Closed Left-R.E. Sets*

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Abstract. A set is called r-closed left-r.e. iff every set r-reducible to it is also a left-r.e. set. It is shown that some but not all left-r.e. cohesive sets are many-one closed left-r.e. sets. Ascending reductions are many-one reductions via an ascending function; left-r.e. cohesive sets are also ascening closed left-r.e. sets. Furthermore, it is shown that there is a weakly 1-generic many-one closed left-r.e. set.

1 Introduction

When studying the limits of computation, one often looks at recursively enumerable (r.e.) and left-r.e. sets. Natural examples of the r.e. sets are Diophantine sets and the word problem of a finitely generated group [8, 11, 13]. The best-known left-r.e. set is Chaitin's Ω [1, 14]. The present work focuses on a special subclass of the left-r.e. sets, namely those which are closed downwards with respect to the many-one or ascending reducibilities. While all r.e. sets exhibit closure under various reducibilities — one-one, many-one, conjunctive, disjunctive, positive truth-table and enumeration [8, 11, 13] — some left-r.e. sets, such as Chaitin's Ω, fail to do so.

We show that the classes of many-one closed left-r.e. sets and r.e. sets do not coincide: there exist both, cohesive and weakly 1-generic sets, which are many-one closed left-r.e. but not recursively enumerable, see Theorems 4, 15 and Remark 16. We also show that there are cohesive left-r.e. sets which are not many-one closed left-r.e., see Theorem 12.

We introduce the more restrictive notion of ascending reducibility. We show that cohesive and even r-cohesive left-r.e. sets are already ascending closed left-r.e. sets, see Theorem 17.

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Kolmogorov complexity measures the information content of strings; the applications of this notion range from quantifying the amount of algorithmic randomness [2, 7] to establishing lower bounds on the average running time of an algorithm [5]. An important tool to measure the complexity of a set A is the initial segment complexity which maps each n to the Kolmogorov complexity of $A(0)A(1)\ldots A(n)$. We show that the initial segment complexity of ascending closed left-r.e. sets has to be sublinear, see Proposition 13. We also show that the initial segment complexity of an ascending closed left-r.e. sets can be $\Omega(n/f(n))$ for any unbounded increasing recursive function f, which is close to optimal, see Theorem 14.

2 Many-One Closed Left-R.E. Sets

Post [9] introduced many-one reducibility by defining that a set B *many-one reduces* to a set A, denoted $A \leq_{\mathrm{m}} B$, if there exists a recursive function f such that $x \in A \iff f(x) \in B$. Below, we formally define a left-r.e. set and many-one closed left-r.e. set.

Definition 1. A set A is *left-r.e.* iff there is a uniformly recursive approximation $A_0, A_1, \ldots$ to A such that $A_s \leq_{\mathrm{lex}} A_{s+1}$ for all s. Here $A_s \leq_{\mathrm{lex}} A_{s+1}$ means that either $A_s = A_{s+1}$ or the least element x of the symmetric difference satisfies $x \in A_{s+1}$. If every set many-one reducible to A is left-r.e. then we say that A is a *many-one closed left-r.e.* set.

It is well-known that every set which is many-one reducible to an r.e. set is also itself r.e. [11]; hence every r.e. set is a many-one closed left-r.e. set. Furthermore, a set is recursive iff it is a bounded truth-table (btt) closed left-r.e. set because the complement of any set btt-reduces to the set itself, see [8] for discussion of btt-reductions.

Definition 2 (Friedberg [3], Lachlan [4], Myhill [6] and Robinson [10]). An infinite set A is *cohesive* iff for every r.e. set B either $B \cap A$ or $\overline{B} \cap A$ is finite. An infinite set A is *r-cohesive* iff for every recursive set B either $A \cap B$ or $A \cap \overline{B}$ is finite.

Cohesive sets have been studied widely in recursion theory; they emerged as the culmination of Post's unsuccessful attempts to generate a Turing incomplete r.e. set [13]. The next result gives a cohesive many-one closed left-r.e. set. We remark that Soare [12] already discovered a cohesive left-r.e. set.

The following notational conventions will be useful. Let

$$\varphi_{e,s}(x) = \begin{cases} \varphi_e(x), & \text{if } \varphi_e \text{ halts on input } y \text{ within } s \text{ steps for all } y \leq x; \\ \uparrow, & \text{otherwise.} \end{cases}$$

Note that if φ_e is total, then $\bigcup_s \varphi_{e,s} = \varphi_e$. Otherwise, the domain of $\bigcup_s \varphi_{e,s}$ is some initial segment of $\mathbb{N}$. Let $\varphi_e^{-1}(x) = \min\{y : \varphi_e(y) = x\}$ and $\varphi_{e,s}^{-1}(x) = \min\{y : \varphi_{e,s}(y) = x\}$.

Lemma 3. *Suppose $\varphi_{e_1}, \varphi_{e_2}, \ldots, \varphi_{e_k}$ are total. Furthermore, suppose that the set $S = \mathrm{range}(\varphi_{e_1}) \cap \mathrm{range}(\varphi_{e_2}) \cap \ldots \cap \mathrm{range}(\varphi_{e_k})$ is infinite. Then, for all a, r, there exist $a_1, a_2, \ldots, a_r \in S$ such that $a < a_1 < a_2 < \ldots < a_r$ and, for n, m with $1 \le n < r$ and $1 \le m \le k$ it holds that $\varphi_{e_m}^{-1}(a_n) < \varphi_{e_m}^{-1}(a_{n+1})$.*

Proof. Let a_1 be any member of S which is greater than a. For i with $2 \le i \le r$, let $a_i \in S$ be chosen such that $a_i > a_{i-1}$ and for m with $1 \le m \le k$, $\varphi_{e_m}^{-1}(a_{i-1}) < \varphi_{e_m}^{-1}(a_i)$. Note that there exist such $a_i \in S$, as S is infinite and only finitely many elements x can have $\varphi_{e_m}^{-1}(x) \le \varphi_{e_m}^{-1}(a_{i-1})$. $\Box$

Theorem 4. *There is a cohesive many-one closed left-r.e. set A.*

Proof. We will use moving markers, $a_0, a_1, \ldots$; let $a_{m,s}$ denote the value of marker a_m as at the beginning of stage s. Let $l_0 = 0$, $l_{d+1} = r_d + 1$, $r_d = l_d + 3^{d+2} + 1$. We let $I_{d,s} = \{a_{m,s} : l_d \le m \le r_d\}$.
For all m, s, we will have the following property:

(R1): $\qquad a_{m,s} < a_{m+1,s}$.

