROOF. Follows from definition of pseudo-subsequence, Proposition 4, and Proposition 5. ■

Proposition 9 Suppose $Q = (q_1, q_2, \dots, q_k)$ and $Q' = (q'_1, q'_2, \dots, q'_l)$, where each $q_i, q'_i \in \text{BASIC}$.

(a) If $q_1 = q'_1$, then (q_2, \dots, q_k) is a pseudo-subsequence of (q'_2, \dots, q'_l) , iff Q is a pseudo-subsequence of Q' .

(b) Suppose $q_1, q'_1 \in \{\text{INIT}, \text{COINIT}\}$, but $q_1 \neq q'_1$. Then, Q is a pseudo-subsequence of Q' iff Q is a pseudo-subsequence of $Q'' = (q'_2, q'_3, \dots, q'_l)$.

(c) Suppose $q_1 = \text{HALF}$. Then, Q is a pseudo-subsequence of Q' implies $(\text{INIT}, q_2, \dots, q_k)$ is a pseudo-subsequence of Q' .

(d) Suppose $q_1 = \text{HALF}$. Then, Q is a pseudo-subsequence of Q' implies $(\text{COINIT}, q_2, \dots, q_k)$ is a pseudo-subsequence of Q' .

(e) Suppose $q'_1 = \text{HALF}$. Then, Q is a pseudo-subsequence of Q' iff (q_2, q_3, \dots, q_k) is a pseudo-subsequence of $(q'_2, q'_3, \dots, q'_l)$.

PROOF. Follows from the definition of pseudo-subsequence. ■

Proposition 10 Suppose $Q = (q_1, q_2, \dots, q_k)$ and $Q' = (q'_1, q'_2, \dots, q'_l)$, where each $q_i, q'_i \in \text{BASIC}$. Suppose Q is not a pseudo-subsequence of Q' .

(a) If $q_1 = q'_1 \in \{\text{INIT}, \text{COINIT}\}$, $q_2 \in \{\text{INIT}, \text{COINIT}\}$, and $q_2 \neq q_1$, then $Q'' = (q_2, q_3, \dots, q_k)$ is not a pseudo-subsequence of Q' .

(b) If $q_1 = \text{HALF}$, and $q'_1 = \text{INIT}$, then $Q'' = (\text{COINIT}, q_2, \dots, q_k)$ is not a pseudo-subsequence of Q' .

(c) If $q_1 = \text{HALF}$, and $q'_1 = \text{COINIT}$, then $Q'' = (\text{INIT}, q_2, \dots, q_k)$ is not a pseudo-subsequence of Q' .

(d) If $q_1 = q'_1 \in \{\text{INIT}, \text{COINIT}\}$, $q_2 = \text{HALF}$, $l \geq 2$, and $q'_2 \in \{\text{INIT}, \text{HALF}\}$, then $Q'' = (q_1, \text{COINIT}, q_3, q_4, \dots, q_k)$ is not a pseudo-subsequence of Q' .

(e) If $q_1 = q'_1 \in \{\text{INIT}, \text{COINIT}\}$, $q_2 = \text{HALF}$, $l \geq 2$, and $q'_2 \in \{\text{COINIT}, \text{HALF}\}$, then $Q'' = (q_1, \text{INIT}, q_3, q_4, \dots, q_k)$ is not a pseudo-subsequence of Q' .

PROOF. We show part (a), (b) and (e). Proof of part (c) is similar to part (b) and proof of part (e) is similar to part (d).

(a) Let $Q''' = (q'_2, q'_3, \dots, q'_l)$. Now by Proposition 9(a) Q is a pseudo-subsequence of Q' iff Q'' is pseudo-subsequence of Q''' . By Proposition 9(b) Q'' is pseudo-subsequence of Q''' iff Q'' is pseudo-subsequence of Q' . Thus, since Q is not a pseudo-subsequence of Q' it follows that Q'' is not a pseudo-subsequence of Q' .

(b) Suppose by way of contradiction that $(\text{COINIT}, q_2, \dots, q_k)$ is a pseudo-subsequence of Q' . Then by applying Proposition 9(b), we have that $(\text{COINIT}, q_2, \dots, q_k)$ is a pseudo-subsequence of $(q'_2, q'_3, \dots, q'_l)$. Thus by Proposition 9(a) we have $(\text{INIT}, \text{COINIT}, q_2, \dots, q_k)$ is a pseudo-subsequence of $(q'_1, q'_2, \dots, q'_l)$. But since $(\text{HALF}, q_2, \dots, q_k)$ is a pseudo-subsequence of $(\text{INIT}, \text{COINIT}, q_2, \dots, q_k)$ we have, $Q = (\text{HALF}, q_2, \dots, q_k)$ is a pseudo-subsequence of $Q' = (q'_1, q'_2, q'_3, q'_4, \dots, q'_l)$. A contradiction to the hypothesis. Part (b) follows.

(d) Let $Q''' = (q'_2, q'_3, \dots, q'_l)$. We consider two cases.

Case 1: $q'_2 = \text{HALF}$.

By Proposition 9(a), Q is a pseudo-subsequence of Q' iff (q_2, \dots, q_k) is a pseudo-subsequence of Q''' . By Proposition 9(e), (q_2, q_3, \dots, q_k) is a pseudo-subsequence of Q''' , iff $(\text{COINIT}, q_3, \dots, q_k)$ is a pseudo-subsequence of Q''' . Thus, since Q is not a pseudo-subsequence of Q' it follows that $(\text{COINIT}, q_3, \dots, q_k)$ is not a pseudo-subsequence of Q''' . Thus, by Proposition 9(a), it follows that $(q_1, \text{COINIT}, q_3, \dots, q_k)$ is not a pseudo-subsequence of Q' .

Case 2: $q'_2 = \text{INIT}$.

By hypothesis and Proposition 9(a) we have that (q_2, q_3, \dots, q_k) is not a pseudo-subsequence of $(q'_2, q'_3, \dots, q'_l)$. Thus by part (b) $(COINIT, q_3, \dots, q_k)$ is not a pseudo-subsequence of $(q'_2, q'_3, \dots, q'_l)$. Thus by Proposition 9(a) $(q_1, COINIT, q_3, \dots, q_k)$ is not a pseudo-subsequence of Q' . \blacksquare

The following theorem shows that reducing any Q' -vector to its proper pseudo-subsequence Q results in the degree of learnability that is properly below the degree defined by the Q' -class.

Theorem 2 *Suppose $Q = (q_1, \dots, q_k) \in BASIC^k$ and $Q' = (q'_1, \dots, q'_l) \in BASIC^l$. Let $R = R_1 \times R_2 \times \dots \times R_k$, $R' = R'_1 \times R'_2 \times \dots \times R'_l$, where each R_i (R'_i) is an infinite subset of N , if $q_i \in \{INIT, COINIT\}$ ($q'_i \in \{INIT, COINIT\}$), and R_i (R'_i) is a subset of Z , with infinite intersection with both N and Z^- , if $q_i = HALF$ ($q'_i = HALF$).*

If Q is not a pseudo-subsequence of Q' then $\mathcal{L}^{Q,R} \not\leq^{TxE} \mathcal{L}^{Q',R'}$.

Proof of the above Theorem is given in the Appendix.

Corollary 1 *Suppose $Q \in BASIC^k$ and $Q' \in BASIC^l$. Then, $\mathcal{L}^Q \leq^{TxE} \mathcal{L}^{Q'}$ iff Q is a pseudo-subsequence of Q' .*

6 Definitions for Open Semi-Hull and some Propositions

Let \mathbf{rat} denote the set of non-negative rationals. $\mathbf{rat}^+ = \mathbf{rat} - \{0\}$, denotes the set of positive rationals.

Any language $SEMI_HULL^n$ defined below is a geometrical figure semi-hull, collection of points in the first quadrant of the plane bounded by the y -axis and a broken-line that consists of a straight fragment l_0 of the x -axis (starting from origin) followed by a straight fragment l_1 that makes an angle $\delta_1 < 90^\circ$ with the x -axis, followed by a fragment l_2 that makes an angle $\delta_2 > \delta_1$ with the x -axis, etc. (In above the angle is being measured anti-clockwise from the positive x -axis).

Definition 13 Suppose $a_1, \dots, a_n \in N$ and $b_1, \dots, b_n \in \mathbf{rat}^+$, where $0 < a_1 < a_2 < \dots < a_n$.

$$SEMI_HULL^n_{a_1, b_1, a_2, b_2, \dots, a_n, b_n} = \{(x, y) \in N^2 \mid y \geq \sum_{1 \leq i \leq n} b_i * (x \div a_i)\}.$$

Note that $SEMI_HULL^0 = N^2$. Also, note that though $SEMI_HULL^n$ above are subsets of N^2 , one can easily consider them as languages $\subseteq N$, by using pairing function. We assume such implicit coding whenever we are dealing with sets $\subseteq N^2$.

Parameters a_i in the above definition specify x -coordinates of break points of the border line, while the b_i specify the slopes that are being added to the slope of the border line after every break point.

To make our classes of languages learnable, we have to impose certain restrictions on the parameters a_i, b_i . First, we want both coordinates a and c of break points (a, c) to be integers. Secondly, for all languages in our classes, we fix a subset S from which slopes b_i may come from. (In the following definition S may be an arbitrary subset of \mathbf{rat}^+ ; however, later we will impose additional restrictions on S). The definition of *valid* sequences of parameters a_i, b_i accomplishes this goal.

Definition 14 Suppose $a_1, \dots, a_n \in N$ and $b_1, \dots, b_n \in \mathbf{rat}^+$, where $0 < a_1 < a_2 < \dots < a_n$. Suppose $S \subseteq \mathbf{rat}^+$.

We say that $(a_1, b_1, \dots, a_n, b_n)$ is *valid* iff for $1 \leq j \leq n$, $[\sum_{1 \leq i \leq n} b_i * (a_j \div a_i)] \in N$. Additionally, if each $b_i \in S$, then we say that $(a_1, b_1, \dots, a_n, b_n)$ is *S-valid*.

Let $VALID = \{(a_1, b_1, \dots, a_n, b_n) \mid (a_1, b_1, \dots, a_n, b_n) \text{ is valid}\}$.

Let $VALID_S = \{(a_1, b_1, \dots, a_n, b_n) \mid (a_1, b_1, \dots, a_n, b_n) \text{ is } S\text{-valid}\}$.

Note that the empty sequence $()$ is both valid and S -valid. Also we require $a_1 > 0$. This is for technical convenience, and crucial for some of our results.

Now we define the class of languages we are going to explore.

Definition 15 Suppose $S \subseteq \mathbf{rat}^+$. Then $SEMI_HULL^{n,S} = \{SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \mid (a_1, b_1, \dots, a_n, b_n) \in VALID_S\}$.

Now we formulate and prove a number of useful technical propositions (few of them immediately follow from the relevant definitions).

Proposition 11 (a) Suppose $(a_1, b_1, \dots, a_j, b_j, a_{j+1}, b_{j+1}) \in VALID$. Then $SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, a_{j+1}, b_{j+1}}^{j+1} \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$.
 (b) Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$ and $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in VALID$, and $a_j \leq a'_j$, $b_j \geq b'_j$. Then, $SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j \subseteq SEMI_HULL_{a_1, b_1, \dots, a'_j, b'_j}^j$.

PROOF. Follows from definitions. ■

Definition 16 Suppose $a_1, b_1, \dots, a_j, b_j$ are given such that $(a_1, b_1, \dots, a_j, b_j) \in VALID$. Then, let $INTER(a_1, b_1, \dots, a_j, b_j) = \bigcap \{SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n}^n \mid n \geq j \wedge (a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n) \in VALID\}$.

Intuitively,

$INTER(a_1, b_1, \dots, a_j, b_j)$, denotes the common portion of all $SEMI_HULL_{a'_1, b'_1, \dots, a'_j, b'_j, \dots}^n$, with $a_i = a'_i$ and $b_i = b'_i$ for $1 \leq i \leq j$.

Proposition 12 (a) Suppose $1 \leq j \leq n$, and $(a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n) \in VALID$. Then $INTER(a_1, b_1, \dots, a_j, b_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n}^n$.
 (b) Suppose $(a_1, b_1, \dots, a_j, b_j, a_{j+1}, b_{j+1})$ is valid. Then, $INTER(a_1, b_1, \dots, a_j, b_j) \subseteq INTER(a_1, b_1, \dots, a_j, b_j, a_{j+1}, b_{j+1})$.

PROOF. Follows from definitions. ■

Definition 17 Suppose $(a_1, b_1, \dots, a_j, b_j) \in VALID$. Then let $maxinter(a_1, b_1, \dots, a_j, b_j)$ denote the least natural number $x > a_j$ such that $\sum_{1 \leq i \leq j} b_i * (x \dot{-} a_i) \in N$.

Proposition 13 Suppose $a_1, b_1, \dots, a_j, b_j$ are given such that $(a_1, b_1, \dots, a_j, b_j) \in VALID$. Then, $INTER(a_1, b_1, \dots, a_j, b_j) = SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^n \cap \{(x, y) \in N^2 \mid x \leq maxinter(a_1, b_1, \dots, a_j, b_j)\}$.

PROOF. Follows from definitions. ■

Proposition 14 Suppose $a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, a'_j, b'_j$ are given, where $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$, and $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ are valid. If $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^n$
 Then,
 (i) $a'_j \leq a_j$ and
 (ii) $b'_j * (maxinter(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \dot{-} a'_j) \geq b_j * (maxinter(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \dot{-} a_j)$.

PROOF. Suppose that $a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, a_j, b_j$ are as given in the hypothesis.

If $a_j < a'_j$, then $(a'_j, \sum_{1 \leq i < j} b_i * (a'_j \div a_i))$ belongs to $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ but does not belong to $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^n$ (since $(a'_j, \sum_{1 \leq i < j} b_i * (a'_j \div a_i))$ is below the last fragment of the border line for $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^n$).

Thus, we must have $a'_j \leq a_j$.

Let $x = \maxinter(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$. Let $y = b'_j * (x \div a'_j) + \sum_{1 \leq i < j} b_i * (x \div a_i)$. Now, (x, y) belongs to $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$. If (x, y) belongs to $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$ then by definition of $SEMI_HULL^j$, we must have $y = b'_j * (x \div a'_j) + \sum_{1 \leq i < j} b_i * (x \div a_i) \geq b_j * (x \div a_j) + \sum_{1 \leq i < j} b_i * (x \div a_i)$. Thus, $b'_j(x \div a'_j) \geq b_j(x \div a_j)$. The Proposition follows. \blacksquare

The following two corollaries have obvious geometrical interpretation.

Corollary 2 Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n)$ and $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j, \dots, a''_n, b''_n)$ are valid.

(a) If $a'_j < a''_j$, then $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n}^n \not\subseteq SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j, \dots, a''_n, b''_n}^n$.

(b) If $a'_j = a''_j$ and $b'_j > b''_j$ then $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n}^n \not\subseteq SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n}^n$.

PROOF. If $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n}^n \subseteq SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j, \dots, a''_n, b''_n}^n$, then it follows that $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j}^n$. Thus it follows from Proposition 14 that $a''_j \leq a'_j$ and if $a''_j = a'_j$, then $b''_j \geq b'_j$. Corollary follows. \blacksquare

Corollary 3 Suppose $(a_1, b_1, \dots, a_n, b_n)$ and $(a'_1, b'_1, \dots, a'_n, b'_n)$ are valid. Suppose $SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \subset SEMI_HULL_{a'_1, b'_1, \dots, a'_n, b'_n}^n$. Let i be the minimum value such that $a_i \neq a'_i$ or $b_i \neq b'_i$. Then, $a_i < a'_i$ or $[a_i = a'_i \text{ and } b_i > b'_i]$.

PROOF. Note that for the least j such that $(a'_j, b'_j) \neq (a_j, b_j)$, we must have $INTER_{a_1, b_1, \dots, a_j, b_j}^n \subseteq SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^n$. Corollary now follows from Proposition 14. \blacksquare

Proposition 15 Suppose $1 \leq j \leq n$. Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, \dots, a_n, b_n)$ and $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ are valid.

(a) If $a'_j \leq a_j$ and $b'_j \geq \sum_{j \leq i \leq n} b_i$, Then, $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n}^n$.

(b) If $a'_j \leq a_j$ and $b'_j > \sum_{j \leq i \leq n} b_i$, Then, $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j \subset SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n}^n$.

PROOF. (a) $SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j = \{(x, y) \in N^2 \mid y \geq b'_j(x \div a'_j) + \sum_{1 \leq i < j} b_i * (x \div a_i)\}$.

$SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n = \{(x, y) \in N^2 \mid y \geq \sum_{j \leq i \leq n} b_i * (x \div a_i) + \sum_{1 \leq i < j} b_i * (x \div a_i)\}$.

Part (a) follows since $b'_j(x \div a'_j) \geq b'_j(x \div a_j) \geq \sum_{j \leq i \leq n} b_i * (x \div a_j) \geq \sum_{j \leq i \leq n} b_i * (x \div a_i)$.

(b) Follows from part (a) and Corollary 2. \blacksquare

Now we are at the point when our results require additional constraint on the set S (of slopes). Intuitively, the set S satisfying the constraints cover the positive rational numbers,

and can be algorithmically listed in a monotonic order on a two-sided-infinite tape. A natural example of set S satisfying the constraint below is the set $N \cup \{1/x \mid x \in N\}$. Although our results below hold for any fixed set S of rationals satisfying the constraint in question, we suggest the reader to keep in mind the above set when reading the proofs.

Definition 18 A set $S \subseteq \mathbf{rat}^+$ is said to be \mathbf{rat}^+ -covering iff there exists a recursive bijection f from Z to S such that,

- (i) for $i, j \in Z$, $i < j$ iff $f(i) < f(j)$.
- (ii) for every $x \in \mathbf{rat}^+$, there exist $y, y' \in S$ such that $y < x < y'$.

A natural choice for a set S (which doesn't satisfy the above constraint) seems to be the set \mathbf{rat}^+ . However, in this case, a complete class of languages $\{L_y = \{x \mid x \in \mathbf{rat}^+, x \geq y\} \mid y \in \mathbf{rat}^+\}$ (see [JKW00]) would be trivially reducible to any class of languages-figures considered in our paper, thus making all of them of the same complexity. The use of \mathbf{rat}^+ -covering sets S gives us opportunity to capture differences in learnability of different geometrical concepts observed in our paper.

Our results below hold for any \mathbf{rat}^+ -covering set S . However it is open at present whether $\mathbf{SEMI_HULL}^{n,S} \equiv_{\mathbf{TextEx}} \mathbf{SEMI_HULL}^{n,S'}$, for arbitrary \mathbf{rat}^+ -covering sets S and S' .

Note that, for any given $n \in N$, and \mathbf{rat}^+ -covering set S , there exists a limiting partial recursive function F such that, if i is a grammar for some $\mathbf{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^{n,S}$, where $(a_1, b_1, \dots, a_n, b_n)$ being S -valid, then $F(i)$ converges to $(a_1, b_1, \dots, a_n, b_n)$. That is one can determine the parameters limit-effectively from any grammar for $\mathbf{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^{n,S}$. This fact will be implicitly used in construction of Ψ in some of our proofs.

Intuitively, the following technical proposition claims that, given a point (x_0, y_0) in a language $\mathbf{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}}^{j-1}$, one can effectively find a lower bound on the boundary slopes b_j of all languages $\mathbf{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$ which do not contain (x_0, y_0) .

Proposition 16 *Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1})$ is valid. Let $(x_0, y_0) \in N^2$ be such that $y_0 > \sum_{1 \leq i < j} b_i(x_0 \div a_i)$. Then, there exists a $B' \in \mathbf{rat}^+$ obtainable effectively from $a_1, b_1, \dots, a_{j-1}, b_{j-1}, x_0$ and y_0 such that for any (a'_j, b'_j) , if $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is valid and $x_0 \leq a'_j$ or $b'_j < B'$, then, $(x_0, y_0) \in \mathbf{INTER}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$.*

PROOF. If $a'_j \geq x_0$, then we would have $y_0 \geq \sum_{1 \leq i < j} b_i(x_0 \div a_i) = b'_j(x_0 \div a'_j) + \sum_{1 \leq i < j} b_i(x_0 \div a_i)$. Thus, $(x_0, y_0) \in \mathbf{INTER}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$.

