Refinements of Inductive Inference by Popperian and Reliable Machines [1]

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Abstract

Restricted and unrestricted algorithmic devices which attempt to arrive in the limit at explanatory computer programs for input functions are studied. The input functions may be interpreted as summaries of the behavior of real world phenomena. A classification of criteria of success for such devices is made based on Karl Popper's refutability principle in philosophy of science. Also considered are criteria of success requiring reliability in the sense that the devices should not mislead us by converging to faulty programs. The criteria in the classifications are compared to one another and some interesting tradeoff results are obtained. The techniques of recursive function theory are employed.

1 Introduction

Consider a real world phenomenon f that is being investigated by an agent **M**. **M** performs discrete experiments x on f. For example, x might be a particle diffraction experiment and $f(x)$ the resultant probable distribution on the other side of the diffraction grating. By a suitable encoding of the experiments and results we may treat f as a function from $N = \{0, 1, 2, \ldots\}$, the set of natural numbers, to N. A complete explanation for f is a computer program for f. Such a program for f gives us predictive power about the results of all possible experiments related to f. We are concerned about the theoretical properties of the agents which attempt to arrive at explanations (possibly only nearly correct) for different phenomena. In what follows we will conceptualize such agents as learners (of programs for functions).

An inductive inference machine (IIM) is an algorithmic device which takes as its input a graph of a function: $N \to N$, an ordered pair at a time, and, as it is receiving its input, outputs computer programs from time to time.

There are several ways in which one may define what it means for a learner to succeed in explaining a phenomenon. One such criterion is due to Gold [16]. We say that **M** **Ex**-identifies f iff **M**, fed the graph of f, outputs a non-empty sequence of programs which converges to a program for f. Note that one may not be able to algorithmically determine, if and when, the sequence of programs output by **M** on f converges. The class **Ex** denotes the class of sets of functions, $\mathcal{C}$, such that some machine **Ex**-identifies each function in $\mathcal{C}$. Another such criterion, due to [2, 11] is as follows. We say that **M** **Bc**-identifies a function f iff **M** fed the graph of f, either outputs a finite sequence of programs, the last of which computes f, or outputs an infinite sequence of programs, all but finitely many of which compute f. The class **Bc** denotes the class of sets of functions, $\mathcal{C}$, such that some machine **Bc**-identifies each function in $\mathcal{C}$. **Ex** and **Bc** are examples of what we refer to below as *learning classes.*

The above criteria have been extended by allowing anomalies in the final program(s) [5, 11, 20].

Karl Popper has enunciated the principle that scientific explanations ought to be subject to refutation [23]. Most of this paper concerns restrictions on the machines requiring them to output explanations which *can* be refuted. Precise mathematical definitions capturing this idea may be found in Section 3.3 below. Our results may be found in Section 4 below. Corollary 2, also in Section 4, shows interesting tradeoffs in learning power obtained as one varies the number of anomalies allowed and the number of mind changes to convergence. For several criteria of success defined with restrictions suggested by Popper's refutability principle, Corollary 2 provides an approximate *product* tradeoff formula by which anomalies can be traded for mind changes without loss of learning power.

Section 5 introduces prediction machines from [1, 3, 5, 22], which machines extrapolate *next values* for functions. Section 6 shows how the power of prediction compares to that of the other success criteria with Popperian restrictions.

Roughly, a machine $\mathbf{M}$ is said to be *reliable* on a class, $\mathcal{C}$, of functions: $N \to N$, iff whenever, on $f \in \mathcal{C}$, the sequence of programs output by $\mathbf{M}$ converges to a final program, that final program "does a good job" of computing f. Intuitively, reliable machines don't mislead by converging to bad programs. [5, 19] introduce the concept of reliability, but in the latter reference reliable is called *strong*. Section 7 provides the precise definitions. Section 8 deals with comparing learning classes obtained by requiring machines to be reliable with the other learning classes of this paper.

Some sharp contrasts in learning power appear when the class of functions on which machines must obey some restriction is varied. For example, taken together Theorem 21, Corollary 4 and Theorems 17 and 22 from Section 4 point to a tremendous loss of learning power if machines are required to obey a weak Popperian restriction on the class of all total functions instead of on the class of all computable functions. Corollary 9 from Section 8 points to a similar result for machines required to be reliable.

This paper grew out of the much earlier [10]. Carl Smith and Jun Tarui nicely pointed out some mistakes in [10].

2 Preliminaries

Recursion-theoretic concepts not explained below are treated in [27]. N denotes the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$.

$*$ denotes a non-member of N and is assumed to satisfy $(\forall n)[n < * < \infty]$. a and b, with or without decorations (decorations are subscripts, superscripts and the like), range over $N \cup \{*\}$. e, i, j, k, l, m, n, q, r, s, t, u, w, x, y, z, with or without decorations, range over N. $\lfloor \frac{x}{y} \rfloor$ denotes the largest natural number n, such that $n \leq \frac{x}{y}$. $x \dot{-} y$ denotes $\max(\{0, x - y\})$. In some contexts, p, with or without decorations, ranges over N, being construed as program for a (partial) function. In other contexts, p, ranges over total functions, with the range of p being construed as programs for (partial) functions.