Define the predicate $P_{e,s}(d)$ as

$$P_{e,s}(d):\ (\exists a_{m,s}, a_{n,s} \in I_{d,s})\ [a_{m,s} < a_{n,s} \text{ and } \varphi_{e,s}^{-1}(a_{m,s}) > \varphi_{e,s}^{-1}(a_{n,s})].$$

For $e \le d$, let

$$i_{e,s}(d) = \begin{cases} 0, & \text{if } I_{d,s} \not\subseteq \mathrm{range}(\varphi_{e,s}); \\ 1, & \text{if } I_{d,s} \subseteq \mathrm{range}(\varphi_{e,s}) \text{ and } P_{e,s}(d); \\ 2, & \text{if } I_{d,s} \subseteq \mathrm{range}(\varphi_{e,s}) \text{ and not } P_{e,s}(d). \end{cases}$$

For $e \le d$, let $Q_{e,s}(d) = (i_{0,s}(d), i_{1,s}(d), \ldots, i_{e,s}(d))$. Note that one can consider $Q_{e,s}(d)$ as a number (base 3), with $i_{0,s}(d)$ as being the most significant bit. So one can talk about $Q_{e,s}(d) > Q_{e',s'}(d')$ etc.

We let $a_m = \lim_{s\to\infty} a_{m,s}$, $I_d = \lim_{s\to\infty} I_{d,s}$, $i_e(d) = \lim_{s\to\infty} i_{e,s}(d)$, and $Q_e(d) = \lim_{s\to\infty} Q_{e,s}(d) = (i_0(d), i_1(d), \ldots, i_e(d))$ (we will show later that these limits exist).

Intuitively, the aim of the construction of the moving markers a_m is to maximise the values of $Q_e(e)$ with higher priority given for lower values of e. The required set A will be defined later by choosing one element from each I_e. We define $a_{m,s}$ via the staging construction below. Stage s defines $a_{m,s+1}$.

Initially, let $a_{m,0} = m$.
Stage s: Check if, there exists $e \le s$, such that, by using $a_{m,s+1} = a_{m,s}$ for $m < l_e$, some values of $a_{m,s+1} \le s$ for $l_e \le m \le r_e$, and any values for $a_{m,s+1}$ for $m > r_e$ such that (R1) is satisfied, we have $Q_{e,s+1}(e) > Q_{e,s}(e)$. If so, then update the values of $a_{m,s+1}$ to the values witnessing above for the least such e. If no such e exists, then $a_{m,s+1} = a_{m,s}$, for all m.
End Stage s

Claim 5. For all e,
(a) for all m with $l_e \le m \le r_e$, $\lim_{e\to\infty} Q_{e,s}(e)$ and $\lim_{s\to\infty} a_{m,s}$ converge.
(b) $\lim_{s\to\infty} I_{e,s}$ converges.
(c) for all $d \ge e$, $\lim_{s\to\infty} i_{e,s}(d)$ converges.

(a) Follows by induction on e and the fact that $Q_{e,s}(e)$ is bounded. Now (b) and (c) follow by definitions. We let a_m, I_e, $i_e(d)$, and $Q_e(d)$ respectively denote $\lim_{s\to\infty} a_{m,s}$, $\lim_{s\to\infty} I_{e,s}$, $\lim_{s\to\infty} i_{e,s}(d)$, and $\lim_{s\to\infty} Q_{e,s}(d)$.

Claim 6. For all d and all $e \leq d$, $Q_e(d+1) \leq Q_e(d)$.

To prove the claim, suppose by way of contradiction that some least d and a corresponding least $e \leq d$ does not satisfy the claim. Let s be large enough such that for all $d' \leq d+1$, $s' > s$, $I_{d',s'} = I_{d',s}$ and $Q_{d',s'}(d') = Q_{d',s}(d')$. Then, in stage s, one could choose $a_{l_d,s+1}, \ldots, a_{r_d,s+1}$ to be $a_{l_{d+1}}, \ldots, a_{r_d+l_{d+1}-l_d}$, which makes $Q_{e,s+1}(d) > Q_{e,s}(d)$, and thus $Q_{d,s+1}(d) > Q_{d,s}(d)$, in contradiction to the choice of s. It follows from Claim 6 that, for all e, for all but finitely many $d \geq e$, $Q_e(d) = Q_e(d+1)$. Thus we get the following:

Claim 7. For all e, for all but finitely many $d > e$, $i_e(d+1) = i_e(d)$. We let $j_e = \lim_{d\to\infty} i_e(d)$.

Claim 8. For all e, $j_e \in \{0, 2\}$.