Now fix any $a'_j < x_0$. If $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is valid, but $(x_0, y_0) \notin \mathbf{INTER}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$, then either (a) $\maxinter(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) < x_0$, or (b) $y_0 < b'_j(x_0 \div a'_j) + \sum_{1 \leq i < j} b_i(x_0 \div a_i)$.

Let $B(a'_j) = (y_0 - \sum_{1 \leq i < j} b_i(x_0 \div a_i)) / (x_0 \div a'_j)$. Now for (b) to be true b'_j must be greater than $B(a'_j)$. We now consider which $b'_j \leq B(a'_j)$ can satisfy (a). Let $x_1 = \maxinter(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, and $y_1 = b'_j * (x_1 \div a'_j) + \sum_{1 \leq i < j} b_i * (x_1 \div a_i)$. For $b'_j \leq B(a'_j)$ to satisfy (a), (x_1, y_1) must lie in the intersection of the following three regions:

- (A) $y > \sum_{1 \leq i < j} b_i(x \div a_i)$,
- (B) $x \leq x_0$, and
- (C) $y \leq B(a'_j) * (x \div a'_j) + \sum_{1 \leq i < j} b_i(x \div a_i)$.

Since the above intersection is finite, there are only finitely many possibilities for (x_1, y_1) , and thus for b'_j . Thus, let $B_1(a'_j) \leq B(a'_j)$, be a positive rational number, such that $\maxinter(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) < x_0$ implies $b'_j > B_1(a'_j)$. Note that such a $B_1(a'_j)$ exists and can be obtained effectively from $a_1, b_1, \dots, a_{j-1}, b_{j-1}, x_0, y_0$.

Now taking $B' = \min(\{B_1(a'_j) \mid a_{j-1} < a'_j < x_0\})$, witnesses the Proposition. \blacksquare

Corollary 4 *Suppose $(a_1, b_1, \dots, a_j, b_j)$ is valid. Then, one can effectively (in $a_1, b_1, \dots, a_j, b_j$) obtain a $B' \in \mathbf{rat}^+$ such that following property is satisfied.*

For any $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ which is valid, if $\text{INTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^n$, then $b'_j \geq B'$ and $a'_j \leq a_j$.

PROOF. Pick $(x_0, y_0) \in N^2$ such that $x_0 > a_j$, and $\sum_{1 \leq i < j} b_i(x_0 \dot{-} a_i) < y_0 < \sum_{1 \leq i \leq j} b_i(x_0 \dot{-} a_i)$. Corollary, now follows from Proposition 14 and 16. \blacksquare

Proposition 18 below will play an important role in some of our proofs. Imagine that, in the process of learning, $\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, \dots, a_n, b_n}^n$, the learner has already learned the parameters $a_1, b_1, \dots, a_{j-1}, b_{j-1}$, and is now trying to learn the parameters a_j, b_j . Our aim is to use a $(\text{INIT}, \text{COINIT})$ type strategy to learn (a_j, b_j) (instead of the trivial $(\text{INIT}, \text{HALF})$ type strategy mentioned in the introduction).

Since we intend to use $(\text{INIT}, \text{COINIT})$ type strategy, we need to be safe in choosing the parameters. That is, choosing (a_j, b_j) must imply that any (a'_j, b'_j) smaller than (a_j, b_j) must be inconsistent with the input, (smaller in the sense of some $(\text{INIT}, \text{COINIT})$ -type ordering of all pairs (a''_j, b''_j)).

This is achieved in two steps. First we order (a''_j, b''_j) , with $b''_j \leq B$, for some fixed positive rational number B in a INIT like fashion. Then, we order $b''_j > B$, in a COINIT like fashion. This latter ordering is easily achievable (as will be seen in the proof of Theorem 3 below). For the former ordering, if $b'_j, b_j \leq B$, then we make sure that if $\text{INTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is a subset of $\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^n$, then ordering places (a'_j, b'_j) below (a_j, b_j) . Proposition 18 (which is based on Corollary 5 and Proposition 17 below) shows that this is achievable. Note that this ordering depends on $(a_1, b_1, \dots, a_{j-1}, b_{j-1})$.

Corollary 5 *Suppose $1 \leq j \leq n$. Let S be any \mathbf{rat}^+ -covering set, and $B \in S$.*

Suppose $a_1, b_1, \dots, a_{j-1}, b_{j-1}$ are given, where $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$ is S -valid.

Then, there exist only finitely many (a'_j, b'_j) , such that

(i) $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is S -valid,

(ii) $b'_j \leq B$, and

(iii) $\text{INTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$.

Moreover, canonical index for the finite set of a'_j, b'_j satisfying above three conditions can be obtained effectively from $B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j$.

Furthermore, for any a'_j, b'_j satisfying the above three conditions, $a'_j \leq a_j$, and if $a'_j = a_j$, then $b'_j \geq b_j$.

PROOF. Furthermore clause follows from Proposition 14.

By Corollary 4, one can effectively find a $B' \in \mathbf{rat}^+$ such that any (a'_j, b'_j) satisfying (i) and (iii) must satisfy $b'_j \geq B'$ and $a'_j \leq a_j$. Corollary now follows since there are only finitely many (a'_j, b'_j) such that $a'_j \leq a_j$, $B' \leq b'_j \leq B$, and $b'_j \in S$ and for any (a'_j, b'_j) , one can effectively test whether (a'_j, b'_j) satisfies clauses (i) to (iii) in the corollary. \blacksquare

The above corollary together with the following technical proposition will enable us to impose a suitable ordering on $(a_1, b_1, \dots, a_j, b_j)$ with an upper bound B on slopes b_j . Think of $(a_1, b_1, \dots, a_{j-1}, b_{j-1})$ as already fixed.

Proposition 17 *Suppose R is a partial order over an r.e. set A , and F is a partial recursive function with domain A such that, for any $x \in A$,*

(i) $\{x' \mid x'Rx\}$ is finite, and

(ii) $F(x)$ is the canonical index for $\{x' \in A \mid x'Rx\}$.

Then, effectively from a program for F , one can obtain a program for a 1-1, partial recursive function h with domain A and range $\subseteq N$, such that $x'Rx$ implies $h(x') \leq h(x)$. Moreover, if A is infinite, then range of h is N .

PROOF. Let f be a recursive function such that $\text{range}(f) = A$. Define h as follows. Let $cur = 0$. Go to stage 0.

Stage s

(* In this stage we make sure that $h(f(s))$, along with $h(x)$, for all x such that $xRf(s)$, are defined appropriately. *)

If $h(f(s))$ has already been defined, then go to stage $s + 1$.

Otherwise let $X = \{x \mid xRf(s)\}$ (note that one can effectively obtain X from $f(s)$ by assumption (ii) of the hypothesis; also note that $f(s) \in X$, since $f(s)Rf(s)$).

Let x_0, x_1, \dots, x_m be listing of members of X such that $x_iRf(s)$ implies $i \leq j$ (note that one can easily get such an ordering since X is finite).

For $j = 0$ to m do

If $h(x_j)$ has not been defined then
 let $h(x_j) = cur$; $cur = cur + 1$.

EndFor

(* Note that $h(f(s))$ along with $h(x)$, for all x such that $xRf(s)$, have been defined. *)

End Stage s

It is easy to verify that h satisfies the properties required in the proposition. ■

Proposition 18 *Let S be any \mathbf{rat}^+ -covering set. Then, there exists a recursive function code with domain $\mathbf{rat} \times \text{VALID}_S$, and range $\subseteq N$ such that following is satisfied.*

Suppose $1 \leq j \leq n$, $B \in S$, and $(a_1, b_1, \dots, a_{j-1}, b_{j-1})$ is S -valid.

(A) *Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$ and $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ are S -valid.*

(A.1) *If $b_j, b'_j \leq B$ and $\text{INTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$, then $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \leq \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$.*

(A.2) *If $b_j, b'_j \leq B$ and $(a_j, b_j) \neq (a'_j, b'_j)$, then $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \neq \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$.*

(A.3) *$\{\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j) \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j) \in \text{VALID}_S \text{ and } b''_j \leq B\} = N$.*

(B) *Suppose $(a_1, b_1, \dots, a_j, b_j)$ is S -valid. If $b_j > B$, then $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B)$.*

PROOF. Fix $a_1, b_1, \dots, a_{j-1}, b_{j-1}$ and B as in the hypothesis. Let us first define a relation Rel on $N \times S \cap \{r \in \mathbf{rat} \mid r \leq B\}$ as follows:

$(a', b')\text{Rel}(a, b)$ iff

$\text{INTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a', b') \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a, b}^j$

Note that (by Corollary 5) for each (a, b) , there are only finitely many (a', b') such that $(a', b')\text{Rel}(a, b)$. Also, for each of these (a', b') , $a' \leq a$ and either $a' < a$ or $b' \geq b$. Furthermore, canonical index for $\{(a', b') \mid (a', b')\text{Rel}(a, b)\}$, can be effectively obtained from (a, b) .

Let Rel^* be transitive closure of Rel . It is easy to verify that Rel^* is a partial order, where for each (a, b) , there exists at most finitely many (a', b') such that $(a', b')\text{Rel}^*(a, b)$. Moreover, canonical index for $\{(a', b') \mid (a', b')\text{Rel}^*(a, b)\}$, can be effectively obtained from (a, b) .

Existence of code as required now follows from Proposition 17. ■

7 Q -classes to Which $\text{SEMI_HULL}^{n,S}$ is Reducible

Our goal in this section is to establish an upper bound on the $\text{SEMI_HULL}^{n,S}$ degrees in terms of the Q -hierarchy. To find such a bound, we actually have to design a learning strategy for languages in $\text{SEMI_HULL}^{n,S}$ that consists of q_i -type strategies for some $Q = (q_1, q_1, \dots, q_k)$, and a grammar learned by every q_i is used by q_{i+1} . A natural strategy of this type would be the following $(q_1, q_2, \dots, q_{2n-1}, q_{2n})$ -strategy, where $q_{2i+1} = \text{INIT}$ and $q_{2i+2} = \text{HALF}$ for $i < n$: learn the first break point a_1 using an INIT -type strategy; once a_1 has been learned, learn the first slope b_1 at the point $(a_1, 0)$ using a HALF -type strategy; then learn the second break point $(a_2, b_1 * (a_2 - a_1))$ using an INIT -type strategy, etc. However, a much more efficient learning strategy is suggested by the following. Informally one can visualize this strategy as follows. Assume that slope values come from the set $N \cup \{1/n \mid n \in N\}$. It may happen that in the beginning the learner receives points (x, y) indicating that the slope to be learned is greater or equal 1. Then the learner uses an INIT -like strategy to learn a break and a COINIT -like strategy (not a HALF -strategy as above!) to learn the slope: the slopes tend to 1 from above, and the learner uses this assumption. If the slope gets smaller than 1, the learner then uses a combined INIT -like strategy to learn the break point and the slope together: both of them change now in INIT -fashion.

There is a slight problem though in the above strategy. It may be possible that slope at some point seems less than 1, but later on when lots of new points $(i, 0)$ come from the input, slope again seems larger than 1. To prevent this from harming the learning process, the learner uses the combined INIT -strategy in a *safe* fashion: Informally, suppose one has learned the parameters, $a_1, b_1, \dots, a_{j-1}, b_{j-1}$, and is now trying to determine a_j, b_j . Now we need to make sure that the combined INIT -strategy does not commit to (a_j, b_j) being (a'_j, b'_j) before it has been able to determine that input data cannot be from $\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots}^n$, for any other a'_j, b'_j, \dots , which satisfies $\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots}^n \subset \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, a'_{j+1}, b'_{j+1}, \dots}^n$, for some value of the parameters $a'_{j+1}, b'_{j+1}, \dots$. It is possible to achieve this by using Proposition 18.

The actual proof of the theorem technically looks somewhat different, and the above method is a bit hidden.

Theorem 3 *Suppose S is rat^+ -covering. Suppose $Q = (q_1, q_2, \dots, q_{2n-1}, q_{2n})$, where $q_{2i+1} = \text{INIT}$ and $q_{2i+2} = \text{COINIT}$, for $i < n$. Then $\text{SEMI_HULL}^{n,S} \leq^{\text{TxEEx}} \mathcal{L}^Q$.*

PROOF. Fix S which is rat^+ -covering. Let h be a recursive bijection from Z to S such that $h(i) < h(i+1)$. Let $B = h(0)$.

The intuitive idea of the learning strategy is as follows. Suppose we have already learned $(a_1, b_1, \dots, a_{j-1}, b_{j-1})$. Then, we use INIT like strategy to learn any pair (a_j, b_j) , if $b_j \leq B = h(0)$, and use COINIT type strategy to learn b_j if $b_j > B$. The former is done using code as in Proposition 18. The latter can be done easily by using h to form a COINIT like strategy.

Going through the proof below (and the proofs for other upper bounds in this paper) one must be aware of the fact that, though intuitively we do use *INIT/COINIT* (or whatever - in other proofs) strategy for learning each successive parameter, in the actual proof we do it by choosing an appropriate minimal consistent *SEMI_HULL* on every step and *INIT/COINIT* strategy is in some sense hidden.

Now we proceed with the formal construction.

Fix code as in Proposition 18. Let g be a function from S to N such that $g(b) = 0$, if $b \leq B$, and $g(b) = h^{-1}(b)$, otherwise. (Note that g is monotonically non-decreasing).

The following claim utilizes the construction of proposition 18 to show that the values ($\text{code}(B, a_1, b_1, \dots, a_j, b_j)$) can be used as conjectures by *INIT*-type substrategies, and the values $g(b_j)$ can be used as conjectures by *COINIT*-type substrategies.

Claim 1 Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j), (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in \text{VALID}_S$, where $(a_j, b_j) \neq (a'_j, b'_j)$. Suppose $\text{INTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j) \subset \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^n$.

Then,

- (i) $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j) < \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, or
- (ii) $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, and $g(b_j) > g(b'_j)$.

PROOF. By Proposition 14 we have that $a_j \leq a'_j$.

We now consider following cases:

Case 1: $b_j, b'_j \leq B$.

Since, $\text{INTER}(a_1, b_1, \dots, a_j, b_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$, we have $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j) < \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18; for getting $<$ instead of \leq use the fact that $(a_j, b_j) \neq (a'_j, b'_j)$).

Case 2: $b_j \leq B < b'_j$.

In this case $a_j < a'_j$ by Proposition 14.

Also, $\text{INTER}(a_1, b_1, \dots, a_j, b_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, B}^j$.

Thus, $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j) < \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, B) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18; for getting $<$ instead of \leq use the fact that $(a_j, b_j) \neq (a'_j, B)$).

Case 3: $b'_j \leq B < b_j$.

In this case, $\text{INTER}(a_1, b_1, \dots, a_j, B) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B}^j \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$. Thus, $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B) \leq \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18). Also $g(b'_j) = 0 < g(b_j)$.

Case 4: $B < b_j \leq b'_j$.

In this case $a_j < a'_j$ by Proposition 14. Thus, $\text{INTER}(a_1, b_1, \dots, a_j, B) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B}^j \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, B}^j$. Thus, $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B) < \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, B) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18; for getting $<$ instead of \leq use the fact that $(a_j, B) \neq (a'_j, B)$).

Case 5: $B < b'_j \leq b_j$.

If $a_j < a'_j$, then $\text{INTER}(a_1, b_1, \dots, a_j, B) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B}^j \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, B}^j$.

Thus, $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B) < \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, B) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18; for getting $<$ instead of \leq use the fact that $(a_j, B) \neq (a'_j, B)$).

If $a_j = a'_j$, then $b'_j < b_j$ (by Proposition 14). Thus, $g(b_j) > g(b'_j)$. Also, $\text{INTER}(a_1, b_1, \dots, a_j, B) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, B}^j$. Thus, $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, B) \leq \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, B) = \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18). \square

Let map be a mapping from VALID to N^* such that

$$\begin{aligned} \text{map}(a_1, b_1, \dots, a_n, b_n) &= \\ (\text{code}(B, a_1, b_1), g(b_1), \text{code}(B, a_1, b_1, a_2, b_2), g(b_2), \dots, \text{code}(B, a_1, b_1, \dots, a_n, b_n), g(b_n)). \end{aligned}$$

Now we suggest the reader to recall the properties of the ordering $<_Q$. For example, note that if $Q = (\text{INIT}, \text{COINIT})$ then $(i, j) <_Q (i', j')$ would mean that $i < i'$ or $(i = i'$ and $j > j')$.

Claim 2 Suppose $(a_1, b_1, \dots, a_n, b_n)$ and $(a'_1, b'_1, \dots, a'_n, b'_n)$ are S -valid.

(A) If $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$, then, for the least j such that $(a_j, b_j) \neq (a'_j, b'_j)$, $\text{INTER}(a_1, b_1, \dots, a_j, b_j) \not\subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$.

(B) If $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$, Then, $\text{INTER}(a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n) \not\subseteq \text{SEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$.

(C) If $\text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n \subset \text{SEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$, then $\text{map}(a_1, b_1, \dots, a_n, b_n) <_Q \text{map}(a'_1, b'_1, \dots, a'_n, b'_n)$.

PROOF. (A) Let j be least number such that $(a_j, b_j) \neq (a'_j, b'_j)$. Note that, for $i < j$, we must have $\text{code}(B, a_1, b_1, \dots, a_i, b_i) = \text{code}(B, a'_1, b'_1, \dots, a'_i, b'_i)$, and $g(b_i) = g(b'_i)$. If $\text{INTER}(a_1, b_1, \dots, a_j, b_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$, then, by Claim 1 we would have $\text{code}(B, a_1, b_1, \dots, a_j, b_j) < \text{code}(B, a'_1, b'_1, \dots, a'_j, b'_j)$ or $\text{code}(B, a_1, b_1, \dots, a_j, b_j) = \text{code}(B, a'_1, b'_1, \dots, a'_j, b'_j)$ and $g(b_j) > g(b'_j)$. Thus, $\text{map}(a_1, b_1, \dots, a_n, b_n) <_Q \text{map}(a'_1, b'_1, \dots, a'_n, b'_n)$, a contradiction to the hypothesis.

(B) Let j be the least number such that $(a_j, b_j) \neq (a'_j, b'_j)$. Now (B) follows using part (A) and the fact that $\text{INTER}(a_1, b_1, \dots, a_j, b_j) \subseteq \text{INTER}(a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n)$ and $\text{SEMI_HULL}_{a'_1, b'_1, \dots, a'_j, b'_j, \dots, a'_n, b'_n}^n \subseteq \text{SEMI_HULL}_{a'_1, b'_1, \dots, a'_j, b'_j}^j$.

(C) Follows from part (B) and $\text{INTER}(a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$. \square

We now continue with the proof of the theorem. The aim is to construct Θ which maps $\text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$ to $L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$ to a grammar sequence converging to a grammar for $\text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$ would be trivial. We thus just define Θ .

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $\text{Prop}(X, a_1, b_1, \dots, a_n, b_n)$ be true iff following two properties are satisfied.

(A) $(a_1, b_1, \dots, a_n, b_n) \in \text{VALID}_S$.

(B) For all $(a'_1, b'_1, \dots, a'_n, b'_n) \in \text{VALID}_S$ such that $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$, $X \not\subseteq \text{SEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$.

Note that condition (B) above is equivalent to

(B') For all j , $1 \leq j \leq n$, (B'.1) and (B'.2) are satisfied.

(B'.1) For all $a'_j \in N, b'_j \in S, b'_j \leq B$,

if $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) < \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$, then

$$[X \not\subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j].$$

(B'.2) If $B \leq b_j$, then for all $b'_j \in S$ such that $b_j < b'_j$, $X \not\subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j}^j$

Note that B'.2 above is equivalent to

(B''.2) If $B \leq b_j$, then for least $b'_j \in S$ such that $b_j < b'_j$, $X \not\subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j}^j$.

Note that whether $X, a_1, b_1, \dots, a_n, b_n$, satisfy (A) and (B'.1) and (B''.2) for all j , $1 \leq j \leq n$, is effectively testable.

Thus, for any finite set $X \subseteq N^2$, let

$$\Theta(X) = \bigcup \{L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q \mid \text{Prop}(X, a_1, b_1, \dots, a_n, b_n)\}.$$

For infinite X' , $\Theta(X') = \bigcup_{X \subseteq X', \text{card}(X) < \infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$,

$\Theta(X) \subseteq L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$ (due to clause (B) in definition of *Prop* above, and the fact that for any valid I and I' , $\text{map}(I) <_Q \text{map}(I')$, implies $L_{\text{map}(I)}^Q \subseteq L_{\text{map}(I')}^Q$), and

(2) for any finite set $X \subseteq N^2$ such that $\{(x, y) \in N^2 \mid x \leq \text{maxinter}(a_1, b_1, \dots, a_n, b_n) \text{ and } y = \min(\{y' \mid (x, y') \in \text{INTER}(a_1, b_1, \dots, a_n, b_n)\})\} \subseteq X \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$,

$$\Theta(X) \supseteq L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q.$$

(By Claim 2(B), and definition of *Prop* and Θ).

Thus, we have that $\Theta(\text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n) = L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$. ■

Now we will show that the above theorem is in some sense optimal. That is, for $Q = (q_1, q_2, \dots, q_{2n-1}, q_{2n})$, where $q_{2i+1} = \text{INIT}$, and $q_{2i+2} = \text{COINIT}$, for $i < n$, and any $Q' \in \text{BASIC}^*$, if $\mathcal{L}^Q \not\leq^{\text{TxE}} \mathcal{L}^{Q'}$, then $\text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^{n, S} \not\leq^{\text{TxE}} \mathcal{L}^{Q'}$. Thus, Q in above theorem cannot be improved if we use components only from *BASIC* (whether we can improve it by using some other basic components is open).

Theorem 4 Suppose $n \in N^+$, $1 \leq j \leq n$. Suppose S is any rat^+ -covering set.

(1) Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1})$ is S -valid.

Let $Q = (q_1, q_2, \dots, q_{2(n-j+1)})$, where $q_{2i+1} = \text{INIT}$ and $q_{2i+2} = \text{COINIT}$ (for $i \leq n - j$).

Suppose $Q' \in \text{BASIC}^*$ is such that $\mathcal{L}^Q \not\leq \mathcal{L}^{Q'}$.

Then $\{\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n) \text{ is } S\text{-valid}\} \not\leq^{\text{TxE}} \mathcal{L}^{Q'}$.

(2) Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j)$ is such that there exists a b_j , such that $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$ is S -valid.

Let $Q = (q_1, \dots, q_{2(n-j+1)})$, where $q_{2i+1} = \text{COINIT}$, for $i \leq n - j$, and $q_{2i+2} = \text{INIT}$, for $i < n - j$. Suppose $Q' \in \text{BASIC}^*$ is such that $\mathcal{L}^Q \not\leq \mathcal{L}^{Q'}$.

Then $\{\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n) \text{ is } S\text{-valid}\} \not\leq^{\text{TxE}} \mathcal{L}^{Q'}$.

PROOF. The intuitive idea of the proof is to use a 's to diagonalize against sequences of *COINIT* and b 's to diagonalize against sequences of *INIT*.

For a fixed n , we prove the above theorem by reverse induction on j (from $j = n$ to $j = 1$). For each such j , we first show (2) and then (1).

Base Case: $j = n$, for (2):

In this case $Q = (\text{COINIT})$.

Now $COINIT \leq^{\mathbf{TxtEx}} \{SEMI_HULL_{a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n, b'_n}^n \mid (a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n, b'_n) \text{ is } S\text{-valid}\}$ as witnessed by Θ and Ψ defined as follows.

Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

For any set $Y \subseteq N$, let $\Theta(Y) = \bigcup_{i \in Y} SEMI_HULL_{a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n, h(i)}^n$.

It is easy to verify that $\Theta(L_i^Q) = SEMI_HULL_{a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n, h(i)}^n$. Let Ψ be defined as follows. If a sequence α of grammars converges to a grammar for $SEMI_HULL_{a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n, h(i)}^n$, then $\Psi(\alpha)$ converges to a grammar for L_i^Q .

It is easy to verify that Θ and Ψ witness that $COINIT \leq^{\mathbf{TxtEx}} \{SEMI_HULL_{a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n, b'_n}^n \mid (a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n, b'_n) \text{ is } S\text{-valid}\}$.

Thus, since $COINIT \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$, (2) is proven for $j = n$.

Induction Case (for (1)):

Suppose for $j > j'$, we have shown both (1) and (2), and for $j = j'$, we have shown (2). Then we show (1) for $j = j'$.

Note that $Q = (q_1, q_2, \dots, q_{2(n-j+1)})$, where q_{2i+1} is *INIT* and q_{2i+2} is *COINIT*, for $i \leq n-j$. Without loss of generality assume that Q' contains at least one *INIT* or *HALF* (otherwise just add *INIT* at the end of Q'). Suppose $Q' = (q'_1, \dots, q'_{k-1}, q'_k, q'_{k+1}, \dots, q'_l)$, where q'_1, \dots, q'_{k-1} are *COINIT*, q'_k may be either *INIT* or *HALF*, and for $k+1 \leq i \leq l$, $q'_i \in \{\text{INIT}, \text{COINIT}, \text{HALF}\}$. Without loss of generality assume that q'_k is *INIT* (since otherwise we could just replace $q'_k = \text{HALF}$ with $(\text{COINIT}, \text{INIT})$ and consider $Q' = (q'_1, \dots, q'_{k-1}, \text{COINIT}, \text{INIT}, q'_{k+1}, \dots, q'_l)$).

Now, by hypothesis we know that $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$. Thus, by Corollary 1, Q is not a pseudo-subsequence of Q' . Now consider Q'' obtained from Q by dropping $q_1 = \text{INIT}$, Q''' obtained from Q' by dropping $q'_1, q'_2, \dots, q'_{k-1}$. Now, Q is not a pseudo-subsequence of Q''' by repeated use of Proposition 9(b), and thus Q'' is not a pseudo-subsequence of Q''' by Proposition 10(a). Thus, by Corollary 1, $\mathcal{L}^{Q''} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$.

Now, suppose by way of contradiction that Θ along with Ψ witnesses the reduction

$$\{SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n) \text{ is } S\text{-valid}\} \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$$

Recall that $SEMI_HULL^0$ is N^2 . Let $v_1 = \min(\{u_1 \mid (\exists u_2, \dots, u_l)[\langle u_1, u_2, \dots, u_l \rangle \in \Theta(SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}}^{j-1})]\})$

For $1 < i < k$, let $v_i = \min(\{u_i \mid (\exists u_{i+1}, \dots, u_l)[\langle v_1, \dots, v_{i-1}, u_i, \dots, u_l \rangle \in \Theta(SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}}^{j-1})]\})$.

Let σ be such that $\text{content}(\sigma) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}}^{j-1}$, and there exist u_k, \dots, u_l , such that $\langle v_1, v_2, \dots, v_{k-1}, u_k, \dots, u_l \rangle \in \Theta(\sigma)$.

Let $a_j > \max(\{x \in N \mid (\exists y \in N)[\langle x, y \rangle \in \text{content}(\sigma)]\})$ be such that, for some b_j , $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$ is *S*-valid. Note that, for all $b''_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n$, such that $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b''_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n)$ is *S*-valid, $\text{content}(\sigma) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b''_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n}$. Thus, Θ (along with Ψ) essentially witnesses that

$$\{SEMI_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b''_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n}^n \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b''_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n) \text{ is } S\text{-valid},\} \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$$

(since parameters for the first $k-1$ *COINIT*s in Q' are fixed to be v_1, \dots, v_{k-1}). This is not possible by induction hypothesis (since $\mathcal{L}^{Q''} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q'''}$).

Induction Case (for (2)): Suppose for $j > j'$, we have shown both (1) and (2). Then we show (2) for $j = j'$.

In this case $Q = (q_1, q_2, \dots, q_{2(n-j+1)})$, where q_{2i+1} is *COINIT*, for $i \leq n-j$, and q_{2i+2} is *INIT*, for $i < n-j$. Without loss of generality assume that Q' contains at least one *COINIT* or *HALF* (otherwise just add *COINIT* to the end of Q'). Suppose Q' is of form

$(q'_1, \dots, q'_{k-1}, q'_k, \dots, q'_l)$, where q'_1, \dots, q'_{k-1} are *INIT*, q'_k may be either *COINIT* or *HALF*, and for $k+1 \leq i \leq l$, $q'_i \in \{\text{INIT}, \text{COINIT}, \text{HALF}\}$. Without loss of generality assume that q'_k is *COINIT* (since otherwise we could just replace $q'_k = \text{HALF}$ with $(\text{INIT}, \text{COINIT})$ and consider $Q' = (q'_1, \dots, q'_{k-1}, \text{INIT}, \text{COINIT}, q'_{k+1}, \dots, q'_l)$).

Now, by hypothesis we know that $\mathcal{L}^Q \not\leq^{\text{TextEx}} \mathcal{L}^{Q'}$. Thus, by Corollary 1, Q is not a pseudo-subsequence of Q' . Thus, for Q'' obtained from Q by dropping $q_1 = \text{COINIT}$, Q''' obtained from Q' by dropping $q'_1, q'_2, \dots, q'_{k-1}$ (note that q'_k is *COINIT*, while the first q in Q'' is *INIT*), Q'' is not a pseudo-subsequence of Q''' (this case is similar to the one in Induction Case for (1) with *INIT*s and *COINIT*s reversed). Thus, by Corollary 1, $\mathcal{L}^{Q''} \not\leq^{\text{TextEx}} \mathcal{L}^{Q'''}$.

Suppose by way of contradiction that Θ along with Ψ witnesses the reduction

$$\{ \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n) \text{ is } S\text{-valid} \} \leq^{\text{TextEx}} \mathcal{L}^{Q'}.$$

Let $X = \{ \langle u_1, u_2, \dots, u_l \rangle \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n) \text{ is } S\text{-valid, and } \Theta(\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n}^n) = L_{u_1, u_2, \dots, u_l}^{Q'} \}$.

Let $v_1 = \min(\{u_1 \mid (\exists u_2, \dots, u_l) \langle u_1, u_2, \dots, u_l \rangle \in X\})$.

For $1 < i < k$, let $v_i = \min(\{u_i \mid (\exists u_{i+1}, \dots, u_l) \langle v_1, \dots, v_{i-1}, u_i, \dots, u_l \rangle \in X\})$.

Let b'_j, \dots, a'_n, b'_n be such that $\Theta(\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n}^n) = L_{v_1, \dots, v_{k-1}, u_k, \dots, u_l}^{Q'}$ for some u_k, \dots, u_l .

Let us fix a $b_j > \sum_{k=j}^{k=n} b'_k$, $b_j \in S$. Then, for any a''_w, b''_w , $j < w \leq n$, such that $(a_1, b_1, \dots, a_j, b_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n)$ is S -valid, we have that

$$\Theta(\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n}^n) = L_{v_1, \dots, v_{k-1}, u_k, \dots, u_l}^{Q'}, \text{ for some values}$$

of u_k, \dots, u_l (due to monotonicity of Θ ; note that $\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n}^n$ by Proposition 15, and, therefore, $\text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n}^n \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b'_j, \dots, a'_n, b'_n}^n$, but v_1, \dots, v_{k-1} are chosen to be the minimum possible values, thus only u_k, \dots, u_l can vary for $a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n$).

Thus, Θ (along with Ψ) essentially witnesses that

$$\{ \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n}^n \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, a''_{j+1}, b''_{j+1}, \dots, a''_n, b''_n) \text{ is } S\text{-valid} \} \leq^{\text{TextEx}} \mathcal{L}^{Q'''} \text{ (since parameters for the first } k-1 \text{ INITs are fixed in } Q' \text{ to be } v_1, \dots, v_{k-1}\text{). This is not possible by induction hypothesis (since } \mathcal{L}^{Q''} \not\leq^{\text{TextEx}} \mathcal{L}^{Q'''}\text{).} \blacksquare$$

Corollary 6 *Suppose S is rat^+ -covering. Let $Q = (q_1, \dots, q_{2n})$, where $q_{2i+1} = \text{INIT}$ and $q_{2i+2} = \text{COINIT}$, for $i < n$. Suppose $Q' \in \text{BASIC}^*$ is such that $\mathcal{L}^Q \not\leq \mathcal{L}^{Q'}$. Then, $\text{SEMI_HULL}^{n,S} \not\leq^{\text{TextEx}} \mathcal{L}^{Q'}$.*

8 Q -classes Which Are Reducible to $\text{SEMI_HULL}^{n,S}$

In this section we establish the best possible lower bound on the complexity of $\text{SEMI_HULL}^{n,S}$ in terms of the Q -classes. One can ask the question: using a learner powerful enough to learn $\text{SEMI_HULL}^{n,S}$, can a learner learn languages from the hierarchy based on *BASIC*? The next result shows that, using a learner able to learn $\text{SEMI_HULL}^{n,S}$, one can learn all languages in $(\text{HALF}, \text{INIT}, \dots, \text{INIT})$, where *INIT* is taken $n-1$ times.

We begin with two useful technical propositions.

Proposition 19 *Suppose S is rat^+ -covering. Then there exists an $S' \subseteq S$, such that S' is rat^+ -covering, and for all $b, b' \in S'$, if $b < b'$ then $2b < b'$.*

PROOF. Suppose h is an isomorphism from Z to S such that $h(i) < h(i+1)$. Define h' as follows: $h'(0) = h(0)$. Suppose we have defined $h'(i)$ and $h'(-i)$. Then define $h'(i+1)$ and $h'(-i-1)$ as follows. $h'(i+1) = h(j)$, such that $h(j) > 2h'(i)$, and $h'(-i-1) = h(j')$ such that $h(j') < h'(-i)/2$. Note that such j and j' exist since S is \mathbf{rat}^+ -covering. It immediately follows that $S' = \text{range}(h')$ satisfies the requirements of the Proposition. \blacksquare

Proposition 20 Suppose $S' \subseteq S$. Then $\text{SEMI_HULL}^{n,S'} \leq^{\mathbf{TxtEx}} \text{SEMI_HULL}^{n,S}$.

PROOF. Follows trivially, since $\text{SEMI_HULL}^{n,S'} \subseteq \text{SEMI_HULL}^{n,S}$. \blacksquare

Theorem 5 Suppose S is \mathbf{rat}^+ -covering. Let $n \in \mathbb{N}^+$, and $Q = (q_1, q_2, \dots, q_n)$, where $q_1 = \text{HALF}$, and for $2 \leq i \leq n$, $q_i = \text{INIT}$.

Then, $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} \text{SEMI_HULL}^{n,S}$.

PROOF. The intuitive idea of the reduction is as follows. Fix the first break point to be $a_1 = 1$. The first HALF -component can be reduced to the first slope b_1 . Then the $(j-1)$ -th INIT -component i_j is reduced to a_j . Then b_j is fixed based on a_j in such a way that $i_j < i'_j$ would imply $a_j < a'_j$ and $b_j > b'_j + \sum_{j < i \leq n} b'_i$, for any potential values of b'_i , $i > j$. This ensures the desired monotonicity in the definition of the reducing operator Θ .

Now we proceed with the formal proof.

Without loss of generality (using Propositions 19 and 20) we can assume that,

(PropertyH) for each $b, b' \in S$, if $b < b'$, then $2b < b'$.

Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

For $i_1 \in Z$, and $i_2, \dots, i_n \in N$, let $\text{map}(i_1, i_2, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n)$, where a_j, b_j , $1 \leq j \leq n$ are defined as follows. $a_1 = 1$, $b_1 = h(i_1 - a_1)$. Suppose we have defined $a_1, b_1, \dots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1 \leq i \leq k} b_i * (x \div a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1} + 1)$ -th least element in A_{k+1} . Let $b_{k+1} = h(i_1 - a_{k+1})$.

Claim 3 $(i_1, i_2, \dots, i_n) \leq_Q (i'_1, i'_2, \dots, i'_n)$ implies $\text{SEMI_HULL}_{\text{map}(i_1, \dots, i_n)}^{n,S} \subseteq \text{SEMI_HULL}_{\text{map}(i'_1, \dots, i'_n)}^{n,S}$.

PROOF. Suppose $(i_1, i_2, \dots, i_n) \leq_Q (i'_1, i'_2, \dots, i'_n)$. Suppose $\text{map}(i_1, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n)$ and $\text{map}(i'_1, \dots, i'_n) = (a'_1, b'_1, \dots, a'_n, b'_n)$.

We consider the following cases.

Case 1: $i_1 > i'_1$.

In this case

(1) $b_1 > b'_1$, and

(2) $a_1 = a'_1 = 1$.

From (1) and (PropertyH) it follows that

(3) $b_1 > 2 * b'_1$.

Now, for $1 \leq i < n$, since $a'_i < a'_{i+1}$, and $b'_i = h(i'_1 - a'_i)$, we have $b'_i > b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S .

Thus, $\sum_{1 \leq i \leq n} b'_i \leq 2b'_1$. Along with (3), we have that $b_1 > \sum_{1 \leq i \leq n} b'_i$. This along with (2), Proposition 11 and Proposition 15 gives $\text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^{n,S} \subseteq \text{SEMI_HULL}_{a_1, b_1}^{1,S} \subseteq \text{SEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^{n,S}$.

Case 2: For some j , $1 < j \leq n$, for $1 \leq k < j$, $i_k = i'_k$, but $i_j < i'_j$.

In this case

(4) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leq i < j$.

(5) $b_j > b'_j$, and

(6) $a_j < a'_j$.

From (5) and (PropertyH) it follows that

(7) $b_j > 2 * b'_j$.

Now, for $1 \leq i < n$, since $a'_i < a'_{i+1}$, and $b'_i = h(i'_1 - a'_i)$, we have $b'_i > b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S .

Thus, $\sum_{j \leq i \leq n} b'_i \leq 2b'_j$. This, along with (7) gives us that $b_j > \sum_{j \leq i \leq n} b'_i$. Thus using (6), Proposition 11 and Proposition 15 we have $SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^{n, S} \subseteq SEMI_HULL_{a'_1, b'_1, \dots, a'_n, b'_n}^{j, S} \subset SEMI_HULL_{a'_1, b'_1, \dots, a'_n, b'_n}^{n, S}$.

Claim follows from above cases. \square

For any $X \subseteq N$, let $\Theta(X) = \bigcup_{\langle i_1, \dots, i_n \rangle \in X} SEMI_HULL_{map(i_1, \dots, i_n)}^n$.

Thus, it follows that $\Theta(L_{i_1, i_2, \dots, i_n}^Q) = SEMI_HULL_{map(i_1, \dots, i_n)}^{n, S}$.

Define Ψ as follows. If a sequence α of grammar converges to a grammar for $SEMI_HULL_{map(i_1, \dots, i_n)}^{n, S}$, then $\Psi(\alpha)$ converges to a grammar for $L_{i_1, i_2, \dots, i_n}^Q$.

It is now easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{T\text{xtEx}} SEMI_HULL^{n, S}$. \blacksquare

We now show that the above result is in some sense the best possible with respect to the Q -classes considered in this paper. For example, as Corollary 9 and 10 show, being able to learn the classes $SEMI_HULL^{n, S}$ cannot help to learn languages even in the classes $(COINIT, COINIT)$ and $(INIT, COINIT)$.

The following technical result will be used to show that the Q -classes with $n + 1$ $INIT$ s in Q cannot be reduced to $SEMI_HULL^{n, S}$.

Theorem 6 *Let $n \in N^+$. Suppose $0 \leq j \leq n$. $Q = (q_1, q_2, \dots, q_{n+1-j})$, where $q_i = INIT$, for $1 \leq i \leq n - j + 1$. Let $R = R_1 \times R_2 \times \dots \times R_{n+1-j}$, where R_1 is of cardinality at least 2 and, for $2 \leq i \leq n + 1 - j$, R_i is an infinite subset of N . Suppose S is \mathbf{rat}^+ -covering, and $(a_1, b_1, \dots, a_j, b_j)$ is S -valid.*

Then,
 $\mathcal{L}^{Q, R} \not\leq^{T\text{xtEx}} \{ SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n) \text{ is } S\text{-valid} \}$.

PROOF. We prove the theorem by reverse induction on j (from n to 0). For $j = n$, theorem clearly holds (since $\mathcal{L}^{Q, R}$ contains at least two languages, whereas $\{ SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n) \text{ is } S\text{-valid} \}$ contains only one language).