To prove the claim, suppose by way of contradiction that $j_e = 1$, for some least e. Choose d large enough such that, for all $e' \leq e$, for all $d' \geq d$, $i_{e'}(d') = j_{e'}$. Consider a large enough stage s such that, for all $d' \leq d$, for all $s' \geq s$, $I_{d',s'} = I_{d',s}$ and $Q_{d',s'}(d') = Q_{d',s}(d')$. Then we could make $Q_{e,s'}(d) > Q_{e,s}(d)$, for large enough $s' > s$ by choosing $a_{l_d,s'}, \ldots, a_{r_d,s'}$ (with $a_{l_d,s'} > a_{l_d}$) appropriately such that for all $e' \leq e$, if $I_d \subseteq \mathrm{range}(\varphi_{e'})$, then $\varphi_{e'}^{-1}(a_{m,s'}) < \varphi_{e'}^{-1}(a_{n,s'})$ for $l_d \leq m < n \leq r_d$. (It is possible to choose such values as, for $e' \leq e$, if $I_d \subseteq \mathrm{range}(\varphi_{e'})$, then $I_{d'} \subseteq \mathrm{range}(\varphi_{e'})$ for all $d' > d$, and then we can use Lemma 3.) But this contradicts the choice of s.

Claim 9. For all e, for all but finitely many $d \geq e$, $i_e(d) = 0$ implies, for all but finitely many d, $\mathrm{range}(\varphi_e) \cap I_d = \emptyset$.

To prove the claim, suppose by way of contradiction that e is such that for all but finitely many $d \geq e$, $i_e(d) = 0$, but for infinitely many d, $\mathrm{range}(\varphi_e) \cap I_d \neq \emptyset$. Fix least such e, and let d be such that (i) for all $e' \leq e$, for all $d' \geq d$, $Q_e(d') = Q_e(d)$, and (ii) for all $e' < e$, if $i_{e'}(d) = 0$, then for all $d' \geq d$, $\mathrm{range}(\varphi_{e'}) \cap I_{d'} = \emptyset$. Let s be such that for all $d' \leq d$, for all $s' \geq s$, $I_{d',s'} = I_{d',s}$ and $Q_{d',s'}(d') = Q_{d',s}(d')$. Let $E = \{e' : e' < e, i_{e'}(d) = 2\} \cup \{e\}$. Then, clearly, $\bigcap_{e' \in E} \mathrm{range}(\varphi_{e'})$ is infinite, and thus using Lemma 3, for large enough $s' > s$, we can find, $a_{l_d,s'}, \ldots, a_{r_d,s'}$ such that $i_{e',s'}(d) = 2$ for $e' \in E$, which makes $Q_{d,s'}(d) > Q_{d,s}(d)$, contradicting the choice of s. The claim follows.

Note above that $r_e - l_e \geq Q_{e+1}(e+1)$ for all possible values of $Q_{e+1}(e+1)$, and thus $a_{r_e - Q_{e+1}(e+1)} \in I_e$. Let

$$A = \{a_{r_e - Q_{e+1}(e+1)} : e \in \mathbb{N}\}.$$

Claim 10. A is cohesive.

To prove the claim, consider any total φ_e. If for all but finitely many $d > e$, $i_e(d) = 0$, then by Claim 9 range(φ_e) contains elements from only finitely many $I_{e'}$, and thus only finitely many elements of A. On the other hand, if, for all but finitely many $d > e$, $i_e(d) = 2$, then range(φ_e) contains all but finitely many $I_{e'}$, and thus all but finitely many elements of A. The claim follows.

Claim 11. Suppose $B \leq_{\mathrm{m}} A$ as witnessed by φ_e. Then, B is a left-r.e. set.

To prove the claim, first suppose that range(φ_e) $\cap A$ is finite. In this case $B = \{y : \varphi_e(y) \in S\}$ for some finite set S. Thus, B is recursive and a left-r.e. set.

Now suppose that range(φ_e) $\cap A$ is infinite. It follows that, for all but finitely many $d > e$, $i_e(d)$ has value 2 (by Claims 8 and 9). Let d be large enough such that $Q_e(d) = Q_e(d')$, for all $d' \geq d$. Consider a stage s_0 such that for all $d' \leq d$, for all $s \geq s_0$, $I_{d',s} = I_{d',s_0}$ and $Q_{d',s}(d') = Q_{d',s_0}(d')$. Define $s_{k+1} > s_k$ such that, for $d \leq d' \leq d + k + 1$, $Q_{e,s_{k+1}}(d') = (j_0, j_1, \ldots, j_e)$. Let

$$\alpha(m, k) = a_{r_m - Q_{m+1,s_k}(m+1)},$$

and define B_k as the characteristic function of $\{y : \varphi_e(y) \in A_{s_k} \cap \bigcup_{r<d+k} I_{r,s_k}\}$ where $A_{s_k} = \{\alpha(m, k) : m < d + k\}$.

The characteristic value of B_k as above converges to characteristic function of B. To show that B is left-r.e., we need to show that $B_k \leq_{\mathrm{lex}} B_{k+1}$. For this consider least d' such that for $m \leq d'$, $I_{m,s_{k+1}} = I_{m,s_k}$ and $Q_{m,s_{k+1}}(m) = Q_{m,s_k}(m)$, but

$$[I_{d'+1,s_{k+1}} \neq I_{d'+1,s_k} \text{ or } Q_{d'+1,s_{k+1}}(d'+1) \neq Q_{d'+1,s_k}(d'+1) \text{ or } d' = d + k + 1].$$

Note that $d' \geq d$. If $d' \geq d + k$, then clearly $B_k \leq_{\mathrm{lex}} B_{k+1}$. Otherwise, for $m < d'$, we have that $\alpha(m, k) = \alpha(m, k + 1)$. Also, $Q_{d'+1,s_k} < Q_{d'+1,s_k+1}$ and $\alpha(d', k + 1) < \alpha(d', k)$, which implies that $\varphi_e^{-1}(\alpha(d', k + 1)) < \varphi_e^{-1}(\alpha(d', k))$ (as φ_e^{-1} is monotonic on I_{d',s_k}, due to $Q_{e,s_k}(d') = Q_{e,s_{k+1}}(d') = (j_0, j_1, \ldots, j_e)$, where $j_e = 2$). Thus, $B_k \leq_{\mathrm{lex}} B_{k+1}$. It follows that B is a left-r.e. set. □

Not every left-r.e. set is many-one closed left-r.e.: Besides Ω, a quite easy example can be found by taking an r.e. and nonrecursive set A and considering the set

$$B = \{2x : x \in A\} \cup \{2x + 1 : x \notin A\}.$$

Then the complement of A is many-one reducbible to B but not a left-r.e. set. In contrast to Theorem 4, one can also find cohesive sets with this property.