So suppose the theorem holds for $j' < j \leq n$. Then, we show that the theorem holds for $j = j'$.

Suppose by way of contradiction that Θ (along with some Ψ) witnesses that $\mathcal{L}^{Q, R} \leq^{T\text{xtEx}} \{ SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n) \text{ is } S\text{-valid} \}$, for some S -valid $(a_1, b_1, \dots, a_j, b_j)$.

Let w be the minimal element in R_1 . For $i \in R_2$, suppose $\Theta(L_{w, i, \min(R_3), \min(R_4), \dots}^Q) = SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, a_{j+1}(i), b_{j+1}(i), \dots, a_n(i), b_n(i)}^{n, b}$. If, as i varies, $a_{j+1}(i)$ takes arbitrarily large value then, for $w' > w$, $w' \in R_1$, $\Theta(L_{w', \min(R_2), \min(R_3), \dots}^Q) = SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$ and is thus not a member of $\{ SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n) \text{ is } S\text{-valid} \}$.

So suppose $i_1 \in R_2$ maximizes $a_{j+1}(i_1)$. Now for $i > i_1$, if $b_{j+1}(i)$ takes arbitrarily small value, then for $w' > w$,

$\Theta(L_{w', \min(R_2), \min(R_3), \dots}^Q) \supseteq \bigcup_{b'_{j+1} \in S} \text{SEMI_HULL}_{a_1, b_1, \dots, a_j, b_j, a_{j+1}(i_1), b'_{j+1}}^j$. However, any member of $\{\text{SEMI_HULL}_{a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n) \text{ is } S\text{-valid}\}$, misses out infinitely many elements in $\bigcup_{b'_{j+1} \in S} \text{SEMI_HULL}_{a_1, b_1, \dots, a_j, b_j, a_{j+1}(i_1), b'_{j+1}}^j$. Thus, $\Theta(L_{w', \min(R_2), \min(R_3), \dots}^Q) \notin \{\text{SEMI_HULL}_{a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_j, b_j, a'_{j+1}, b'_{j+1}, \dots, a'_n, b'_n) \text{ is } S\text{-valid}\}$.

So, suppose $i_2 > i_1$, $i_2 \in R_2$, minimizes $b_1(i_2)$.

Thus, Θ (along with Ψ) essentially witnesses that that $\mathcal{L}^{Q', R'} \leq^{\text{TxtEx}} \{\text{SEMI_HULL}_{a_1, b_1, \dots, a_j, b_j, a_{j+1}(i_2), b_{j+1}(i_2), a'_{j+2}, b'_{j+2}, \dots, a'_n, b'_n}^n \mid (a_1, b_1, \dots, a_j, b_j, a_{j+1}(i_2), b_{j+1}(i_2), a'_{j+2}, b'_{j+2}, \dots, a'_n, b'_n) \text{ is } S\text{-valid}\}$, where Q' is obtained by dropping q_1 from Q . R' is obtained from R by dropping R_1 and changing R_2 to $R_2 - \{x \mid x \leq i_2\}$.

This is a contradiction to induction hypothesis. \blacksquare

Corollary 7 Suppose $Q = (q_1, q_2, \dots, q_{n+1})$, where $q_i = \text{INIT}$, for $1 \leq i \leq n+1$. Suppose S is rat^+ -covering.

Then, $\mathcal{L}^Q \not\leq^{\text{TxtEx}} \text{SEMI_HULL}^{n, S}$.

Proof similar to the one used in proving Theorem 6 above can be used to show the following theorem (we just need to interchange the role of w and w' in the proof, for $j = 0$ case).

Theorem 7 Let $n \in \mathbb{N}^+$, and $Q = (q_1, q_2, \dots, q_{n+1})$, where $q_1 = \text{COINIT}$, and $q_i = \text{INIT}$, for $2 \leq i \leq n+1$. Let $R = R_1 \times R_2 \times \dots \times R_{n+1}$, where R_1 is of cardinality at least 2 and, for $2 \leq i \leq n+1$, R_i is an infinite subset of N . Suppose S is any rat^+ -covering set.

Then, $\mathcal{L}^{Q, R} \not\leq^{\text{TxtEx}} \text{SEMI_HULL}^{n, S}$.

Corollary 8 Suppose $Q = (q_1, q_2, \dots, q_{n+1})$, where $q_1 = \text{COINIT}$, and $q_i = \text{INIT}$, for $2 \leq i \leq n+1$. Suppose S is rat^+ -covering.

Then, $\mathcal{L}^Q \not\leq^{\text{TxtEx}} \text{SEMI_HULL}^{n, S}$.

Do there exist Q -classes reducible to $\text{SEMI_HULL}^{n, S}$ with COINIT on the second or greater positions in Q ? Our next results show that even $(\text{COINIT}, \text{COINIT})$ and $(\text{INIT}, \text{COINIT})$ are not reducible to $\text{SEMI_HULL}^{n, S}$.

Based on Corollary 3, let us define $<_{\text{valid}}$ as follows.

Definition 19 Suppose $(a_1, b_1, \dots, a_n, b_n)$ and $(a'_1, b'_1, \dots, a'_n, b'_n)$ are valid. Then, $(a_1, b_1, \dots, a_n, b_n) <_{\text{valid}} (a'_1, b'_1, \dots, a'_n, b'_n)$ iff there exists an i , $1 \leq i \leq n$ such that, for $1 \leq j < i$, $a_j = a'_j$, $b_j = b'_j$ and $a_i < a'_i$ or $a_i = a'_i$ and $b_i > b'_i$.

Note that $<_{\text{valid}}$ imposes a total order among valid sequences of form $(a_1, b_1, \dots, a_n, b_n)$. Moreover, $\text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n \subset \text{SEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$, implies $(a_1, b_1, \dots, a_n, b_n) <_{\text{valid}} (a'_1, b'_1, \dots, a'_n, b'_n)$.

Similarly define $(a_1, b_1, \dots, a_n, b_n) \leq_{\text{valid}} (a'_1, b'_1, \dots, a'_n, b'_n)$ iff $(a_1, b_1, \dots, a_n, b_n) <_{\text{valid}} (a'_1, b'_1, \dots, a'_n, b'_n)$ or $(a_1, b_1, \dots, a_n, b_n) = (a'_1, b'_1, \dots, a'_n, b'_n)$.

\geq_{valid} and $>_{\text{valid}}$ can be similarly defined.

For any fixed $n \in \mathbb{N}$, $X \subseteq \{(a_1, b_1, \dots, a_n, b_n) \mid (a_1, b_1, \dots, a_n, b_n) \in \text{VALID}_S\}$, let $\max_{\text{valid}}(X) = (a_1, b_1, \dots, a_n, b_n) \in X$, if any, such that $(\forall (a'_1, b'_1, \dots, a'_n, b'_n) \in X)[(a'_1, b'_1, \dots, a'_n, b'_n) \leq_{\text{valid}} (a_1, b_1, \dots, a_n, b_n)]$.

Similarly, let $\min_{\text{valid}}(X) = (a_1, b_1, \dots, a_n, b_n) \in X$, if any, such that $(\forall (a'_1, b'_1, \dots, a'_n, b'_n) \in X)[(a_1, b_1, \dots, a_n, b_n) \leq_{\text{valid}} (a'_1, b'_1, \dots, a'_n, b'_n)]$.

Proposition 21 Suppose $n \in N$, S is \mathbf{rat}^+ -covering, and $X \subseteq \{(a_1, b_1, \dots, a_n, b_n) \mid (a_1, b_1, \dots, a_n, b_n) \in \text{VALID}_S\}$. Then, if $\min_{\text{valid}}(X)$ does not exist, then no subset of $(\bigcap_{I \in X} \text{SEMI_HULL}_I^n)$ belongs to $\text{SEMI_HULL}^{n,S}$.

PROOF. Suppose S, X is given as above, and $\min_{\text{valid}}(X)$ does not exist. Then, there exists a j , such that for some $a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j$, there exist arbitrarily large b_j such that $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, \dots) \in X$, for some $a_{j+1}, b_{j+1}, \dots, a_n, b_n$, (which may depend on b_j). Thus, $\bigcap_{I \in X} \text{SEMI_HULL}_I^n \subseteq \{(x, y) \mid x \leq a_j\}$. But no subset of $\{(x, y) \in N^2 \mid x \leq a_j\}$ belongs to $\text{SEMI_HULL}^{n,S}$ (any language L in $\text{SEMI_HULL}^{n,S}$ satisfies: for all x , there exists a y such that $(x, y) \in L$). \blacksquare

Theorem 8 Suppose S is \mathbf{rat}^+ -covering and $n \in N$. Let $Q = (\text{COINIT}, \text{COINIT})$, $R = R_1 \times R_2$, where R_2 is infinite, and R_1 contains at least two elements. Then $\mathcal{L}^{Q,R} \not\leq^{\text{TxtEx}} \text{SEMI_HULL}^{n,S}$.

PROOF. Suppose by way of contradiction that $\mathcal{L}^{Q,R} \leq^{\text{TxtEx}} \text{SEMI_HULL}^{n,S}$ as witnessed by Θ (along with Ψ).

Let $w = \min(R_1)$. Suppose $\Theta(L_{w,x}^Q) = \text{SEMI_HULL}_{a_1(x), b_1(x), \dots, a_n(x), b_n(x)}^n$, for $x \in R_2$.

Note that, for $x < x'$, $x, x' \in R_2$, $\Theta(L_{w,x}^Q) \supset \Theta(L_{w,x'}^Q)$. Thus, for $x < x'$, $x, x' \in R_2$, $(a_1(x), \dots, b_n(x)) >_{\text{valid}} (a_1(x'), \dots, b_n(x'))$, by Corollary 3.

Let $X = \{(a_1(x), b_1(x), \dots, a_n(x), b_n(x)) \mid x \in R_2\}$.

If $\min_{\text{valid}}(X)$ does not exist, then for any $w_1 > w$, $w_1 \in R_1$ and $w_2 \in R_2$, $\Theta(L_{w_1, w_2}^Q) \subseteq \bigcap_{x \in R_2} \text{SEMI_HULL}_{a_1(x), b_1(x), \dots}^n$, does not belong to $\text{SEMI_HULL}^{n,S}$ by Proposition 21.

So suppose $x_1 \in R_2$ is such that $\min_{\text{valid}}(X) = (a_1(x_1), b_1(x_1), \dots, a_n(x_1), b_n(x_1))$. But then, for all $x \geq x_1$, $(a_1(x), \dots, b_n(x)) \leq_{\text{valid}} (a_1(x_1), \dots, b_n(x_1))$ (due to monotonicity of Θ). Thus, for all $x \geq x_1$, $x \in R_2$, $(a_1(x), \dots, b_n(x)) = (a_1(x_1), \dots, b_n(x_1))$. A contradiction to Θ (along with Ψ) witnessing that $\mathcal{L}^{Q,R} \leq^{\text{TxtEx}} \text{SEMI_HULL}^{n,S}$. \blacksquare

Corollary 9 Suppose S is \mathbf{rat}^+ -covering, and $n \in N$. Let $Q = (\text{COINIT}, \text{COINIT})$. Then $\mathcal{L}^Q \not\leq^{\text{TxtEx}} \text{SEMI_HULL}^{n,S}$.

Similarly, to Theorem 8 we also have the following theorem (for proving it, we just need to interchange the roles of $L_{w,x}^Q$ and $L_{w_1,x}^Q$ in the proof of Theorem 8).

Theorem 9 Suppose $n \in N$ and S is \mathbf{rat}^+ -covering. Let $Q = (\text{INIT}, \text{COINIT})$, $R = R_1 \times R_2$, where R_2 is infinite, and R_1 contains at least two elements. Then $\mathcal{L}^{Q,R} \not\leq^{\text{TxtEx}} \text{SEMI_HULL}^{n,S}$.

Corollary 10 Suppose S is \mathbf{rat}^+ -covering. Let $Q = (\text{INIT}, \text{COINIT})$. Then $\mathcal{L}^Q \not\leq^{\text{TxtEx}} \text{SEMI_HULL}^{n,S}$.

9 Definitions for Complements of Open Semi Hull

In this section we define the classes of complements of SEMI_HULL s and establish some useful propositions following from appropriate definitions.

Definition 20 Suppose $a_1, \dots, a_n \in N$ and $b_1, \dots, b_n \in \mathbf{rat}^+$, where $0 < a_1 < a_2 < \dots < a_n$.
 $\text{coSEMI_HULL}_{a_1, b_1, a_2, b_2, \dots, a_n, b_n}^n = \{(x, y) \in N^2 \mid y < \sum_{1 \leq i \leq n} b_i * (x \div a_i)\} = N^2 - \text{SEMI_HULL}_{a_1, b_1, a_2, b_2, \dots, a_n, b_n}^n$.

Definition 21 Suppose $S \subseteq \mathbf{rat}^+$ is \mathbf{rat}^+ -covering.
 $coSEMI_HULL^{n,S} = \{coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \mid (a_1, b_1, \dots, a_n, b_n) \in VALID_S\}$.

Proposition 22 Suppose $(a_1, b_1, \dots, a_n, b_n)$ and $(a'_1, b'_1, \dots, a'_m, b'_m)$ are valid.

Then, $SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \subseteq SEMI_HULL_{a'_1, b'_1, \dots, a'_m, b'_m}^m$ iff $coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \supseteq coSEMI_HULL_{a'_1, b'_1, \dots, a'_m, b'_m}^m$.

PROOF. Follows from the definitions. ■

Definition 22 Suppose $a_1, b_1, \dots, a_j, b_j$ are given such that $(a_1, b_1, \dots, a_j, b_j) \in VALID$.
Then, let $coINTER(a_1, b_1, \dots, a_j, b_j) = \bigcup \{coSEMI_HULL_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n}^n \mid n \geq j \wedge (a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n) \in VALID\} = N^2 - INTER(a_1, b_1, \dots, a_j, b_j)$.

Proposition 23 Suppose $1 \leq j \leq n$, and $(a_1, b_1, \dots, a_n, b_n)$ and $(a'_1, b'_1, \dots, a'_m, b'_m)$ are valid.
Then $coINTER(a'_1, b'_1, \dots, a'_m, b'_m) \supseteq coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n$ iff $INTER(a'_1, b'_1, \dots, a'_m, b'_m) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n$.

PROOF. Follows from the definitions. ■

Proposition 24 Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1})$ is valid. Let $(x, y) \in N^2$ be such that $y > \sum_{1 \leq i < j} b_i(x \div a_i)$. Then, there exists a $B' \in \mathbf{rat}^+$ obtainable effectively from $a_1, b_1, \dots, a_{j-1}, b_{j-1}, x$ and y such that for any (a'_j, b'_j) , if $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is valid and $(x, y) \in coINTER_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$, then $a'_j < x$ and $b'_j \geq B'$.

PROOF. Follows from Proposition 16. ■

10 Classes to Which $coSEMI_HULL^{n,S}$ is Reducible

In this section we obtain nearly the best possible upper bound on the complexity of $coSEMI_HULL$ s in terms of Q -degrees. Intuitive upper bound \mathcal{L}^Q for $Q = (q_1, q_2, \dots, q_{2n-1}, q_{2n})$ with $q_{2i+1} = COINIT$ and $q_{2i+2} = HALF$ can be easily established using the following learning strategy for $coSEMI_HULL^{n,S}$: apply a $COINIT$ -type strategy to learn the first break point a_1 , then apply a $HALF$ -type strategy to learn the first slope b_1 , etc. However, the upper bound established below contains only $n + 1$ components!

Theorem 10 Suppose $n \in N$ and S is \mathbf{rat}^+ -covering. Suppose $Q = (q_1, q_2, \dots, q_{n+1})$, where $q_1 = INIT$ and $q_i = COINIT$, for $2 \leq i \leq n + 1$. Then $coSEMI_HULL^{n,S} \leq^{T\text{xtEx}} \mathcal{L}^Q$.

PROOF. The intuitive strategy for learning $coSEMI_HULL^{n,S}$ providing the desired upper bound operates as follows: first, it applies an $INIT$ -type strategy to learn a bound on the maximum slope: $\sum_i b_i$; then, using this bound, it applies $COINIT$ -type strategies to learn every pair of parameters a_i, b_i . This $COINIT$ -type strategy is in some sense mirror image of the $INIT$ -type strategy used in the proof of Theorem 3. The technical details though become somewhat more complicated.

Now we proceed with the formal proof.

Fix S which is \mathbf{rat}^+ -covering. Let h be a recursive bijection from Z to S such that $h(i) < h(i + 1)$. Fix code as in Proposition 18.

Claim 4 Suppose $B \in S$. Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j), (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in \text{VALID}_S$, where $b_i, b'_i \leq B$, and $(a_j, b_j) \neq (a'_j, b'_j)$. Suppose $\text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$.

Then, $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) < \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$.

PROOF. By Proposition 14 and definitions of SEMI_HULL , coSEMI_HULL , INTER , coINTER , we have that $a'_j \leq a_j$.

Now, since, $\text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$, we have $\text{INTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$. Thus, we have $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) < \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$ (by Proposition 18; for getting $<$ instead of \leq use the fact that $(a_j, b_j) \neq (a'_j, b'_j)$). \square

Let map be a mapping from VALID to N^* such that

$$\text{map}(a_1, b_1, \dots, a_n, b_n) = (h^{-1}(B), \text{code}(B, a_1, b_1), \text{code}(B, a_1, b_1, a_2, b_2), \dots, \text{code}(B, a_1, b_1, \dots, a_n, b_n)),$$

where B is the least element of S such that $\max(h(0), \sum_{1 \leq i \leq n} b_i) \leq B$.

Claim 5 Suppose $(a_1, b_1, \dots, a_n, b_n)$ is S -valid. Let $B = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b_i)\})$.

(A) Suppose $1 \leq j \leq n$. Suppose $B \geq b'_j$, and $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is S -valid. If $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) > \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$. Then $\text{coINTER}(a_1, b_1, \dots, a'_j, b'_j) \not\supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$.

(B)

Suppose $(a'_1, b'_1, \dots, a'_n, b'_n)$ is S -valid, and $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$. Then either $B > \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\})$ or for the least j , $1 \leq j \leq n$, such that $(a_j, b_j) \neq (a'_j, b'_j)$, $\text{coINTER}(a'_1, b'_1, \dots, a'_j, b'_j) \not\supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$.

(C) Suppose $(a'_1, b'_1, \dots, a'_n, b'_n)$ is S -valid,

and $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$. Then either $B > \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\})$ or $\text{coINTER}(a'_1, b'_1, \dots, a'_j, b'_j, \dots, a'_n, b'_n) \not\supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$.

(D) If $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n \subset \text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$, then

$\text{map}(a_1, b_1, \dots, a_n, b_n) <_Q \text{map}(a'_1, b'_1, \dots, a'_n, b'_n)$.

(E) Suppose $1 \leq j \leq n$. There exists a finite $X_j \subseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_j, b_j}^n$ such that, for all S -valid $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j)$, if $B \geq b''_j$ and $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j) > \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$, then $\text{coINTER}(a_1, b_1, \dots, a''_j, b''_j) \not\supseteq X_j$.

(F) There exists a finite $X \subseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$ such that, for all S -valid $(a'_1, b'_1, \dots, a'_n, b'_n)$, if $B = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\})$, and $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$, then $\text{coINTER}(a'_1, b'_1, \dots, a'_n, b'_n) \not\supseteq X$.

PROOF. (A) Follows from Claim 4.

(B) Let $B' = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\})$. Since $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$, $h^{-1}(B') \leq h^{-1}(B)$. Therefore, $B' \leq B$, and we must have $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\}) \leq \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b_i)\}) = B$. If $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\}) = B$, then let j be the least value such that $1 \leq j \leq n$, and $(a'_j, b'_j) \neq (a_j, b_j)$. Thus, $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) > \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$. Now part (B) follows from part (A).

(C) Since $\text{coINTER}(a'_1, b'_1, \dots, a'_j, b'_j) \supseteq \text{coINTER}(a'_1, b'_1, \dots, a'_j, b'_j, \dots, a'_n, b'_n)$ (by Proposition 12 and definitions of INTER and coINTER), and $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j \subseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$ (Proposition 11 and definitions of SEMI_HULL , coSEMI_HULL), part (C) follows from part (B).

(D) Suppose $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n \subset \text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$. Thus, we must have $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\}) \geq \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b_i)\})$ (since otherwise, for all but finitely many x , $\sum_{1 \leq i \leq n} b_i(x \dot{-} a_i) > 1 + \sum_{1 \leq i \leq n} b'_i(x \dot{-} a'_i)$, which would contradict $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n \subseteq \text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$).

If $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\}) > \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b_i)\})$, then clearly, $\text{map}(a_1, b_1, \dots, a_n, b_n) <_Q \text{map}(a'_1, b'_1, \dots, a'_n, b'_n)$. If $\min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\}) = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b_i)\})$, then (D) follows from part (C), and the fact that $\text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n \subseteq \text{coINTER}(a'_1, b'_1, \dots, a'_n, b'_n)$.

(E) Let $(x, y) \in N^2$ be such that $(x, y) \in \text{coSEMI_HULL}_{a_1, b_1, \dots, a_j, b_j}^n \subseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$, but $y > \sum_{1 \leq i < j} b_i(x \dot{-} a_i)$. Note that there exists such (x, y) .

By

Proposition 24, there exists a $B' \in \text{rat}^+$ such that, if $(x, y) \in \text{coINTER}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$, then $a'_j < x$ and $b'_j \geq B'$. Thus, it follows that for any (a'_j, b'_j) , if $a'_j \geq x$, or $b'_j < B'$, $(x, y) \notin \text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$. Now, for each $a'_j < x$, and $b'_j \in S$ such that $B' \leq b'_j \leq B$, if $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_j, b_j}^j \not\subseteq \text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, then pick $x_{a'_j, b'_j}, y_{a'_j, b'_j}$ such that $(x_{a'_j, b'_j}, y_{a'_j, b'_j}) \in \text{coSEMI_HULL}_{a_1, b_1, \dots, a_j, b_j}^j - \text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$; if $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_j, b_j}^j \subseteq \text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, then let $(x_{a'_j, b'_j}, y_{a'_j, b'_j}) = (x, y)$.

Now let $X_j = \{(x, y)\} \cup \{(x_{a'_j, b'_j}, y_{a'_j, b'_j}) \mid a'_j < x, B' \leq b'_j \leq B, b'_j \in S\}$. Using part (A), it is easy to verify that X_j witnesses the claim of part (E).

(F) Let $X = \bigcup_{1 \leq j \leq n} X_j$ where X_j is as in part (E). Now, if $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$, and $B = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b'_i)\})$, then there exists a j , $1 \leq j \leq n$ such for $1 \leq i < j$, $a_i = a'_i$ and $b_i = b'_i$ and $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) > \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$. Part (F) now follows from part (E) and definition of X . \square

We now continue with the proof of the theorem. The aim is to construct Θ which maps $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$ to $L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$ to a grammar sequence converging to a grammar for $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$ would be trivial. We thus just define Θ .

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $\text{Prop}(X, a_1, b_1, \dots, a_n, b_n)$ be true iff following three properties are satisfied. Let $B = \min(\{b \in S \mid b \geq h(0) \wedge (\forall (x, y) \in X)[b \geq \frac{y}{x}]\})$.

(A) $(a_1, b_1, \dots, a_n, b_n) \in \text{VALID}_S$,

(B) $B \geq \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b_i)\})$

and

(C) If $B = \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leq i \leq n} b_i)\})$, then for all j , $1 \leq j \leq n$, for all $a'_j \in N, b'_j \in S$ such that $(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is S -valid, $B \geq b'_j$ and $\text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) > \text{code}(B, a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$, $[X \not\subseteq \text{coINTER}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}]$.

Note that whether $X, a_1, b_1, \dots, a_n, b_n$, satisfy (A) and (B) and (C) is effectively testable.

Moreover, (C) along with definition of B above implies that

(D) For all $(a'_1, b'_1, \dots, a'_n, b'_n) \in \text{VALID}$ such that $b'_i \in S$ and $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$, $X \not\subseteq \text{coSEMI_HULL}(a'_1, b'_1, \dots, a'_n, b'_n)$.

Thus for finite $X \subseteq N^2$, let

$$\Theta(X) = \bigcup \{L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q \mid \text{Prop}(X, a_1, b_1, \dots, a_n, b_n)\}.$$

For infinite X' , $\Theta(X') = \bigcup_{X \subseteq X', \text{card}(X) < \infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$,
 $\Theta(X) \subseteq L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$ (by (D) and the fact that for any valid I and I' , $\text{map}(I) <_Q \text{map}(I')$, implies $L_{\text{map}(I)}^Q \subseteq L_{\text{map}(I')}^Q$), and
(2) For all S -valid $(a_1, b_1, \dots, a_n, b_n)$, by Claim 5(F), there exists a finite $X \subseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$, such that $\Theta(X) \supseteq L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$.
Thus, we have that $\Theta(\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n) = L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q$. ■

The construction in the proof of above theorem can be slightly changed along the following lines: instead of learning first the bound on the maximum slope, one can first apply a *COINIT*-type strategy trying to learn the parameters a_1, b_1 under assumption that b_1 is smaller than some fixed bound $B \in \mathbf{rat}^+$, and then apply an *INIT*-type strategy to learn both the bound on the maximum slope and b_1 if the latter becomes greater than the bound B . Thus we obtain the following theorem.

Theorem 11 *Suppose $n \in \mathbf{N}$ and S is \mathbf{rat}^+ -covering. Suppose $Q = (q_1, q_2, \dots, q_{n+1})$, where $q_2 = \text{INIT}$ and $q_i = \text{COINIT}$, for $1 \leq i \leq n+1$, $i \neq 2$. Then $\text{coSEMI_HULL}^{n,S} \leq^{\mathbf{TxtEx}} \mathcal{L}^Q$.*

11 Classes Which are Reducible to $\text{coSEMI_HULL}^{n,S}$

In this section we will get a lower bound for $\text{coSEMI_HULL}^{n,S}$ having n components, and thus being very close to the upper bounds obtained in the previous section.

The proof of the following theorem is similar to the proof of the lower bound for *SEMI_HULLS*, with *COINIT*s replacing *INIT*s.

Theorem 12 *Suppose S is \mathbf{rat}^+ -covering. Let $n \in \mathbf{N}^+$, and $Q = (q_1, q_2, \dots, q_n)$, where $q_1 = \text{HALF}$, and for $2 \leq i \leq n$, $q_i = \text{COINIT}$.
Then, $\mathcal{L}^Q \leq^{\mathbf{TxtEx}} \text{coSEMI_HULL}^{n,S}$.*

The proof of the above theorem is given in the Appendix.

Note that upper and lower bounds for $\text{coSEMI_HULL}^{n,S}$ given by Theorems 10, 11, and 12 do not match. The lower bound in Theorem 12 above is the best possible (for Q -classes involving components from *BASIC*). However it is open whether the upper bound can be improved for general n . For $n = 1$, we do know that the upper bound can be improved to show that $\text{coSEMI_HULL}^{1,S} \leq^{\mathbf{TxtEx}} \text{HALF}$ (which is optimal by Theorem 12).

12 Open Hulls - Intersections of Semi-Hulls

Now consider the class of language-figures that are intersections of *SEMI_HULLS* adjacent to the x -axis (that is with the first break point $(a_1, 0)$) and reverse *SEMI_HULLS* adjacent to the y -axis (with the first break point $(0, a'_1)$). These figures are the open hulls.

We give the formal definition below (preceded by the formal definition of the reverse *SEMI_HULLS* adjacent to the y -axis).

Definition 23 $\text{REV_SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n = \{(x, y) \mid (y, x) \in \text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n\}$.
 $\text{REV_SEMI_HULL}^{n,S} = \{\text{REV_SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n \mid \text{SEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n \in \text{SEMI_HULL}^{n,S}\}$.

Definition 24

$$\begin{aligned}
OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m} &= SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \cap REV_SEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^m. \\
OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m, S} &= \{OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m, S} \mid SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \in \\
&SEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^{n, S}, REV_SEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^m \in REV_SEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^{m, S}, \text{ and } \sum_{1 \leq i \leq n} b_i < \\
&\frac{1}{\sum_{1 \leq i \leq m} d_i}\}.
\end{aligned}$$

The latter condition, $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, ensures that the languages in OP_HULLs are infinite, and thus the corresponding geometrical figures are *open hulls*.²

Surprisingly, unlike $SEMI_HULLs$ and $coSEMI_HULL$, upper and lower bounds for OP_HULLs match. The following theorem establishes the lower bound for the OP_HULLs . Somewhat surprising is also the fact that the learnability degree of open hulls is below the learnability degree of semi-hulls. However, the reader must note that while slopes of segments of the border-line for semi-hulls are bounded only by both axes, the slopes of the segments of border-lines in a open hull bound each other (the reader should note that, as we only consider (reversed) semi-hulls with at least one angle in the theorems below, the (reversed) semi-hull cannot contain the entire domain). Consequently, while learning the angles along both border-lines in a open hull, the learning algorithm becomes aware of these bounds and, as it turns out, can use a shorter sequence of the primitive strategies to learn the concept in question.

Theorem 13 *Suppose S is \mathbf{rat}^+ -covering. Suppose $n \geq 1, m \geq 1$. Let $Q = (q_1, \dots, q_n)$, where each $q_i = INIT$. Then, (a) $\mathcal{L}^Q \leq^{\mathbf{TextEx}} OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m, S}$, and (b) $\mathcal{L}^Q \leq^{\mathbf{TextEx}} OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{m, n, S}$.*

Proof of the above Theorem is given in the Appendix. We next show the upper bound for OP_HULLs .

Theorem 14 *Suppose S is \mathbf{rat}^+ -covering. Suppose $n \geq m \geq 1$. Let $Q = (q_1, \dots, q_n)$, where each $q_i = INIT$. Then, (a) $OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m, S} \leq^{\mathbf{TextEx}} \mathcal{L}^Q$, and (b) $OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{m, n, S} \leq^{\mathbf{TextEx}} \mathcal{L}^Q$.*

Proof of the above Theorem is given in the Appendix.

13 Complements of Open Hulls

In this section we define and explore the classes of complements of OP_HULLs .

Definition 25

$$\begin{aligned}
REV_coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n &= \{(x, y) \mid (y, x) \in coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n\}. \\
REV_coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, S} &= \{REV_coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \mid coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \in \\
&coSEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^{n, S}\}.
\end{aligned}$$

Definition 26 $coOP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m} = coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \cup REV_coSEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^m = N^2 - OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m}$.

$$\begin{aligned}
coOP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m, S} &= \{coOP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m, S} \mid coSEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^n \in \\
&coSEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^{n, S}, REV_coSEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^m \in REV_coSEMI_HULL_{c_1, d_1, \dots, c_m, d_m}^{m, S}, \text{ and } \\
&\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}\}.
\end{aligned}$$

²If we do not require $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, then the geometrical figure may be finite. In this case, the complexity of learnability shows similar properties as the classes considered (with the above constraint on slopes), however the analysis becomes more complex.

The following theorem gives the lower bound for *coOP_HULLS*.

Theorem 15 *Suppose S is rat^+ -covering. Suppose $n \geq 1, m \geq 1$. Let $Q = (q_1, \dots, q_n)$, where each $q_i = \text{COINIT}$. Then, (a) $\mathcal{L}^Q \leq^{\text{TxtEx}} \text{coOP_HULL}^{n,m,S}$, and (b) $\mathcal{L}^Q \leq^{\text{TxtEx}} \text{coOP_HULL}^{m,n,S}$.*

Proof of the above Theorem is given in the Appendix. The following theorem gives the upper bound for *coOP_HULLS*.

Theorem 16 *Suppose $n \geq m \geq 1$. Let $Q = (q_1, \dots, q_n)$, where each $q_i = \text{COINIT}$. Then, (a) $\text{coOP_HULL}^{n,m,S} \leq^{\text{TxtEx}} \mathcal{L}^Q$, and (b) $\text{coOP_HULL}^{m,n,S} \leq^{\text{TxtEx}} \mathcal{L}^Q$.*

Proof of the above Theorem is given in the Appendix.

14 Conclusions

A new complexity scale has been successfully applied for evaluating the complexity of learning various geometrical figures from texts. Many upper bounds obtained by us are surprisingly lower than the ones suggested by intuitive learning strategies. Another surprising result is that upper and lower bounds match for *OP_HULL* and their complements, while there is a gap between upper and lower bounds for *SEMI_HULLS* that cannot be narrowed. One more interesting aspect of this picture is that upper bounds for *OP_HULLS*, the intersection of *SEMI_HULLS*, are much lower than that for *SEMI_HULLS* themselves! In general, the picture of upper and lower bounds for *OP_HULLS* and their complements is much more uniform than for *SEMI_HULLS* and their complements: bounds for *coOP_HULLS* can be obtained from the bounds for *OP_HULLS* by just replacing *INITs* by *COINITs*, while bounds for *SEMI_HULLS* and *coSEMI_HULLS* differ even in the number of components in Q -vectors.

There are many other interesting types of geometrical concepts whose complexity can be explored in terms of the Q -classes. For example, one can evaluate the complexity of learning *SEMI_HULLS* and all other figures observed in our paper dropping requirement of the first angle being adjacent to x or y -axis. Even more promising seems to be the class of finite unions of *OP_HULLS* (though proofs may become technically messy). In general, we are convinced that the Q -classes (possibly using some other basic classes/strategies) are very promising tools for exploring the complexity of learning hard languages from texts.

15 Acknowledgements

We thank Mohan Kankanhalli who suggested the name *SEMI_HULL*. Sanjay Jain was supported in part by NUS grant number R252-000-127-112. Efim Kinber was supported in part by the URCG grant of Sacred Heart University. Preliminary version of this paper appeared in COLT'2001.

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16 Appendix

16.1 Proof of Theorem 1

Theorem 1 $HALF \equiv^{\mathbf{TxtEx}} INIT \times COINIT$.

PROOF. For $a \in Z$, let $L_a = \{x \in Z \mid x \geq a\}$. For $i, j \in N$, let $X_{i,j} = \{x \in N \mid x \leq i\} \times \{x \mid x \geq j\}$.

We now define Θ and Ψ witnessing that $HALF \leq^{\mathbf{TxtEx}} INIT \times COINIT$.

For any finite subset Y of Z , let $\Theta(Y) = \bigcup_{a \geq 0, a \in Y} X_{0,a} \cup \bigcup_{a < 0, a \in Y} X_{-a,0}$.

It can be easily verified that $\Theta(L_a) = X_{0,a}$ (if $a \geq 0$), and $\Theta(L_a) = X_{-a,0}$ (if $a < 0$).

Ψ is defined as follows. Suppose a sequence α of grammars converges to grammar p . Suppose $i = \max(\{x \in N \mid (\exists y \in N)[\langle x, y \rangle \in W_p]\})$, and $j = \min(\{y \in N \mid (\exists x \in N)[\langle x, y \rangle \in W_p]\})$. Then, if $i = 0$, then $\Psi(\alpha)$ converges to a program for L_j . If $i > 0$, then $\Psi(\alpha)$ converges to a program for L_{-i} .

It is easy to verify that Θ and Ψ witness that $HALF \leq^{\mathbf{TxtEx}} INIT \times COINIT$.

Now, we show that $INIT \times COINIT \leq^{\mathbf{TxtEx}} HALF$.

Define h as follows:

for $i, j \in N$,

$h(i, j) = -[(i+1)(i+2)/2] + 1 + j$, if $i \geq j$;

$h(i, j) = [j(j+1)/2] - i$, if $i < j$.

Intuitively picture $h(i, j)$ as follows:

$$\begin{array}{ccccccc} \dots & h(2,0) & h(2,1) & h(2,2) & h(1,0) & h(1,1) & h(0,0) \\ \dots & -5 & -4 & -3 & -2 & -1 & 0 \end{array}$$

$$\begin{array}{ccccccc} h(0,1) & h(1,2) & h(0,2) & h(2,3) & h(1,3) & h(0,3) & \dots \\ 1 & 2 & 3 & 4 & 5 & 6 & \dots \end{array}$$

Note that for all $i, j, k, l \in N$, if $i \geq k$ and $j \leq l$, then $h(i, j) \leq h(k, l)$. Now, for any set $Y \subseteq N^2$, let $\Theta(Y) = \bigcup_{(i,j) \in Y} L_{h(i,j)}$. It is easy to verify that $\Theta(X_{i,j}) = L_{h(i,j)}$.

$\Psi(\alpha)$ is defined as follows. If a sequence α of grammars converges to a grammar p for $L_{h(i,j)}$, then $\Psi(\alpha)$ converges to a grammar for $X_{i,j}$. (Note that, if p is a grammar for some $L_{h(i,j)}$, then such i, j can be determined in the limit from p .) It is easy to verify that Θ and Ψ witness that $INIT \times COINIT \leq^{\mathbf{TxtEx}} HALF$. ■

16.2 Proof of Theorem 2

Theorem 2 Suppose $Q = (q_1, \dots, q_k) \in BASIC^k$ and $Q' = (q'_1, \dots, q'_l) \in BASIC^l$. Let $R = R_1 \times R_2 \times \dots \times R_k$, $R' = R'_1 \times R'_2 \times \dots \times R'_l$, where each R_i (R'_i) is an infinite subset of N , if $q_i \in \{INIT, COINIT\}$ ($q'_i \in \{INIT, COINIT\}$), and R_i (R'_i) is a subset of Z , with infinite intersection with both N and Z^- , if $q_i = HALF$ ($q'_i = HALF$).

If Q is not a pseudo-subsequence of Q' then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} \mathcal{L}^{Q',R'}$.

PROOF. We prove the theorem by double induction (first on k and then on l). For $k = 0$ or $l = 0$ theorem clearly holds. Suppose by induction that the theorem holds for $k \leq m$, $l \in N$, and for $k = m + 1$, $l \leq r$. We then show that the theorem holds for $k = m + 1$ and $l = r + 1$. Suppose by way of contradiction that Θ (along with Ψ) witnesses that $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}} \mathcal{L}^{Q',R'}$.

We consider the following cases:

Case 1: $q_1 = INIT$.

Case 1.1: $q'_1 = COINIT$.

Consider σ , which minimizes $i \in N$ such that $\langle i, \dots \rangle \in \text{content}(\Theta(\sigma))$. Let $j \in N$ be the maximum number such that $\langle j, \dots \rangle \in \text{content}(\sigma)$. It follows that, for any $j' > j$, $j' \in R_1$, $\Theta(L_{j', \dots}^Q)$, for any values of other parameters, is of the form $L_{i, \dots}^{Q'}$, for some values of the other parameters. Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{Q, RR} \leq^{\text{TextEx}} \mathcal{L}^{QQ', RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq j\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1 . Now we are done by induction hypothesis.

Case 1.2: $q'_1 = \text{INIT}$.

In this case $k \geq 2$.

Case 1.2.1: $q_2 = \text{INIT}$.

Fix $i_1 \in R_1$, and consider the set $\bigcup_{i_2 \in R_2, \dots} \Theta(L_{i_1, i_2, \dots}^Q)$. Suppose this set contains $\langle i'_1, \dots \rangle$, for arbitrarily large i'_1 . Then for any $ii_1 > i_1$ (since $L_{ii_1, \dots}^Q \supseteq L_{i_1, \dots}^Q$ for all possible values of other parameters) we have that $\Theta(L_{ii_1, \min(R_2), \min(R_3), \dots}^Q)$ contains elements of form $\langle i'_1, \dots \rangle$ for arbitrarily large i'_1 . Thus, $\Theta(L_{ii_1, \min(R_2), \min(R_3), \dots}^Q) \notin \mathcal{L}^{Q'}$.

So let i'_1 be maximum value such that some element of form $\langle i'_1, \dots \rangle$ is in $\bigcup_{i_2 \in R_2, \dots} \Theta(L_{i_1, i_2, \dots}^Q)$.