Theorem 12. *There is a left-r.e. cohesive set A which is not a many-one closed left-r.e. set.*

Proof. In the following let $W_{d,s}$ denote the set of elements of W_d below s which are enumerated within s steps into W_d. Partition $\mathbb{N}$ into intervals I_i of length 2^i: $I_i = \{2^i - 1, 2^i, 2^i + 1, \ldots, 2^{i+1} - 2\}$. Furthermore, assign to every x the e-state given as

$$q_{e,s}(x) = \sum_{d<e} 2^{e-1-d} * W_{d,s}(x).$$

We say that

$q_{e,s}(I_i) = c$ iff $c < 2^e$ is the largest number satisfying $q_{e,s}(x) \geq c$ for at least $2^i - 2^{i-e-1} \cdot (c+1)$ elements of I_i.

Here we let $J_{e,i,s}$ be a witness for the above fact in the way such that $J_{e,i,s} \subseteq I_i$, $|J_{e,i,s}| = 2^i - 2^{i-e-1} \cdot (c+1)$ and $q_{e,s}(x) \geq c$ for all $x \in J_{e,i,s}$. Here we assume that $J_{e,i,s+1} \neq J_{e,i,s}$ implies that $q_{e,s+1}(I_i) > q_{e,s}(I_i)$. It is easy to verify that $\lim_{s\to\infty} q_{e,s}(I_i)$ converges for each e, i and thus, $\lim_{s\to\infty} J_{e,i,s}$ converges for each e, i.

Define $i_{0,s}, i_{1,s}, \ldots$ such that the following properties are satisfied:
(a) for all e, s: $i_{e,s} < i_{e+1,s}$ and $i_{e,s+1} \geq i_{e,s} > 2e + 2$;
(b) for all e, s, j with $i_{e,s} \leq j \leq s$ it holds that $q_{e,s}(I_{i_{e,s}}) \geq q_{e,s}(I_j)$.
(c) for all s, for the least e (if any) such that $i_{e,s} \neq i_{e,s+1}$ or $J_{e,i_{e,s},s} \neq J_{e,i_{e,s+1},s+1}$: $q_{e,s+1}(I_{i_{e,s+1}}) > q_{e,s}(I_{i_{e,s}})$.

Note that such $i_{j,s}$ can be recursively defined. It is easy to verify by induction that $i_e = \lim_{s\to\infty} i_{e,s}$ converges. Furthermore, note that $q_{0,s}(I_{i_{0,s}}) = 0$ for all s and $J_{0,i_{0,s},s} = I_{i_{0,s}}$ for all s. Hence, $i_{0,s} = i_{0,0}$ for all s. Now we are ready to define A.

Definition of A_s:
Let $H_{e,s} = \{x \in J_{e,i_{e,s},s} : q_{e,s}(x) = q_{e,s}(I_{i_{e,s}})\}$ for all e.
Let $x_{e,s}$ be the $(q_{e+1,s}(I_{i_{e+1,s}}) + 1)$-th element from above of $H_{e,s}$ for all e.
Let $A_s = \{x_{0,s}, x_{1,s}, \ldots\}$.
End Definition of A_s

Let $A(x) = \lim_{s\to\infty} A_s(x)$. One can verify that $\lim_{s\to\infty} i_{e,s}$, $\lim_{s\to\infty} q_{e,s}(I_{i_{e,s}})$ and $\lim_{s\to\infty} J_{e,i_{e,s},s}$ converge. Thus it is easy to verify that A is well defined. We also let $i_e, J_{e,i_e}, H_e, q_e(x), q_e(I_j)$ denote the limiting values of $i_{e,s}$, $J_{e,i_{e,s},s}$, $H_{e,s}$, $q_{e,s}(x)$, $q_{e,s}(I_j)$, respetively.

Here, it should be noted that $H_{e,s}$ has at least $2^{i_{e,s}-e-1}$ elements. To see this, let $c = q_{e,s}(I_{i_{e,s}})$ and note that $J_{i_{e,s},s}$ has at least $2^i - 2^{i-e-1} \cdot (c+1)$ elements of which less than $2^{i_{e,s}} - 2^{i_{e,s}-e-1} \cdot (c+2)$ many x satisfy $q_{e,s}(x) > c$ while all x satisfy $q_{e,s}(x) \geq c$. So at least $2^{i_{e,s}-e-1}$ elements x of $J_{e,i_{e,s},s}$ satisfy $q_{e,s}(x) = c$ and these are in $H_{e,s}$. As $i_{e,s} \geq 2e+2$, it follows that $|H_{e,s}| \geq 2^{e+1}$ and so there is, for each possible value c' of $q_{e+1,s}(I_{e+1,s}) < 2^{e+1}$, a $(c'+1)$-th largest element of $H_{e,s}$. Thus every $x_{e,s}$ as defined above really exists. For each e, the sequence of the $x_{e,s}$ converges to some value x_e.