Let σ be such that $\text{content}(\sigma) \subseteq L_{i_1, i_2, \dots}^Q$, and $\Theta(\sigma)$ contains an element of form $\langle i'_1, \dots \rangle$. Let i_2 be maximum value such that some element of form $\langle i_1, i_2, \dots \rangle$ is in $\text{content}(\sigma)$. It follows that, for all $ii_2 > i_2$, $ii_2 \in R_2$, $\Theta(L_{i_1, ii_2, \dots}^Q)$, for any value of other parameters, is of form $L_{i'_1, \dots}^{Q'}$, for some value of other parameters. Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{QQ, RR} \leq^{\text{TextEx}} \mathcal{L}^{QQ', RR'}$, where QQ is obtained from Q by dropping q_1 , QQ' is obtained from Q' by dropping q'_1 , RR' is obtained from R' by dropping R'_1 and RR is obtained from R by dropping R_1 plus changing R_2 to $R_2 - \{x \mid x \leq i_2\}$. Now we are done by induction hypothesis.

Case 1.2.2: $q_2 = \text{COINIT}$.

In this case, by Proposition 10(a) QQ obtained from Q by dropping q_1 is not a pseudo-subsequence of Q' . Thus we are done by induction hypothesis.

Case 1.2.3: $q_2 = \text{HALF}$.

If $l = 1$, then we are done. So assume $l \geq 2$.

Case 1.2.3.1: $q'_2 = \text{INIT}$ or HALF .

Then, by replacing q_2 by COINIT , using Proposition 10(d) we still have that Q is not a pseudo-subsequence of Q' . Thus, we can use Case 1.2.2.

Case 1.2.3.2: $q'_2 = \text{COINIT}$.

Then, by replacing q_2 by INIT , using Proposition 10(e) we still have that Q is not a pseudo-subsequence of Q' . Thus, we can use Case 1.2.1.

Case 1.3 $q'_1 = \text{HALF}$.

In this case, let $i \in R_1$. Suppose $\Theta(L_{i, \dots}^Q) = L_{j, \dots}^{Q'}$, for some values of other parameters. But then for all $i' > i$, $\Theta(L_{i', \dots}^Q)$, for any value of other parameters, must be of form $L_{j', \dots}^{Q'}$, for some value of other parameters, where $j' \leq j$. Thus, one could essentially consider Θ (along with Ψ) as a reduction from $\mathcal{L}^{Q, RR}$ to $\mathcal{L}^{QQ', RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq i\}$, QQ' is obtained from Q' by replacing q'_1 with INIT , and RR' is obtained from R' by replacing R'_1 by $\{x \in N \mid -x + j \in R'_1 - \{y \mid y \geq j\}\}$. Thus we can use Case 1.2.

Case 2: $q_1 = \text{COINIT}$. This case is very similar to Case 1. We give the analysis for completeness sake.

Case 2.1: $q'_1 = \text{INIT}$.

Let $i \in N$ be minimum value such that $\Theta(L_{i, \dots}^Q) = L_{i, \dots}^{Q'}$, for some values of the other parameters. Let $j \in R_1$ be such that $\Theta(L_{j, \dots}^Q) = L_{i, \dots}^{Q'}$, for some values of the parameters. It follows

that for all $j' > j$, $j' \in R_1$, $\Theta(L_{j', \dots}^Q)$, for any value of other parameters, is of form $L_{i, \dots}^{Q'}$, for some value of other parameters.

Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{Q, RR} \leq \text{TxtEx } \mathcal{L}^{QQ', RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq j\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1 . Now we are done by induction hypothesis.

Case 2.2: $q'_1 = \text{COINIT}$.

In this case $k \geq 2$.

Case 2.2.1: $q_2 = \text{INIT}$.

In this case, by Proposition 10(a) QQ obtained from Q by dropping q_1 is not a pseudo-subsequence of Q' . Thus we are done by induction hypothesis.

Case 2.2.2: $q_2 = \text{COINIT}$. Fix $i_1 \in R_1$, and consider $\Theta(L_{i_1, \dots}^Q) = L_{i'_1, \dots}^{Q'}$. If i'_1 achieves arbitrary high value (for some values of other parameters) then, for $ii_1 > i_1$, since $L_{ii_1, 0, 0, \dots}^Q \subseteq L_{i_1, \dots}^Q$, and $\bigcap_{i'_1 \in R'_1} L_{i'_1, \dots}^{Q'} = \emptyset$, $\Theta(L_{ii_1, 0, 0, \dots}) = \emptyset \notin \mathcal{L}^{Q', R'}$.

So let i'_1 be maximum value such that for some value of other parameters, $\Theta(L_{i_1, \dots}^{Q, R}) = L_{i'_1, \dots}^{Q', R'}$. let i_2 be such that, for some value of other parameters, $\Theta(L_{i_1, i_2, \dots}^{Q, R}) = L_{i'_1, \dots}^{Q', R'}$. It follows that, for all $ii_2 > i_2$, $\Theta(L_{i_1, ii_2, \dots}^{Q, R})$ is of form $L_{i'_1, \dots}^{Q', R'}$. Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{QQ, RR} \leq \text{TxtEx } \mathcal{L}^{QQ', RR'}$, where QQ is obtained from Q by dropping q_1 , QQ' is obtained from Q' by dropping q'_1 , RR' is obtained from R' by dropping R'_1 and RR is obtained from R by dropping R_1 plus changing R_2 to $R_2 - \{x \mid x \leq i_2\}$. Now we are done by induction hypothesis.

Case 2.2.3: $q_2 = \text{HALF}$.

If $l = 1$, then we are done. So assume $l \geq 2$.

Case 2.2.3.1: $q'_2 = \text{INIT}$ or HALF .

Then, by replacing q_2 by COINIT , using Proposition 10(d), we still have that Q is not a pseudo-subsequence of Q' . Thus, we can use Case 2.2.2.

Case 2.2.3.2: $q'_2 = \text{COINIT}$.

Then, by replacing q_2 by INIT , using Proposition 10(e), we still have that Q is not a pseudo-subsequence of Q' . Thus, we can use Case 2.2.1.

Case 3: $q_1 = \text{HALF}$.

Case 3.1: $q'_1 = \text{INIT}$.

Then by Proposition 10(b) replacing q_1 by COINIT , still gives us that Q is not a pseudo-subsequence of Q' . Thus, we can use Case 2.

Case 3.2: $q'_1 = \text{COINIT}$.

Then by Proposition 10(c) replacing q_1 by INIT , still gives us that Q is not a pseudo-subsequence of Q' . Thus, we can use Case 1.

Case 3.3: $q'_1 = \text{HALF}$.

In this case, let $i \in R_1$ and consider $\Theta(L_{i, \dots}^Q)$, for some value of other parameters. Suppose it is $L_{j, \dots}^{Q'}$, for some value of other parameters. Then for all $i' < i$, $i' \in R_1$, $\Theta(L_{i', \dots}^Q)$ must be of form $L_{j', \dots}^{Q'}$, where $j' \leq j$. Thus, one could essentially consider this as a reduction from $\mathcal{L}^{QQ, RR}$ to $\mathcal{L}^{QQ', RR'}$, where QQ is obtained from Q by replacing q_1 by INIT , RR is obtained from R by replacing R_1 by $\{x \in N \mid -x + i \in R_1 - \{y \mid y \geq i\}\}$, QQ' is obtained from Q' by replacing q'_1 with INIT , and RR' is obtained from R' by replacing R'_1 by $\{x \in N \mid -x + j \in R'_1 - \{y \mid y \geq j\}\}$. Now we can use Case 1.

It follows from above cases that $\mathcal{L}^{Q, R} \not\leq \text{TxtEx } \mathcal{L}^{Q', R'}$. ■

16.3 Proof of Theorem 12

Proposition 25 Suppose $S' \subseteq S$. Then $\text{coSEMI_HULL}^{n,S'} \leq^{\text{TxtEx}} \text{coSEMI_HULL}^{n,S}$.

Theorem 12 Suppose S is rat^+ -covering. Let $n \in N^+$, and $Q = (q_1, q_2, \dots, q_n)$, where $q_1 = \text{HALF}$, and for $2 \leq i \leq n$, $q_i = \text{COINIT}$.

Then, $\mathcal{L}^Q \leq^{\text{TxtEx}} \text{coSEMI_HULL}^{n,S}$.

PROOF. Without loss of generality (using Propositions 19 and 25) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $\text{map}(i_1, i_2, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n)$, where a_j, b_j , $1 \leq j \leq n$ are defined as follows. $a_1 = 1$, $b_1 = h(-i_1 - a_1)$. Suppose we have defined $a_1, b_1, \dots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1 \leq i \leq k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1} + 1)$ -th least element in A_{k+1} . Let $b_{k+1} = h(-i_1 - a_{k+1})$.

Claim 6 $(i_1, i_2, \dots, i_n) <_Q (i'_1, i'_2, \dots, i'_n)$ implies $\text{coSEMI_HULL}_{\text{map}(i_1, \dots, i_n)}^{n,S} \subset \text{coSEMI_HULL}_{\text{map}(i'_1, \dots, i'_n)}^{n,S}$.

PROOF. Suppose $(i_1, i_2, \dots, i_n) <_Q (i'_1, i'_2, \dots, i'_n)$. Suppose $\text{map}(i_1, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n)$ and $\text{map}(i'_1, \dots, i'_n) = (a'_1, b'_1, \dots, a'_n, b'_n)$.

We consider the following cases.

Case 1: $i_1 > i'_1$.

In this case

- (1) $b_1 < b'_1$, and
- (2) $a_1 = a'_1$.

From (1) it follows that

- (3) $2b_1 < b'_1$.

Now, for $1 \leq i < n$, since $a_i < a_{i+1}$, and $b_i = h(-i_1 - a_i)$, we have $b_i > b_{i+1}$. Thus, $b_i > 2 * b_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S . Thus, $\sum_{1 \leq i \leq n} b_i \leq 2b_1$. Along with (3), we have that $b'_1 \geq \sum_{1 \leq i \leq n} b_i$. This along with (2), Proposition 11, Proposition 15 and Proposition 22 gives $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^{n,S} \subseteq \text{coSEMI_HULL}_{a'_1, b'_1}^{1,S} \subseteq \text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^{n,S}$.

Case 2: For some j , $1 < j \leq n$, $i_k = i'_k$, for $1 \leq k < j$, but $i_j > i'_j$.

In this case

- (4) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leq i < j$.
- (5) $b_j < b'_j$, and
- (6) $a_j > a'_j$.

From (5) it follows that

- (7) $2 * b_j < b'_j$.

Now, for $1 \leq i < n$, since $a_i < a_{i+1}$, $b_i = h(-i_1 - a_i)$, we have $b_i > b_{i+1}$. Thus, $b_i > 2 * b_{i+1}$, for $1 \leq i < n$, by hypothesis about elements of S .

Thus, $\sum_{j \leq i \leq n} b_i \leq 2b_j$. This, along with (7) gives us that $b'_j > \sum_{j \leq i \leq n} b_i$. Thus using (6), Proposition 11, Proposition 15, and Proposition 22 we have $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^{n,S} \subseteq \text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^{n,S}$.

Claim follows from above cases. \square

We let $\Theta(X) = \bigcup_{(i_1, \dots, i_n) \in X} \text{coSEMI_HULL}_{\text{map}(i_1, \dots, i_n)}^n$.

Thus, it follows that $\Theta(L_{i_1, i_2, \dots, i_n}^Q) = \text{coSEMI_HULL}_{\text{map}(i_1, \dots, i_n)}^{n,S}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $\text{coSEMI_HULL}_{\text{map}(i_1, \dots, i_n)}^{n,S}$, then $\Psi(\alpha)$ converges to a grammar for $L_{i_1, i_2, \dots, i_n}^Q$.

It is now easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{\text{TxtEx}} \text{coSEMI_HULL}^{n,S}$. \blacksquare

16.4 Proofs of Theorem 13 and Theorem 14

The following proposition is an obvious corollary of the definition of *OP_HULLS*.

Proposition 26 *Suppose $S \subseteq S'$. Then $OP_HULL^{n,m,S} \leq^{\text{TextEx}} OP_HULL^{n,m,S'}$.*

Theorem 13 *Suppose S is rat^+ -covering. Suppose $n \geq 1$, $m \geq 1$. Let $Q = (q_1, \dots, q_n)$, where each $q_i = \text{INIT}$. Then, (a) $\mathcal{L}^Q \leq^{\text{TextEx}} OP_HULL^{n,m,S}$, and (b) $\mathcal{L}^Q \leq^{\text{TextEx}} OP_HULL^{m,n,S}$.*

PROOF. We only show part (a). Part (b) can be proved similarly.

The desired reduction works very similarly to the analogous reduction in Theorem 5: we just fix a *REV_SEMI_HULL* and try to reduce a language in \mathcal{L}^Q to the *SEMI_HULL* part of the *OP_HULL*; the slope of the *REV_SEMI_HULL* provides a starting point for learning the first slope b_1 , thus *HALF* being replaced with *INIT* in the first component of Q .

Now we proceed with the formal proof.

Without loss of generality (using Propositions 19 and 26) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $(c_1, d_1, \dots, c_m, d_m)$ be S -valid such that $\sum_{1 \leq i \leq m} d_i < 1/h(1)$ (note that there clearly exist such $c_1, d_1, \dots, c_m, d_m$).

Let $\text{map}(i_1, i_2, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$, where a_j, b_j , $1 \leq j \leq n$ are defined as follows. $a_1 = i_1 + 1$, $b_1 = h(-a_1)$. Suppose we have defined $a_1, b_1, \dots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1 \leq i \leq k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1} + 1)$ -th least element in A_{k+1} . Let $b_{k+1} = h(-a_{k+1})$.

Note

that,

for all $(i_1, i_2, \dots, i_n) \in N^n$, if $\text{map}(i_1, i_2, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$, then since $a_i < a_{i+1}$, by definition of b_i , we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. This along with requirement on S gives $b_i > 2b_{i+1}$. Thus, $\sum_{1 \leq i \leq n} b_i \leq 2b_1 \leq 2h(-a_1) \leq 2h(0) < h(1)$. Since, $\sum_{1 \leq i \leq m} d_i < 1/h(1)$, we immediately have that $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, for all (i_1, i_2, \dots, i_n) .

Claim 7 $(i_1, i_2, \dots, i_n) <_Q (i'_1, i'_2, \dots, i'_n)$ implies $OP_HULL_{\text{map}(i_1, \dots, i_n)}^{n,m,S} \subset OP_HULL_{\text{map}(i'_1, \dots, i'_n)}^{n,m,S}$.

PROOF. Suppose $(i_1, i_2, \dots, i_n) <_Q (i'_1, i'_2, \dots, i'_n)$. Suppose $\text{map}(i_1, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$ and $\text{map}(i'_1, \dots, i'_n) = (a'_1, b'_1, \dots, a'_n, b'_n; c_1, d_1, \dots, c_m, d_m)$.

Let j be the least number such that $i_k = i'_k$, for $1 \leq k < j$, and $i_j \neq i'_j$.

Thus,

(1) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leq i < j$.

(2) $b_j > b'_j$, and

(3) $a_j < a'_j$.

From (2) it follows that

(4) $b_j > 2 * b'_j$.

Now, for $1 \leq i < n$, since $a'_i < a'_{i+1}$, we have $b'_i = h(-a'_i) > h(-a'_{i+1}) = b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S .

Thus, $\sum_{j \leq i \leq n} b'_i \leq 2b'_j$. This, along with (4) gives us that $b_j > \sum_{j \leq i \leq n} b'_i$.

Thus using (3), Proposition 11 and Proposition 15 we have $SEMI_HULL_{a_1, b_1, \dots, a_n, b_n}^{n,S} \subset SEMI_HULL_{a'_1, b'_1, \dots, a'_n, b'_n}^{n,S}$.

Claim follows. \square

We now define $\Theta(X)$ for any finite set X as follows.

We let $\Theta(X) = \bigcup_{(i_1, \dots, i_n) \in X} OP_HULL_{\text{map}(i_1, \dots, i_n)}^{n,m}$.

Thus, it follows that $\Theta(L^Q_{i_1, i_2, \dots, i_n}) = OP_HULL_{\text{map}(i_1, \dots, i_n)}^{n,m,S}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $OP_HULL_{\text{map}(i_1, \dots, i_n)}^{n,m,S}$, then $\Psi(\alpha)$ converges to a grammar for $L_{i_1, i_2, \dots, i_n}^Q$.

It is now easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{\text{TextEx}} OP_HULL^{n,m,S}$. \blacksquare

Before proving Theorem 14 we need some propositions.

Definition 27 Suppose $(a_1, b_1, \dots, a_j, b_j)$ is valid. $REV_INTER(a_1, b_1, \dots, a_j, b_j) = \{(x, y) \mid (y, x) \in INTER(a_1, b_1, \dots, a_j, b_j)\}$.

Definition 28 Suppose $(a_1, b_1, \dots, a_j, b_j)$ and $(c_1, d_1, \dots, c_k, d_k)$ are valid. Then, $INT_OP_HULL(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k) = \bigcap \{OP_HULL_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n; c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m}^{n,m} \mid n \geq j \text{ and } m \geq k \text{ and } (a_1, b_1, \dots, a_n, b_n) \text{ and } (c_1, d_1, \dots, c_m, d_m) \text{ are valid and } \sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}\}$.

Proposition 27 Suppose $(a_1, b_1, \dots, a_j, b_j)$ and $(c_1, d_1, \dots, c_k, d_k)$ are valid, and $\sum_{1 \leq i \leq j} b_j < \frac{1}{\sum_{1 \leq i \leq k} d_i}$. Then,

(a) $\{(x, y) \mid x \leq \text{maxinter}(a_1, b_1, \dots, a_j, b_j), y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \dots, a_j, b_j)\})\} \subseteq INT_OP_HULL(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k)$.

(b) $\{(x, y) \mid y \leq \text{maxinter}(c_1, d_1, \dots, c_k, d_k), x = \min(\{x' \mid (x', y) \in REV_INTER(c_1, d_1, \dots, c_j, d_j)\})\} \subseteq INT_OP_HULL(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k)$.

PROOF. We show only part (a). Part (b) can be proved similarly. Suppose $(x_0, y_0) \in \{(x, y) \mid x \leq \text{maxinter}(a_1, b_1, \dots, a_j, b_j), y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \dots, a_j, b_j)\})\}$. Then, $1 + \sum_{1 \leq i \leq j} b_i(x_0 \div a_i) > y_0 \geq \sum_{1 \leq i \leq j} b_i(x_0 \div a_i)$. Thus, $1 + \sum_{1 \leq i \leq j} b_i * x_0 \geq y_0$. Hence,

$$(1) \sum_{1 \leq i \leq j} b_i \geq (y_0 - 1)/x_0.$$

Clearly, $(x_0, y_0) \in INTER(a_1, b_1, \dots, a_j, b_j)$.

Thus, $(x_0, y_0) \in SEMI_HULL_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n}^n$, for all valid $(a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n)$ (by definition of $INTER$).

Thus, if $(x_0, y_0) \notin INT_OP_HULL(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k)$, there must exist a valid $(c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m)$, and $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, such that $(x_0, y_0) \notin REV_SEMI_HULL_{c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m}^m$. But this would mean, $x_0 < \sum_{1 \leq i \leq m} d_i(y_0 \div c_i) \leq \sum_{1 \leq i \leq m} d_i(y_0 \div 1)$. Thus,

$$(2) \sum_{1 \leq i \leq m} d_i > x_0/(y_0 - 1).$$

From (1) and (2) we have

$$\sum_{1 \leq i \leq n} b_i > \frac{1}{\sum_{1 \leq i \leq m} d_i}. \text{ A contradiction to the hypothesis. } \blacksquare$$

Proposition 28

Suppose $(a_1, b_1, \dots, a_j, b_j)$, $(a'_1, b'_1, \dots, a'_j, b'_j)$, $(c_1, d_1, \dots, c_k, d_k)$, $(c'_1, d'_1, \dots, c'_k, d'_k)$, are valid. Suppose further that $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq k} d_i}$, and $\sum_{1 \leq i \leq j} b'_i < \frac{1}{\sum_{1 \leq i \leq k} d'_i}$.

If $INT_OP_HULL(a'_1, b'_1, \dots, a'_j, b'_j; c'_1, d'_1, \dots, c'_k, d'_k) \subseteq OP_HULL_{a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k}^{j,k}$, then, $INTER(a'_1, b'_1, \dots, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$ and $REV_INTER(c'_1, d'_1, \dots, c'_k, d'_k) \subseteq REV_SEMI_HULL_{c_1, d_1, \dots, c_k, d_k}^k$.

PROOF. Suppose

$$INT_OP_HULL(a'_1, b'_1, \dots, a'_j, b'_j; c'_1, d'_1, \dots, c'_k, d'_k) \subseteq OP_HULL_{a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k}^{j,k}$$

Thus (using Proposition 27) $\{(x, y) \mid x \leq \text{maxinter}(a'_1, b'_1, \dots, a'_j, b'_j), y = \min(\{y' \mid (x, y') \in INTER(a'_1, b'_1, \dots, a'_j, b'_j)\})\} \subseteq INT_OP_HULL(a'_1, b'_1, \dots, a'_j, b'_j; c'_1, d'_1, \dots, c'_k, d'_k) \subseteq$

$OP_HULL_{a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k} \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$. It follows that
 $INTER(a'_1, b'_1, \dots, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$.
 Similarly, it can be shown that
 $REV_INTER(c'_1, d'_1, \dots, c'_k, d'_k) \subseteq REV_SEMI_HULL_{c_1, d_1, \dots, c_k, d_k}^k$. ■

Corollary 11 *Suppose $1 \leq j \leq n$, $1 \leq k \leq n$. Let S be any \mathbf{rat}^+ -covering set.*

Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$, $(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k)$ are S -valid, and $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq k} d_i}$.

Then, there exist only finitely many (a'_j, b'_j, c'_k, d'_k) such that

(i) $b'_j + \sum_{1 \leq i < j} b_i < \frac{1}{d'_k + \sum_{1 \leq i < k} d_i}$, and

(ii) $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k)$ are S -valid, and

(iii) $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$, and

$REV_INTER(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV_SEMI_HULL_{c_1, d_1, \dots, c_k, d_k}^k$.

Moreover, canonical index for the finite set of (a'_j, b'_j, c'_k, d'_k) satisfying above three conditions can be obtained effectively from $a_1, b_1, \dots, a_j, b_j, c_1, d_1, \dots, c_k, d_k$.

Furthermore, for any (a'_j, b'_j, c'_k, d'_k) satisfying the above three conditions, $a'_j \leq a_j$, $c'_k \leq c_k$, and if $a'_j = a_j$ then $b'_j \geq b_j$, and if $c'_k = c_k$, then $d'_k \geq d_k$.

PROOF. Suppose $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$, and $REV_INTER(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV_SEMI_HULL_{c_1, d_1, \dots, c_k, d_k}^k$. By Corollary 5 and Proposition 14 it follows that $a'_j \leq a_j$, $c'_k \leq c_k$, and there exists only finitely many (a'_j, b'_j) such that $b'_j \leq b_j$ and $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$, and only finitely many (c'_k, d'_k) such that $d'_k \leq d_k$ and $REV_INTER(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV_SEMI_HULL_{c_1, d_1, \dots, c_k, d_k}^k$, and these (a'_j, b'_j) , (c'_k, d'_k) can be obtained effectively from $(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k)$. Let $B'_j = \min(\{b_j\} \cup \{b'_j \mid b'_j \leq b_j \wedge INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j\})$. Let $D'_k = \min(\{d_k\} \cup \{d'_k \mid d'_k \leq d_k \wedge REV_INTER(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV_SEMI_HULL_{c_1, d_1, \dots, c_k, d_k}^k\})$.

Note that clause (i) in the corollary implies that $b'_j < \frac{1}{d'_k}$ and $d'_k < \frac{1}{b'_j}$.

It follows that, for any (a'_j, b'_j, c'_k, d'_k) to satisfy the hypothesis of the corollary we must have $0 \leq a'_j \leq a_j$, $0 \leq c'_k \leq c_k$, $B'_j \leq b'_j \leq 1/D'_k$, and $D'_k \leq d'_k \leq 1/B'_j$. Thus, there exist only finitely many (a'_j, b'_j, c'_k, d'_k) which can satisfy clauses (i), (ii) and (iii) of the corollary. Moreover since for any (a'_j, b'_j, c'_k, d'_k) it is effectively testable whether clauses (i), (ii) and (iii) of the corollary are satisfiable, we can find the canonical index for the set of (a'_j, b'_j, c'_k, d'_k) satisfying the clauses (i), (ii) and (iii) of the corollary effectively from $a_1, b_1, \dots, a_j, b_j, c_1, d_1, \dots, c_k, d_k$.

Furthermore clause of the corollary follows using Corollary 5. ■

Proposition 29 *Let S be any \mathbf{rat}^+ -covering set. Then, there exists a recursive function $Icode$ with domain $VALID_S \times VALID_S$, and range $\subseteq N$ such that following is satisfied.*

Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j)$, $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k)$, $(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k)$ are S -valid, and $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq k} d_i}$ and $b'_j + \sum_{1 \leq i < j} b_i < \frac{1}{d'_k + \sum_{1 \leq i < k} d_i}$. Then

(A) *If $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI_HULL_{a_1, b_1, \dots, a_j, b_j}^j$, and $REV_INTER(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV_SEMI_HULL_{c_1, d_1, \dots, c_k, d_k}^k$ then $Icode(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \leq Icode(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k)$.*

(B) If $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$, then
 $\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \neq$
 $\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k)$.

(C) $\{\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c''_k, d''_k) \mid (a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j) \in$
 $\text{VALID}_S, (c_1, d_1, \dots, c_{k-1}, d_{k-1}, c''_k, d''_k) \in \text{VALID}_S, \text{ and } b''_j + \sum_{1 \leq i < j} b_i < \frac{1}{d''_k + \sum_{1 \leq i < k} d_i}\} = N$.

Proof can be done along similar lines as of Proposition 18.

Now we show the upper bound for OP_HULLs .

Theorem 14 Suppose S is rat^+ -covering. Suppose $n \geq m \geq 1$. Let $Q = (q_1, \dots, q_n)$, where each $q_i = \text{INIT}$. Then, (a) $OP_HULL^{n,m,S} \leq^{\text{TxtEx}} \mathcal{L}^Q$, and (b) $OP_HULL^{m,n,S} \leq^{\text{TxtEx}} \mathcal{L}^Q$.

PROOF. We show only part (a). Part (b) can be done similarly.

Intuitively, we use INIT type strategy to learn every set of parameters (a_i, b_i, c_i, d_i) . This is possible based on Proposition 29 above.

Now we proceed with the formal proof.

Let h be a recursive bijection from Z to S such that $h(i) < h(i+1)$, for $i \in Z$. Let Icode be as in Proposition 29. For $(a_1, b_1, \dots, a_n, b_n)$ and $(c_1, d_1, \dots, c_m, d_m)$ in VALID_S , such that $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, we let $\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m) =$
 $(\text{Icode}(a_1, b_1; c_1, d_1), \text{Icode}(a_1, b_1, a_2, b_2; c_1, d_1, c_2, d_2), \dots,$
 $\text{Icode}(a_1, b_1, \dots, a_m, b_m; c_1, d_1, \dots, c_m, d_m), \text{Icode}(a_1, b_1, \dots, a_{m+1}, b_{m+1}; c_1, d_1, \dots, c_m, d_m), \dots$
 $\text{Icode}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m))$

Claim 8 Suppose $(a_1, b_1, \dots, a_n, b_n)$ and $(a'_1, b'_1, \dots, a'_n, b'_n)$, $(c_1, d_1, \dots, c_m, d_m)$ and $(c'_1, d'_1, \dots, c'_m, d'_m)$, are S -valid, and $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, and $\sum_{1 \leq i \leq n} b'_i < \frac{1}{\sum_{1 \leq i \leq m} d'_i}$.

(A) Suppose $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$. Then, for the least j such that $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$, $\text{INTER}(a_1, b_1, \dots, a_j, b_j) \not\subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$ or $\text{REV_INTER}(c_1, d_1, \dots, c_{\min(j,m)}, d_{\min(j,m)}) \not\subseteq \text{REV_SEMI_HULL}_{c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}}^j$.

(B) Suppose $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$. Then, $\text{INTER}(a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n) \not\subseteq \text{SEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$ or $\text{REV_INTER}(c_1, d_1, \dots, c_m, d_m) \not\subseteq \text{REV_SEMI_HULL}_{c'_1, d'_1, \dots, c'_m, d'_m}^m$.

(C) Suppose $OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n,m} \subset OP_HULL_{a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m}^{n,m}$. Then $\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m) <_Q \text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m)$.

(D) Suppose $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$. Then, $\{(x, y) \in N^2 \mid x \leq \text{maxinter}(a_1, b_1, \dots, a_n, b_n) \text{ and } y = \min(\{y' \mid (x, y') \in \text{INTER}(a_1, b_1, \dots, a_n, b_n)\}) \text{ or } y \leq \text{maxinter}(c_1, d_1, \dots, c_m, d_m) \text{ and } x = \min(\{x' \mid (x', y) \in \text{REV_INTER}(c_1, d_1, \dots, c_m, d_m)\})\} \not\subseteq OP_HULL_{a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m}^{n,m}$.

PROOF. (A)

Let j be least number such that $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$. Note that, for $i < j$, we must have $\text{Icode}(a_1, b_1, \dots, a_i, b_i; c_1, d_1, \dots, c_{\min(i,m)}, d_{\min(i,m)}) = \text{Icode}(a'_1, b'_1, \dots, a'_i, b'_i; c'_1, d'_1, \dots, c'_{\min(i,m)}, d'_{\min(i,m)})$. If $\text{INTER}(a_1, b_1, \dots, a_j, b_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j}^j$ and $\text{REV_INTER}(c_1, d_1, \dots, c_{\min(j,m)}, d_{\min(j,m)}) \subseteq$

$REV_SEMI_HULL_{c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}}^j$,

then, by Proposition 29 we would have $Icode(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_{\min(j,m)}, d_{\min(j,m)}) < Icode(a'_1, b'_1, \dots, a'_j, b'_j; c'_1, d'_1, \dots, c'_{\min(j,m)}, d'_{\min(j,m)})$. Thus,

$map(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_{\min(j,m)}, d_{\min(j,m)}) <_Q map(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_{\min(j,m)}, d'_{\min(j,m)})$, a contradiction to the hypothesis.

(B)

Follows from the fact that $INTER(a_1, b_1, \dots, a_j, b_j) \subseteq INTER(a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n)$,
 $REV_INTER(c_1, d_1, \dots, c_{\min(j,m)}, d_{\min(j,m)}) \subseteq REV_INTER(c_1, d_1, \dots, c_{\min(j,m)}, d_{\min(j,m)}, \dots, c_m, d_m)$,

$SEMI_HULL_{a'_1, b'_1, \dots, a'_n, b'_n}^n \subseteq SEMI_HULL_{a'_1, b'_1, \dots, a'_j, b'_j}^j$ and $REV_SEMI_HULL_{c'_1, d'_1, \dots, c'_m, d'_m}^m \subseteq REV_SEMI_HULL_{c'_1, d'_1, \dots, c'_{\min(j,m)}, d'_{\min(j,m)}}^j$ and part (A).

(C)

If

$OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n,m} \subset OP_HULL_{a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m}^{n,m}$, then by Proposition 28,
 $INTER(a_1, b_1, \dots, a_n, b_n) \subseteq SEMI_HULL_{a'_1, b'_1, \dots, a'_n, b'_n}^n$ and $REV_INTER(c_1, d_1, \dots, c_m, d_m) \subseteq REV_SEMI_HULL_{c'_1, d'_1, \dots, c'_m, d'_m}^m$. Now part (C) follows from part (B).

(D) Follows from (B) and definition of $OP_HULL^{n,m}$. \square

We now continue with the proof of the theorem. The aim is to construct Θ which maps $OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n,m,S}$ to $L_{map(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L_{map(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$ to a grammar sequence converging to a grammar for $OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n,m}$ would be trivial. We thus just define Θ .

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $Prop(X, a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$ be true iff following two properties are satisfied.

(A) $(a_1, b_1, \dots, a_n, b_n) \in VALID_S$, $(c_1, d_1, \dots, c_m, d_m) \in VALID_S$, and $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$.

(B) For all $(a'_1, b'_1, \dots, a'_n, b'_n), (c'_1, d'_1, \dots, c'_m, d'_m) \in VALID_S$, $\sum_{1 \leq i \leq n} b'_i < \frac{1}{\sum_{1 \leq i \leq m} d'_i}$, such that $map(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m) <_Q map(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$, $X \not\subseteq OP_HULL_{a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m}^{n,m}$.

Note that condition (B) above is equivalent to

(B') For all j , $1 \leq j \leq n$, for all $a'_j, c'_{\min(j,m)} \in N, b'_j, d'_{\min(j,m)} \in S$, such that $b'_j + \sum_{1 \leq i < j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1 \leq i < \min(j,m)} d'_i}$,

if $Icode(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) < Icode(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, then

$[X \not\subseteq OP_HULL_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}}^{j, \min(j,m)}]$.

Note that whether $X, a_1, b_1, \dots, a_n, b_n, c_1, d_1, \dots, c_m, d_m$, satisfy (A) and (B'), for all j , $1 \leq j \leq n$, is effectively testable.

Thus, for finite $X \subseteq N^2$, let

$\Theta(X) = \cup \{L_{map(a_1, b_1, \dots, a_n, b_n)}^Q \mid Prop(X, a_1, b_1, \dots, a_n, b_n)\}$.

For infinite X' , $\Theta(X') = \cup_{X \subseteq X', \text{card}(X) < \infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n,m}$,

$\Theta(X) \subseteq L_{map(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$ (due to clause (B) in definition of $Prop$ above, and the fact that for any valid I and I' , $map(I) <_Q map(I')$, implies $L_{map(I)}^Q \subseteq L_{map(I')}^Q$), and

(2) for any finite set X such that $\{(x, y) \in N^2 \mid x \leq \maxinter(a_1, b_1, \dots, a_n, b_n) \text{ and } y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \dots, a_n, b_n)\}) \text{ or } y \leq \maxinter(c_1, d_1, \dots, c_m, d_m) \text{ and } x = \min(\{x' \mid (x', y) \in REV_INTER(c_1, d_1, \dots, c_m, d_m)\})\} \subseteq X \subseteq OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m}$,

$$\Theta(X) \supseteq L_{\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q.$$

(By Claim 8(D), and definition of *Prop* and Θ).

Thus, we have that
 $\Theta(OP_HULL_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m}) = L_{\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$ ■

16.5 Proofs of Theorem 15 and Theorem 16

Proposition 30 *Suppose $S \subseteq S'$. Then $coOP_HULL^{n, m, S} \leq^{\text{TxtEx}} coOP_HULL^{n, m, S'}$.*

The following theorem gives the lower bound for *coOP_HULLs*.

Theorem 15 *Suppose S is rat^+ -covering. Suppose $n \geq 1$, $m \geq 1$. Let $Q = (q_1, \dots, q_n)$, where each $q_i = COINIT$. Then, (a) $\mathcal{L}^Q \leq^{\text{TxtEx}} coOP_HULL^{n, m, S}$, and (b) $\mathcal{L}^Q \leq^{\text{TxtEx}} coOP_HULL^{m, n, S}$.*

PROOF. We only show part (a). Part (b) can be proved similarly. The proof is very similar to the proof of lower bound for *OP_HULLs* with *INITs* being replaced by *COINITs*. Without loss of generality (using Propositions 19 and 30) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $(c_1, d_1, \dots, c_m, d_m)$ be S -valid such that $\sum_{1 \leq i \leq m} d_i < 1/h(1)$ (note that there clearly exist such $c_1, d_1, \dots, c_m, d_m$).

Let $\text{map}(i_1, i_2, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$, where a_j, b_j , $1 \leq j \leq n$ are defined as follows. $a_1 = i_1 + 1$, $b_1 = h(-a_1)$. Suppose we have defined $a_1, b_1, \dots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1 \leq i \leq k} b_i * (x \div a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1} + 1)$ -th least element in A_{k+1} . Let $b_{k+1} = h(-a_{k+1})$.

Note

that,

for all $(i_1, i_2, \dots, i_n) \in N^n$, if $\text{map}(i_1, i_2, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$, then since $a_i < a_{i+1}$, by definition of b_i , we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. This along with requirement on S gives $b_i > 2b_{i+1}$. Thus, $\sum_{1 \leq i \leq n} b_i \leq 2b_1 \leq 2h(-a_1) \leq 2h(0) < h(1)$. Since, $\sum_{1 \leq i \leq m} d_i < 1/h(1)$, we immediately have that $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, for all (i_1, i_2, \dots, i_n) .

Claim 9 $(i_1, i_2, \dots, i_n) <_Q (i'_1, i'_2, \dots, i'_n)$ *implies*
 $coOP_HULL_{\text{map}(i_1, \dots, i_n)}^{n, m, S} \subset coOP_HULL_{\text{map}(i'_1, \dots, i'_n)}^{n, m, S}$.

PROOF. Suppose $(i_1, i_2, \dots, i_n) <_Q (i'_1, i'_2, \dots, i'_n)$. Suppose $\text{map}(i_1, \dots, i_n) = (a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$ and $\text{map}(i'_1, \dots, i'_n) = (a'_1, b'_1, \dots, a'_n, b'_n; c_1, d_1, \dots, c_m, d_m)$.

Let j be the least number such that $i_k = i'_k$, for $1 \leq k < j$, and $i_j \neq i'_j$.

Thus,

(1) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leq i < j$.

(2) $b_j < b'_j$, and

(3) $a_j > a'_j$.

From (2) it follows that

(4) $2b_j < b'_j$.

Now, for $1 \leq i < n$, since $a_i < a_{i+1}$, we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. Thus, $b_i > 2 * b_{i+1}$, for $1 \leq i < n$ by hypothesis about elements of S .

Thus, $\sum_{j \leq i \leq n} b_i \leq 2b_j$. This, along with (4) gives us that $b'_j > \sum_{j \leq i \leq n} b_i$. Thus using (3), Proposition 11, Proposition 15, and Proposition 22, we have $\text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^{n, S} \subset \text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_j, b'_j}^{j, S} \subseteq \text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^{n, S}$.

Claim follows. \square

We now define $\Theta(X)$ for any finite set X as follows.

We let $\Theta(X) = \bigcup_{(i_1, i_2, \dots, i_n) \in X} \text{coOP_HULL}_{\text{map}(i_1, \dots, i_n)}^{n, m}$.

Thus, it follows that $\Theta(L_{i_1, i_2, \dots, i_n}^Q) = \text{coOP_HULL}_{\text{map}(i_1, \dots, i_n)}^{n, m, S}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $\text{coOP_HULL}_{\text{map}(i_1, \dots, i_n)}^{n, m, S}$, then $\Psi(\alpha)$ converges to a grammar for $L_{i_1, i_2, \dots, i_n}^Q$.

It is now easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{\text{TextEx}} \text{coOP_HULL}^{n, m, S}$. \blacksquare

Before proving Theorem 16 we need some propositions.

Definition 29 Suppose $(a_1, b_1, \dots, a_j, b_j)$ is valid. $\text{REV_coINTER}(a_1, b_1, \dots, a_j, b_j) = \{(x, y) \mid (y, x) \in \text{coINTER}(a_1, b_1, \dots, a_j, b_j)\}$.

Definition 30 Suppose $(a_1, b_1, \dots, a_j, b_j)$ and $(c_1, d_1, \dots, c_k, d_k)$ are valid. Then, $\text{coINT_OP_HULL}(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k) = \bigcup \{ \text{coOP_HULL}_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n; c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m}^{n, m} \mid n \geq j \text{ and } m \geq k \text{ and } (a_1, b_1, \dots, a_n, b_n) \text{ and } (c_1, d_1, \dots, c_m, d_m) \text{ are valid and } \sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i} \} = N^2 - \text{INT_OP_HULL}(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k)$.

Proposition 31 Suppose $(a_1, b_1, \dots, a_j, b_j)$ and $(c_1, d_1, \dots, c_k, d_k)$ are valid. Suppose $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq k} d_i}$. Let $(x, y) \in N^2$ be such that $1 + \sum_{1 \leq i < j} b_i(x \div a_i) \geq y \geq \sum_{1 \leq i < j} b_i(x \div a_i)$. Then, if $(x, y) \in \text{coINT_OP_HULL}(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k)$, then $(x, y) \in \text{coINTER}(a_1, b_1, \dots, a_j, b_j)$.