To show that $(A_s)_{s\in\mathbb{N}}$ forms a left r.e. approximation, we need to show that $A_s \leq_{\mathrm{lex}} A_{s+1}$. So consider the least e (if any) such that $x_{e,s+1} \neq x_{e,s}$. Note that $i_{e,s+1} = i_{e,s}$ and $J_{e,i_{e,s+1},s+1} = J_{e,i_{e,s},s}$, as otherwise $e > 0$ and $x_{e-1,s+1} \neq x_{e-1,s}$. Hence $H_{e,s+1} \subseteq H_{e,s}$ and, for $s' = s, s{+}1$, $x_{e,s'}$ is the $(q_{e+1,s'}(I_{i_{e+1,s'}})+1)$-th element of $H_{e,s'}$ from above. As $i_{d,s+1} = i_{d,s}$ and $J_{d,i_{d,s+1},s+1} = J_{d,i_{d,s},s}$ for all $d \leq e$, it follows by rule (c) that $q_{e+1,s+1}(I_{e+1,i_{e+1,s+1},s+1}) \geq q_{e+1,s}(I_{e+1,i_{e+1,s},s})$. Hence $x_{e,s+1} < x_{e,s}$ and that implies that $A_{s+1} >_{lex} A_s$. So A is a left-r.e. set.

Now we show that A is cohesive. So consider any d, e, k such that $d < e$ and $k \geq 0$. Then, we claim that $q_{d+1}(x_e) \geq q_{d+1}(x_{e+k})$. To see this, suppose $2^k c \leq q_{e+k}(I_{i_{e+k}}) \leq 2^k c + 2^k - 1$. Thus, at least $2^i - 2^{i-e-k-1} \cdot (2^k \cdot$

$c + 2^k$), many x in $I_{i_{e+k}}$ have $q_{e+k}(x) \geq 2^k c$. Thus, $2^i - 2^{i-e-1}(c+1)$ of x in $I_{i_{e+k}}$ have $q_e(x) \geq c$ and thus $q_e(I_{e+k}) \geq c$. Now, for $x_{e+k} \in H_{e+k}$ and $x_e \in H_e$, $q_{d+1}(x_{e+k}) = \lfloor q_{e+k}(I_{e+k})/2^{k+e-d-1} \rfloor < (c+1)2^k/2^{k+e-d-1}$, and thus $q_{d+1}(x_{e+k}) \leq c/2^{e-d-1}$. On the other hand, $q_{d+1}(x_e) = \lfloor q_e(I_e)/2^{e-d-1} \rfloor \geq \lfloor c/2^{e-d-1} \rfloor$. Thus, $q_{d+1}(x_{e+k}) \leq q_{d+1}(x_e)$.

Thus, as $A = \{x_0, x_1, \ldots\}$, for all d, $q_{d+1}(x_e)$ is same for all but finitely many e. For each d it follows that $W_d(x_e)$ is the same value for almost all e. Thus A is cohesive.

Now consider $B \leq_m A$ via f where, for all i and $x \in I_i$, $f(x) = \max(I_i) + \min(I_i) - x$. Note that $f(x) = f^{-1}(x)$. Thus, f also witnesses $A \leq_m B$. Let $(A_s)_{s \in \mathbb{N}}$ be the left-r.e. approximation of A as given above and $(B_s)_{s \in \mathbb{N}}$ be a left-r.e. approximation of B. Then, the following holds for all e, s:

> $(*)$ If the least $e+1$ elements $x_{0,s}, x_{1,s}, \ldots, x_{e,s}$ of A_s satisfy that $f(x_{0,s})$, $f(x_{1,s}), \ldots, f(x_{e,s})$ are the unique elements of B_s below $\max(\{I_{i_{e,s}}\})$ then $x_0 = x_{0,s}, x_1 = x_{1,s}, \ldots, x_e = x_{e,s}$.

For a proof, assume that the above would be false for some e, s and let d be the least index such that $x_d \neq x_{d,s}$; by the left-r.e.-ness of the approximation, $x_d < x_{d,s}$. Furthermore, by (c), $i_{d,s} = i_d$ as otherwise $d > 0$ and $x_{d-1} \neq x_{d-1,s}$. So $f(x_{d,s}) < f(x_d)$ and $B \cap \{0, 1, \ldots, \max(I_{i_d})\} = \{f(x_0), f(x_1), \ldots, f(x_d)\}$. But $\{f(x_0), f(x_1), \ldots, f(x_d)\} <_{lex} \{f(x_{0,s}), f(x_{1,s}), \ldots, f(x_{d,s})\}$ and hence $B <_{lex} B_s$, a contradiction to $(B_s)_{s \in \mathbb{N}}$ being a left-r.e. approximation of B. So $(*)$ is true. Now one can determine x_e by searching for the first stage s where $f(x_{0,s})$, $f(x_{1,s}), \ldots, f(x_{e,s})$ are the unique elements of B below $\max(\{I_{i_{e,s}}\})$ and then one knows that $x_e = x_{e,s}$. Thus, we get that A is recursive, in contradiction to A being cohesive. □

3 Ascending Closed Left-R.E. Sets

An *ascending reduction* is a recursive function f which satisfies $f(x) \leq f(x+1)$ for all x; $B \leq_{\mathrm{asc}} A$ iff there is an ascending reduction f with $B(x) = A(f(x))$ for all x. A is called *ascending closed left-r.e.* iff every $B \leq_{\mathrm{asc}} A$ is a left-r.e. set.

Let $A[n]$ denote the string $A(0)A(1)\ldots A(n)$. Let $C(x)$ denote the *plain Kolmogorov complexity* for x. That is, $C(x) = \min\{\log(y) : U(y) = x\}$, where U is a fixed universal Turing machine. The function mapping n to $C(A[n])$ is called the initial segment complexity of A and the next result shows that the initial segment complexity of ascending closed left-r.e. sets is sublinear.

Proposition 13. *If A is an ascending closed left-r.e. set then the initial segment complexity $n \mapsto C(A[n])$ is a function of sublinear order.*

Proof. Let c be any constant, and let G_n denote the interval $\{x : x \leq \lceil n/c \rceil\}$. For $d < c$, define B^d by $B^d(x) = A(cx + d)$. Thus $B^d \leq_{\mathrm{asc}} A$. Let $(B^d_s)_{s \in \mathbb{N}}$ be left-r.e. approximations of B^d. For each n, let $d_n < c$ be the index for which $(B^{d_n}_s \cap G_n)_{s \in \mathbb{N}}$ converges slowest. Then given d_n and $B^{d_n} \cap G_n$, we can determine

$B^d \cap G_n$ for each $d < c$ and therefore $A[n]$ as well. Hence, for some constant b_c and for all n, $C(A[n]) \leq n/c + b_c$. This shows that the complexity function $n \mapsto C(A[n])$ has sublinear order. $\Box$

Theorem 14. *Let g be a recursive and unbounded non-decreasing function. Then there is an ascending closed left-r.e. set A such that $n \mapsto C(A[n])$ has at least the order $n/g(n)$.*

Proof. Without loss of generality assume $1 \leq g(i) \leq i$. Partition $\mathbb{N}$ into intervals I_i of length 2^i: $I_i = \{2^i - 1, 2^i, 2^i + 1, \ldots, 2^{i+1} - 2\}$. For each I_i, we will construct a subset $J_i = \lim_{s \to \infty} J_{i,s}$. Let $J_{i,0} = I_i$. At stage s, if there is an $e < \log(g(i))$ (which has not been handled earlier) and an x such that

$$\varphi_e(0)\downarrow \leq \varphi_e(1)\downarrow \leq \varphi_e(2)\downarrow \leq \ldots \leq \varphi_e(x)\downarrow \quad \text{and} \quad \varphi_e(x) > \max(I_i).$$

Then, choose one such e and the corresponding x. Determine the two subsets $J_{i,s} \cap \{\varphi_e(y) : y \leq x\}$ and $J_{i,s} - \{\varphi_e(y) : y \leq x\}$, and let $J_{i,s+1}$ be that one of these two subsets which has the higher cardinality (in case of tie, choose arbitrarily). Note that during the approximation process $J_{i,s}$ gets halved at most $\log(g(i))$ times and therefore the limit J_i has at least $2^i/g(i)$ many elements.

Define A so that the characteristic function of A on the set J_i, in ascending order, is the binary representation of the least number a_i with $C(a_i) \geq 2^i/g(i) - 2$ (where as many leading zeros are added as needed to use up all bits of J_i); A has no elements outside the sets J_i. Note that there is a recursive approximation $a_{i,s}$ to a_i from below.

The set A is left-r.e. as we can have an approximation A_s which takes on each $J_{i,s}$ the characteristic function of the binary representation of $a_{i,s}$ (with sufficiently many leading zeros added in); A_s is 0 on $I_i - J_{i,s}$. If the interval $J_{i,s}$ shrinks to $J_{i,s+1}$, then the bits of $a_{i,s}$ move to the left and some leading zeros are skipped; if $a_{i,s+1} > a_{i,s}$ then the bits are also ascending in lexicographic manner. Hence the resulting approximation is a left-r.e. approximation which runs independently on each interval I_i.

Now suppose $B \leq_{\text{asc}} A$ via a recursive non-decreasing function φ_e. If the range of φ_e is finite, then B is clearly recursive. Now suppose that range of φ_e is infinite. Let r be the greatest index satisfying $g(r) \leq e$. Let $s_0 = s_1 = s_2 = \ldots = s_r$ be so large that $A_{s_0}(x) = A(x)$ for all $x \leq \max(I_r)$. For $k \geq r$, let $s_{k+1} > s_k$ be such that for all $s \geq s_{k+1}$ either $J_{k+1,s} \subseteq \text{range}(\varphi_e)$ or $J_{k+1,s} \cap \text{range}(\varphi_e) = \emptyset$. Note that s_{k+1} can be computed effectively from k.

Define the approximation $(B_k)_{k \in \mathbb{N}}$ of B as

$$B_k(x) = \begin{cases} A_{s_k}(\varphi_e(x)), & \text{if } \varphi_e(x) \leq \max(I_k); \\ 0, & \text{if } \varphi_e(x) > \max(I_k). \end{cases}$$

This approximation is a left-r.e. approximation to B as it starts to consider the interval I_k, for $k > r$, only after stage s_k such that for all $s \geq s_k$, $J_{k,s} \subseteq \text{range}(\varphi_e)$ or $J_{k,s} \cap \text{range}(\varphi_e) = \emptyset$. In the first case all the bits of J_{k,s_k} are copied order-preservingly into B_k and the left-r.e. approximation to A on I_k is turned into a left-r.e. approximation to B on the preimage of I_k under φ_e; in the second case all x with $\varphi_e(x) \in I_k$ satisfy $\varphi_e(x) \notin J_{k,s_k}$ and therefore $B_s(x) = 0$ for these x and all stages s. So A is an ascending closed left-r.e. set.

Furthermore, $C(a_i) \geq 2^i/g(i) - 2$. Also, we can compute a_i from the number i, the string $A[\max(I_i)]$ and the number of stages s at which $J_{i,s+1} \neq J_{i,s}$. Hence, for some b and almost all i we have

$$C(A[\max(I_i)]) \geq C(a_i) - \log i - \log g(i) - b \geq \frac{2^i}{2g(i)}.$$

Taking now n with $\min(I_{i+1}) \leq n \leq \max(I_{i+1})$ and using that g is non-decreasing, we have that $C(A[n]) \geq \frac{n}{8g(n)}$ for almost all n. This proves the postulated bound. $\square$

Another important type of sets are the 1-generic and weakly 1-generic sets [8]. As one cannot have left-r.e. 1-generic sets, one might ask for which reducibilities r there are r-closed left-r.e. weakly 1-generic sets. The next result shows that one can make such sets for the notion of ascending closed left-r.e. sets.