PROOF. Suppose $(x, y) \in \text{coINT_OP_HULL}(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k)$. Then, $(x, y) \in \bigcup \{ \text{coOP_HULL}_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n; c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m}^{n, m} \mid n \geq j \text{ and } m \geq k \text{ and } (a_1, b_1, \dots, a_n, b_n) \text{ and } (c_1, d_1, \dots, c_m, d_m) \text{ are valid and } \sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i} \}$. Thus $(x, y) \in \bigcup \{ \text{coSEMI_HULL}_{a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n}^n \mid n \geq j \text{ and } (a_1, b_1, \dots, a_n, b_n) \text{ is valid} \} \cup \bigcup \{ \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m}^m \mid m \geq k \text{ and } (c_1, d_1, \dots, c_m, d_m) \text{ is valid and } \sum_{1 \leq i \leq m} d_i < \frac{1}{\sum_{1 \leq i < j} b_i} \}$. Thus, $(x, y) \in \text{coINTER}(a_1, b_1, \dots, a_j, b_j) \cup \bigcup \{ \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m}^m \mid m \geq k \text{ and } (c_1, d_1, \dots, c_m, d_m) \text{ is valid and } \sum_{1 \leq i \leq m} d_i < \frac{1}{\sum_{1 \leq i < j} b_i} \}$.

We claim that $(x, y) \notin \bigcup \{ \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m}^m \mid m \geq k \text{ and } (c_1, d_1, \dots, c_m, d_m) \text{ is valid and } \sum_{1 \leq i \leq m} d_i < \frac{1}{\sum_{1 \leq i < j} b_i} \}$. This would prove the proposition. Suppose by way of contradiction that $(x, y) \in \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_k, d_k, \dots, c_m, d_m}^m$, where $m \geq k$, $(c_1, d_1, \dots, c_m, d_m)$ is valid and $\sum_{1 \leq i \leq m} d_i < \frac{1}{\sum_{1 \leq i < j} b_i}$. But, then $(x, y) \in \text{REV_coSEMI_HULL}_{c_1, \sum_{1 \leq i \leq m} d_i}^m$. Thus, $x < \sum_{1 \leq i \leq m} d_i(y \div 1)$. If $y \leq 1$, then clearly, above cannot happen. So assume $y > 1$. Thus, $x < \sum_{1 \leq i \leq m} d_i(y - 1)$. Hence

$$(1) \ x/(y - 1) < \sum_{1 \leq i \leq m} d_i.$$

However, $y \leq 1 + \sum_{1 \leq i < j} b_i(x \div a_i)$. Thus, $y - 1 \leq \sum_{1 \leq i < j} b_i(x \div a_i)$. Thus, $y - 1 \leq \sum_{1 \leq i < j} b_i x$. Thus,

$$(2) \ (y - 1)/x \leq \sum_{1 \leq i < j} b_i.$$

Multiplying (1) and (2) we have $1 \leq [\sum_{1 \leq i < j} b_i] * [\sum_{1 \leq i \leq m} d_i]$. But then, $\sum_{1 \leq i \leq n} b_i \geq \sum_{1 \leq i < j} b_i \geq 1/[\sum_{1 \leq i \leq m} d_i]$. A contradiction. \blacksquare

Similarly, one can show

Proposition 32 Suppose $(a_1, b_1, \dots, a_j, b_j)$ and $(c_1, d_1, \dots, c_k, d_k)$ are valid. Suppose $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq k} d_i}$. Let $(x, y) \in N^2$ be such that $1 + \sum_{1 \leq i < j} d_i(y \div c_i) \geq x \geq \sum_{1 \leq i < j} d_i(y \div c_i)$. Then, if $(x, y) \in \text{coINT_OP_HULL}(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k)$, then $(x, y) \in \text{REV_coINTER}(c_1, d_1, \dots, c_k, d_k)$.

Proposition 33

Suppose $(a_1, b_1, \dots, a_j, b_j)$, $(a'_1, b'_1, \dots, a'_j, b'_j)$, $(c_1, d_1, \dots, c_k, d_k)$, $(c'_1, d'_1, \dots, c'_k, d'_k)$, are valid. Suppose further that $\sum_{1 \leq i \leq j} b_i < \frac{1}{\sum_{1 \leq i \leq k} d_i}$, and $\sum_{1 \leq i \leq j} b'_i < \frac{1}{\sum_{1 \leq i \leq k} d'_i}$.

If $\text{coINT_OP_HULL}(a'_1, b'_1, \dots, a'_j, b'_j; c'_1, d'_1, \dots, c'_k, d'_k) \supseteq \text{coOP_HULL}_{a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_k, d_k}^{j, k}$, then, $\text{coINTER}(a'_1, b'_1, \dots, a'_j, b'_j) \supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_j, b_j}^j$ and $\text{REV_coINTER}(c'_1, d'_1, \dots, c'_k, d'_k) \supseteq \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_k, d_k}^k$.

PROOF. Follows from Proposition 28. \blacksquare

Theorem 16 Suppose $n \geq m \geq 1$. Let $Q = (q_1, \dots, q_n)$, where each $q_i = \text{COINIT}$. Then, (a) $\text{coOP_HULL}^{n, m, S} \leq_{\text{TextEx}} \mathcal{L}^Q$, and (b) $\text{coOP_HULL}^{m, n, S} \leq_{\text{TextEx}} \mathcal{L}^Q$.

PROOF. We show only part (a). Part (b) can be done similarly.

Intuitively, we use *COINIT* type strategy to learn every set of parameters (a_i, b_i, c_i, d_i) . This is possible based on Proposition 29 above using a method similar to that used in Theorem 14, though technical details become more complicated.

Let h be a recursive bijection from Z to S such that $h(i) < h(i + 1)$, for $i \in Z$. Let *Icode* be as in Proposition 29. For $(a_1, b_1, \dots, a_n, b_n)$ and $(c_1, d_1, \dots, c_m, d_m)$ in *VALID_S*, such that $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$, we let $\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m) = (\text{Icode}(a_1, b_1; c_1, d_1), \text{Icode}(a_1, b_1, a_2, b_2; c_1, d_1, c_2, d_2), \dots, \text{Icode}(a_1, b_1, \dots, a_m, b_m; c_1, d_1, \dots, c_m, d_m), \text{Icode}(a_1, b_1, \dots, a_{m+1}, b_{m+1}; c_1, d_1, \dots, c_m, d_m), \dots, \text{Icode}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m))$

Claim 10 Suppose $(a_1, b_1, \dots, a_j, b_j)$, $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \dots, c_k, d_k)$, $(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k)$, are *S*-valid, where $\sum_{1 \leq i \leq j} b_j < \frac{1}{\sum_{1 \leq i \leq k} d_k}$, and $b'_j + \sum_{1 \leq i < j} b_j < \frac{1}{d'_k + \sum_{1 \leq i < k} d_k}$. Suppose $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$.

Suppose $\text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$ and $\text{REV_coINTER}(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \supseteq \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k}^k$, then $\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) < \text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k)$.

PROOF. By hypothesis, we have $\text{INTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq \text{SEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$ and $\text{REV_INTER}(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq \text{REV_SEMI_HULL}_{c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k}^k$. Thus, we have $\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) < \text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k)$ (by Proposition 29; for getting $<$, use the fact that $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$). \square

Claim 11 Suppose $(a_1, b_1, \dots, a_n, b_n)$, $(c_1, d_1, \dots, c_m, d_m)$ are S -valid, and $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$.

(A) Suppose $1 \leq j \leq n$, $1 \leq k \leq m$. Suppose $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ and $(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k)$ are S -valid, $b'_j + \sum_{1 \leq i < j} b_i < \frac{1}{d'_k + \sum_{1 \leq i < k} d_i}$.

Suppose $\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) > \text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k)$. Then

$[\text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \not\supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$
or $\text{REV_coINTER}(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \not\supseteq \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k}^k]$.
Thus, $\text{coINT_OP_HULL}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \not\supseteq \text{coOP_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k}^{j,k}$.

(B) Suppose $(a'_1, b'_1, \dots, a'_n, b'_n)$ and $(c'_1, d'_1, \dots, c'_m, d'_m)$ are S -valid, $\sum_{1 \leq i \leq n} b'_i < \frac{1}{\sum_{1 \leq i \leq m} d'_i}$.

Suppose $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$.

Then, for the least j such that $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$, $\text{coINTER}(a_1, b_1, \dots, a'_j, b'_j) \not\supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$, or

$\text{REV_coINTER}(c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) \not\supseteq \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}}^j$.

(C) Suppose $(a'_1, b'_1, \dots, a'_n, b'_n)$ and $(c'_1, d'_1, \dots, c'_m, d'_m)$ are S -valid, $\sum_{1 \leq i \leq n} b'_i < \frac{1}{\sum_{1 \leq i \leq m} d'_i}$.

Suppose $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$. Then, $\text{coINTER}(a'_1, b'_1, \dots, a'_j, b'_j, \dots, a'_n, b'_n) \not\supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_n, b_n}^n$ or

$\text{REV_coINTER}(c'_1, d'_1, \dots, c'_m, d'_m) \not\supseteq \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_m, d_m}^m$.

(D) Suppose $(a'_1, b'_1, \dots, a'_n, b'_n)$ and $(c'_1, d'_1, \dots, c'_m, d'_m)$ are S -valid, $\sum_{1 \leq i \leq n} b'_i < \frac{1}{\sum_{1 \leq i \leq m} d'_i}$.

Suppose $\text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n,m} \subset \text{coOP_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m}^{n,m}$. Then $\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m) <_Q \text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m)$.

(E) For $1 \leq j \leq n$, there exists a finite $X_j \subseteq \text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n,m}$ such that,

for all S -valid $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$, such that $b'_j + \sum_{1 \leq i < j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1 \leq i < \min(j,m)} d_i}$, if

$\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$

$> \text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, then,

$\text{coINT_OP_HULL}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) \not\supseteq X_j$.

(F) There

exists a finite $X \subseteq \text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n,m}$ such that, for all j , $1 \leq j \leq n$, for S -valid $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$, such

that $b'_j + \sum_{1 \leq i < j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1 \leq i < \min(j,m)} d_i}$, if

$\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$

$> \text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, then,

$\text{coINT_OP_HULL}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) \not\supseteq X$.

PROOF.

(A) Suppose the hypothesis. Then, it follows

from Claim 10 that $[\text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \not\supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$

or $\text{REV_coINTER}(c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \not\supseteq \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k}^k]$.

Thus, by Proposition 33, we have $\text{coINT_OP_HULL}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c'_k, d'_k) \not\subseteq \text{coOP_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{k-1}, d_{k-1}, c_k, d_k}^{j, k}$.

(B) Let j be least number such that $(a_j, b_j, c_{\min(j, m)}, d_{\min(j, m)}) \neq (a'_j, b'_j, c'_{\min(j, m)}, d'_{\min(j, m)})$. Note that, for $i < j$, we must have $\text{Icode}(a_1, b_1, \dots, a_i, b_i; c_1, d_1, \dots, c_{\min(i, m)}, d_{\min(i, m)}) = \text{Icode}(a'_1, b'_1, \dots, a'_i, b'_i; c'_1, d'_1, \dots, c'_{\min(i, m)}, d'_{\min(i, m)})$. If $\text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq \text{coSEMI_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j}^j$ and $\text{REV_coINTER}(c_1, d_1, \dots, c_{\min(j, m)-1}, d_{\min(j, m)-1}, c'_{\min(j, m)}, d'_{\min(j, m)}) \supseteq \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_{\min(j, m)-1}, d_{\min(j, m)-1}, c_{\min(j, m)}, d_{\min(j, m)}}^j$, then, by part (A) we would have $\text{Icode}(a_1, b_1, \dots, a_j, b_j; c_1, d_1, \dots, c_{\min(j, m)}, d_{\min(j, m)}) > \text{Icode}(B, a'_1, b'_1, \dots, a'_j, b'_j; c'_1, d'_1, \dots, c'_{\min(j, m)}, d'_{\min(j, m)})$. Thus, $\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_{\min(j, m)}, d_{\min(j, m)}) < \text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_{\min(j, m)}, d'_{\min(j, m)})$, a contradiction to the hypothesis.

(C) Follows from the fact that, for $1 \leq j \leq n$, $\text{coINTER}(a_1, b_1, \dots, a_j, b_j) \supseteq \text{coINTER}(a_1, b_1, \dots, a_j, b_j, \dots, a_n, b_n)$, $\text{REV_coINTER}(c_1, d_1, \dots, c_{\min(j, m)}, d_{\min(j, m)}) \supseteq \text{REV_coINTER}(c_1, d_1, \dots, c_{\min(j, m)}, d_{\min(j, m)}, \dots, c_m, d_m)$,

$\text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n}^n \supseteq \text{coSEMI_HULL}_{a'_1, b'_1, \dots, a'_j, b'_j}^j$ and $\text{REV_coSEMI_HULL}_{c'_1, d'_1, \dots, c'_m, d'_m}^m \supseteq \text{REV_coSEMI_HULL}_{c'_1, d'_1, \dots, c'_{\min(j, m)}, d'_{\min(j, m)}}^j$ and part (B).

(D) If $\text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m} \subset \text{coOP_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m}^{n, m}$, then $\text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m} \subset \text{coINT_OP_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m}^{n, m}$. Thus, by Proposition 33, $\text{coSEMI_HULL}(a_1, b_1, \dots, a_n, b_n) \subseteq \text{coINTER}_{a'_1, b'_1, \dots, a'_n, b'_n}^n$ and $\text{REV_coSEMI_HULL}(c_1, d_1, \dots, c_m, d_m) \subseteq \text{REV_coINTER}_{c'_1, d'_1, \dots, c'_m, d'_m}^m$. Now part (D) follows from part (C).

(E) Let $(x_1, y_1) \in \text{coSEMI_HULL}_{a_1, b_1, \dots, a_j, b_j}^j$ be such that $1 + \sum_{1 \leq i < j} b_i(x_1 \div a_i) \geq y_1 > \sum_{1 \leq i < j} b_i(x_1 \div a_i)$. Note that there exists such (x_1, y_1) . By Proposition 31 if $(x_1, y_1) \in \text{coINT_OP_HULL}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \dots, c_{\min(j, m)-1}, d_{\min(j, m)-1}, c''_{\min(j, m)}, d''_{\min(j, m)})$, then $(x_1, y_1) \in \text{coINTER}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j)$. Thus, by Proposition 24, there exists a $B' \in \mathbf{rat}^+$ such that if $(x_1, y_1) \in \text{coINT_OP_HULL}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \dots, c_{\min(j, m)-1}, d_{\min(j, m)-1}, c''_{\min(j, m)}, d''_{\min(j, m)})$, then $a''_j < x_1$ and $b''_j > B'$.

Similarly, let $(x_2, y_2) \in \text{REV_coSEMI_HULL}_{c_1, d_1, \dots, c_{\min(j, m)}, d_{\min(j, m)}}^{\min(j, m)}$ be such that $1 + \sum_{1 \leq i < \min(j, m)} b_i(y_2 \div a_i) \geq x_2 > \sum_{1 \leq i < \min(j, m)} b_i(y_2 \div a_i)$. Note that there exists such (x_2, y_2) . By Proposition 32 if $(x_2, y_2) \in \text{coINT_OP_HULL}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \dots, c_{\min(j, m)-1}, d_{\min(j, m)-1}, c''_{\min(j, m)}, d''_{\min(j, m)})$, then $(x_2, y_2) \in \text{REV_coINTER}(c_1, d_1, \dots, c_{\min(j, m)-1}, d_{\min(j, m)-1}, c''_{\min(j, m)}, d''_{\min(j, m)})$. Thus, by Proposition 24, there exists a $B'' \in \mathbf{rat}^+$ such that if $(x_2, y_2) \in \text{coINT_OP_HULL}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \dots, c_{\min(j, m)-1}, d_{\min(j, m)-1}, c''_{\min(j, m)}, d''_{\min(j, m)})$, then $c''_{\min(j, m)} < y_2$ and $d''_{\min(j, m)} > B''$.

Thus, for (x_1, y_1) and (x_2, y_2) to be in $\text{coINT_OP_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j, m)-1}, d_{\min(j, m)-1}, c'_{\min(j, m)}, d'_{\min(j, m)}}^{j, \min(j, m)}$, where $b'_j + \sum_{1 \leq i < j} b_i < \frac{1}{d'_{\min(j, m)} + \sum_{1 \leq i < \min(j, m)} d_i}$, we must have $a'_j \leq x_1$, $b'_j \geq B'$, $c'_{\min(j, m)} \leq y_2$, $d'_{\min(j, m)} \geq B''$, and $b'_j \leq 1/B''$ and $d'_{\min(j, m)} \leq 1/B'$. Since there are only finitely many such $(a'_j, b'_j, c'_{\min(j, m)}, d'_{\min(j, m)})$ with $b'_j, d'_{\min(j, m)} \in S$, we have part (E) using part (A).

(F) Let $X = \bigcup_{1 \leq j \leq n} X_j$ where X_j is as in part (E). Now, if $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n)$, there exists a j , $1 \leq j \leq n$ such for $1 \leq i < j$, $a_i = a'_i$ and $b_i = b'_i$ and $\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) > \text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$.

Part (F) now follows from part (E), and definition of X . \square

We now continue with the proof of the theorem. The aim is to construct Θ which maps $\text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m, S}$ to $L_{\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L_{\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$ to a grammar sequence converging to a grammar for $\text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m}$ would be trivial. We thus just define Θ .

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $\text{Prop}(X, a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$ be true iff following two properties are satisfied.

(A) $(a_1, b_1, \dots, a_n, b_n) \in \text{VALID}_S$, $(c_1, d_1, \dots, c_n, d_n) \in \text{VALID}_S$, and $\sum_{1 \leq i \leq n} b_i < \frac{1}{\sum_{1 \leq i \leq m} d_i}$.

(B) For all j , $1 \leq j \leq n$,

for all $a'_j, c'_j \in N, b'_j, d'_{\min(j,m)} \in S$, such that $b'_j + \sum_{1 \leq i < j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1 \leq i < \min(j,m)} d_j}$,

If $\text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) > \text{Icode}(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, then

$[X \not\subseteq \text{coINT_OP_HULL}_{a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \dots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}}^{j, \min(j,m)}]$.

Note that whether $X, a_1, b_1, \dots, a_n, b_n, c_1, d_1, \dots, c_m, d_m$, satisfy (A) and (B), for all j , $1 \leq j \leq n$, is effectively testable.

Moreover, (B) above implies that,

(C) For all $(a'_1, b'_1, \dots, a'_n, b'_n), (c'_1, d'_1, \dots, c'_m, d'_m) \in \text{VALID}_S$, $\sum_{1 \leq i \leq n} b'_i < \frac{1}{\sum_{1 \leq i \leq m} d'_i}$, such that $\text{map}(a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m) <_Q \text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)$, $X \not\subseteq \text{coOP_HULL}_{a'_1, b'_1, \dots, a'_n, b'_n; c'_1, d'_1, \dots, c'_m, d'_m}^{n, m}$.

Thus, for finite $X \subseteq N^2$, let

$\Theta(X) = \bigcup \{L_{\text{map}(a_1, b_1, \dots, a_n, b_n)}^Q \mid \text{Prop}(X, a_1, b_1, \dots, a_n, b_n)\}$.

For infinite X' , $\Theta(X') = \bigcup_{X \subseteq X', \text{card}(X) < \infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq \text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m}$

$\Theta(X) \subseteq L_{\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$ (by (C) and the fact that for any valid I and I' , $\text{map}(I) <_Q \text{map}(I')$, implies $L_{\text{map}(I)}^Q \subseteq L_{\text{map}(I')}^Q$), and

(2) for any S -valid $(a_1, b_1, \dots, a_n, b_n)$ and $(c_1, d_1, \dots, c_m, d_m)$, by Claim 11(F) there exists a finite set $X \subseteq \text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m}$, such that $\Theta(X) \supseteq$

$L_{\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$.

Thus, we have that $\Theta(\text{coOP_HULL}_{a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m}^{n, m}) = L_{\text{map}(a_1, b_1, \dots, a_n, b_n; c_1, d_1, \dots, c_m, d_m)}^Q$. \blacksquare