Recall that a set is *weakly* 1-*generic* iff for every recursive function f from numbers to strings there exist n and m with $f(n) = A(n+1)A(n+2)\ldots A(n+m)$. The difference between weakly 1-generic and 1-generic is that here one requires the f to be total and independent of the values of A below n.

Theorem 15. *There is an ascending closed left-r.e. weakly* 1-*generic set A.*

Proof. We will be defining moving markers a_e, b_e and c_e, where $a_e \leq b_e \leq c_e$, $a_0 = 0$ and $a_{e+1} = c_e + 1$. Intuitively, we want to use the part $A(b_e), A(b_e + 1), \ldots, A(c_e)$ to ensure weak 1-genericity (by making $A(b_e)A(b_e + 1)\ldots A(b_e + |\varphi_e(b_e)| - 1) = \varphi_e(b_e)$, if $\varphi_e(b_e)$ is defined). The part $A(a_e), \ldots, A(b_e - 1)$ is used to ensure that A is ascending closed left-r.e.

At the beginning of stage s, the markers have values $a_{e,s}, b_{e,s}$ and $c_{e,s}$ respectively. We will have that $a_e = \lim_{s\to\infty} a_{e,s}$, $b_e = \lim_{s\to\infty} b_{e,s}$, $c_e = \lim_{s\to\infty} c_{e,s}$.

Let $a_{0,s} = 0$ for all s. Let $a_{e+1,s} = c_{e,s} + 1$, for all e, s. Initially $a_{e,0} = b_{e,0} = c_{e,0} = e$, and $A_0 = 0^\infty$. We will also use sets $J_{e',e,s}$, for $e' < e$. These sets are useful for defining A in such a way that, if $\varphi_{e'}$ witnesses an ascending reduction from B to A, then B is left-r.e. Initially, for all e, for $e' < e$, $J_{e',e,0} = \emptyset$. Below, for ease of presentation, we will only describe the changes from stage s to stage $s+1$; all variables which are not explicitly updated will retain the corresponding values from stage s.

Stage s:

1. If there exists an $e \leq s$ such that either Cond e.1 or Cond e.2 hold, then choose least such e and go to step 2. Otherwise go to stage $s + 1$.
   - Cond e.1: There exists $e' < e$ such that, $J_{e',e,s} = \emptyset$ and $\mathrm{range}(\varphi_{e'}) \cap \{x : x > c_{e,s}\}$ contains at least $2e + 2$ elements as can be verified within s steps.
   - Cond e.2: $c_{e,s} = b_{e,s}$ and $\varphi_e(b_{e,s})\downarrow$ within s steps.

2. Fix least e such that Cond e.1 or Cond e.2 holds. If Cond e.1 holds, then go to step 3. Otherwise go to step 4.

3. Fix one e' such that Cond e.1 holds for e'.
Let $J_{e',e,s+1}$ be $2e+2$ elements from range$(\varphi_{e'}) \cap \{x : x > c_{e,s}\}$.
Update $b_{e,s+1} = \max(J_{e',e,s+1}) + 1$, $c_{e,s+1} = b_{e,s+1}$.
For $m > e$, let $a_{m,s+1} = b_{m,s+1} = c_{m,s+1} = c_{m-1,s+1} + 1$.
For $m > e$, and $m' < m$, let $J_{m',m,s+1} = \emptyset$.
Let A_{s+1} be obtained from A_s by (i) deleting all elements $\geq b_{e,s}$, and by (ii) inserting, for each $m < e$ such that $J_{m,e,s} \neq \emptyset$, one new element (which was not earlier in A_s) from $J_{m,e,s}$.
Go to stage $s+1$.

4. Suppose $\varphi_e(b_{e,s}) = y$.
Let $c_{e,s+1} = b_{e,s} + |y| + 1$.
For $m > e$, let $a_{m,s+1} = b_{m,s+1} = c_{m,s+1} = c_{m-1,s+1} + 1$.
For $m > e$, and $m' < m$, let $J_{m',m,s+1} = \emptyset$.
Let A_{s+1} be obtained from A_s by (i) deleting all elements $\geq b_{e,s} + |y| + 1$, and by (ii) inserting, for each $m < e$ such that $J_{m,e,s} \neq \emptyset$, one new element (which was not earlier in A_s) from $J_{m,e,s}$, and by setting (iii) $A_{s+1}(b_{e,s}) \ldots A_{s+1}(b_{e,s} + |y| - 1) = y$,
Go to stage $s+1$.

End stage s.

It can be shown by induction on e that $\lim_{s\to\infty} a_{e,s}$, $\lim_{s\to\infty} b_{e,s}$, $\lim_{s\to\infty} c_{e,s}$ indeed exist. For this, for $e' < e$, after $a_{e'}, b_{e'}$ and $c_{e'}$ have reached their final value, a_e does not get modified any further (a_e is set to $c_{e-1} + 1$, in the last stage in which c_{e-1} gets modified). Furthermore, once a_e reaches its final value, b_e can change at most e times due to Cond e.1 holding for some $e' < e$ (and thus execution of step 3). Once b_e reaches its final value, c_e gets modified at most once due to success of Cond e.2 (and thus execution of step 4). The "$2e+2$" in the algorithm description suffices since each index e has e indices below it, and, after all variables $a_{e'}, b_{e'}, c_{e'}$, with $e' < e$ have stabilised, we encounter Cond e.1 at most once for each $e' < e$, and correspondingly Cond e.2 once in the beginning, and at most once after each modification of b_e via Cond e.1. Also, note that, for $m < e$, $J_{m,e,s} \subseteq \{x : a_{e,s} \leq x < b_{e,s}\}$.

Let $A(x) = \lim_{s\to\infty} A_s(x)$. Now we show that A is weakly 1-generic. Suppose $a_{e'}, b_{e'}, c_{e'}$, for $e' \leq e$, reach their final values before stage s. If $\varphi_e(b_e)$ is defined then Cond e.2 succeeds in some stage $s' \geq s$, and step 4 defines $A_{s'+1}(b_e) \ldots A_{s'+1}(b_e + |y| - 1) = y$, where $\varphi_e(b_e) = y$. Furthermore, A never gets modified on inputs $\leq c_{e,s'+1} = c_e$ after stage s'.

Now suppose $B \leq_{\mathrm{asc}} A$ as witnessed by φ_r. If range(φ_r) is finite, then clearly B is recursive. So assume range(φ_r) is infinite. Thus, for each $e > r$, Cond e.1 will succeed (eventually) for $e' = r$, after a_e has achieved its final value.

Define s_0 such that a_m, b_m, c_m, (for $m \leq r$) as well as $A[c_r]$ have reached their final values by stage s_0. Let $s_{k+1} > s_k$ such that $J_{r,r+j,s_{k+1}} \neq \emptyset$, for all $j \leq k+1$. Let $B_k = \{x : \varphi_r(x) \in A_{s_k}$ and $\varphi_r(x) \leq c_{r+k,s_k}\}$.

Clearly $B(x) = \lim_{k\to\infty} B_k(x)$. Thus, to show that B is left-r.e. it suffices to show that $B_k \leq_{\mathrm{lex}} B_{k+1}$. So consider the least $x \leq c_{r+k,s_k}$, if any, such that in some stage s', $s_k \le s' < s_{k+1}$, Cond e.1 or Cond e.2 succeeds, and $x \ge b_{e,s'}$ (if there is no such x, then we are done). Clearly, $e \ge r$ by hypothesis on s_0. Note that for $j \le k$, $J_{r,r+j,s_k} \ne \emptyset$. Thus, $J_{r,r+j,s'} \ne \emptyset$. Thus, in stage s', $A_{s'+1}(x')$ is set to 1, for some $x' \in J_{r,e,s'}$ such that $A_{s_k}(x') = 0$. Note that $x' < b_{e',s} \le x$. Let y' be least such that $\varphi_r(y') = x'$. Thus,

$$B_k \le_{\text{lex}} A_{s'+1}(\varphi_r(0))A_{s'+1}(\varphi_r(1)) \dots A_{s'+1}(\varphi_r(y')) \le_{\text{lex}} B_{k+1}$$

as desired. $\square$

Remark 16. Note that one can adjust the proof to show that there is a many-one closed left-r.e. and weakly 1-generic set. For this, main change in the construction would be to change Cond e.1 above to:

Cond e.1: There exists $e' < e$ such that $J_{e',e,s} = \emptyset$, and for some z, z', for all $x \le z$, $\varphi_{e'}(x)\downarrow \le z'$ within s steps, and $\{\varphi_{e'}(x) : x \le z\} \cap \{x : c_{e,s} < x \le z'\}$ contains at least $2e + 2$ elements.

Then, setting $J_{e',e,s+1}$ as in step 3, and making b_e to be $> z'$, would achieve the goal, as any element in A which is larger than z' would be able to influence membership in $B = \{x' : \varphi(x') \in A\}$, only for $x > z$. We omit the details.

The next result shows that every r-cohesive set is ascending closed left-r.e. set. Thus, r-cohesive left-r.e. sets form a subclass of ascending closed left-r.e. sets. Recall that every cohesive set is r-cohesive.

Theorem 17. *Every left-r.e. r-cohesive set is an ascending closed left-r.e. set.*

Proof. Suppose A is a left-r.e. r-cohesive set. Suppose $B \le_{\text{asc}} A$. Let $(A_s)_{s\in\mathbb{N}}$ be the left-r.e. approximation of A and f be a non-decreasing recursive function which witnesses that $B \le_{\text{asc}} A$. If range$(f) \cap A$ is finite, then clearly B is recursive. So assume range$(f) \cap A$ is infinite. But then, for some x and for all $y \ge x$, $y \in A$ implies $y \in$ range(f). Fix this x.

Let s_0 be such that for all $s \ge s_0$, for all $y \le x$, $A_s(y) = A(y)$; let $s_{n+1} > s_n$ be such that the least $n + 1$ members of $A_{s_{n+1}} - \{y : y \le x\}$ exist and are in range(f); note that one can effectively find such s_{n+1} from s_n. Let

$$B_n(y) = \begin{cases} A(f(y)), & \text{if } f(y) \le x; \\ 1, & \text{if } f(y) \text{ is among the least } n \text{ members} \\ & \text{of } A_{s_n} \text{ which are greater than } x; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that B_n is an approximation of B. To see that $(B_n)_{n\in\mathbb{N}}$ form a left-r.e. approximation, we need to show that $B_n \le_{\text{lex}} B_{n+1}$ for all n. So consider any n. If $B_n \not\subseteq B_{n+1}$, then there exists least y such that $y \in B_n - B_{n+1}$. Then $f(y)$ is among the first n members of A_{s_n} which are greater than x and not among the first $n+1$ members of $A_{s_{n+1}}$ which are greater than x. As $(A_s)_{s\in\mathbb{N}}$ is left-r.e. approximation, we have that $A_{s_{n+1}}$ must contain a $f(y')$, $x < f(y') < f(y)$, such that $f(y') \notin A_{s_n}$. But, then $y'' \in B_{n+1} - B_n$, for some $y'' \le y'$. Thus, $(B_n)_{n\in\mathbb{N}}$ is a left-r.e. approximation of B. $\square$

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