We let P, S, with or without decorations, range over subsets of N. complement(S) denotes the complement of S, i.e. complement$(S) = N - S$. $\in, \subseteq, \subseteq, \supseteq, \supset$ respectively denote member of, subset, proper subset, superset and proper superset. $\uparrow$ denotes undefined. $\downarrow$ denotes defined. $\langle \cdot, \cdot \rangle$ denotes a 1-1 mapping from pairs of natural numbers onto natural numbers. π_1, π_2 are the corresponding projection functions. $\langle \cdot, \cdot \rangle$ is extended to n-tuples in a natural way. card(P) denotes the cardinality of P. So then, 'card$(P) \leq *$' means that card(P) is finite. min(P) and

$\max(P)$ respectively denote the minimum and maximum element in P. We take $\min(\emptyset)$ to be ∞ and $\max(\emptyset)$ to be 0.

η, with or without decorations, ranges over partial functions. We identify partial functions with their graphs. For $a \in (N \cup \{*\})$, $\eta_1 =^a \eta_2$ means that $\text{card}(\{x \mid \eta_1(x) \neq \eta_2(x)\}) \leq a$. $\text{domain}(\eta)$ and $\text{range}(\eta)$ respectively denote the domain and range of partial function η. For an expression E in x, $\lambda x.\text{E}(x)$ denotes the function f such that, for all x, $f(x) = \text{E}(x)$.

$\mathcal{R}$ denotes the class of all *recursive* functions, i.e., total computable functions with arguments and values from N. $\mathcal{T}$ denotes the class of all total functions. f, g, h, with or without decorations, range over $\mathcal{T}$. $\mathcal{C}$ and $\mathcal{S}$, with or without decorations, range over subsets of $\mathcal{T}$.

0-extension of a partial function η is a total function f such that, for all x,

$$f(x) = \begin{cases} \eta(x), & \text{if } x \in \text{domain}(\eta); \\ 0, & \text{otherwise.} \end{cases}$$

φ denotes a fixed *acceptable* programming system for the partial computable functions: $N \to N$ [26, 27, 17]. (Case showed the acceptable systems are *characterized* as those in which every control structure can be constructed; Royer and later Marcoux examined complexity analogs of this characterization [24, 25, 28, 18].) φ_i denotes the partial computable function computed by program i in the φ-system. We let Φ be an arbitrary Blum complexity measure [6] associated with acceptable programming system φ; such measures exist for any acceptable programming system [6]. We let $W_i = \text{domain}(\varphi_i)$. Let $W_{i,s} = \{x \leq s \mid \Phi_i(x) \leq s\}$.

A set S is said to be *simple* iff (a) S is recursively enumerable, (b) complement(S) is infinite and (c) complement(S) does not have any infinite recursively enumerable subset. A class of functions $\mathcal{C}$ is said to be an *r.e. class* iff either $\mathcal{C}$ is empty or there exists a recursive g such that $\{\varphi_{g(i)} \mid i \in N\} = \mathcal{C}$.

The quantifiers '$\overset{\infty}{\forall}$', '$\overset{\infty}{\exists}$' and '$\exists!$' mean 'for all but finitely many', 'there exist infinitely many,' and 'there exists a unique' respectively. The quantifier $\forall^j$ denotes 'for all but at most j'.

3 Learning Paradigms

For any partial function η and any natural number n such that, for each $x < n$, $\eta(x)\downarrow$, we let $\eta[n]$ denote the finite initial segment $\{(x, \eta(x)) \mid x < n\}$. Let $\text{INIT} = \{f[n] \mid f \in \mathcal{R} \wedge n \in N\}$. We let σ, τ and γ, with or without decorations, range over INIT.

Definition 1 [16] A *learning machine* is an algorithmic device which computes a mapping from INIT into $N \cup \{?\}$ such that, if $\mathbf{M}(f[n]) \neq ?$, then $\mathbf{M}(f[n+1]) \neq ?$.

We let $\mathbf{M}$, with or without decorations, range over learning machines. In Definition 1 above, '?' denotes the situation when $\mathbf{M}$ outputs "no conjecture" on some $\sigma \in$ INIT.

In Definition 2 below we spell out what it means for a learning machine on a function to converge in the limit.

Definition 2 Suppose $\mathbf{M}$ is a learning machine and f is a computable function. $\mathbf{M}(f)\downarrow$ (read: $\mathbf{M}(f)$ *converges*) just in case $(\exists i)(\stackrel{\infty}{\forall} n)$ $[\mathbf{M}(f[n]) = i]$. If $\mathbf{M}(f)\downarrow$, then $\mathbf{M}(f)$ is defined = the unique i such that $(\stackrel{\infty}{\forall} n)[\mathbf{M}(f[n]) = i]$, otherwise we say that $\mathbf{M}(f)$ diverges (written: $\mathbf{M}(f)\uparrow$).

3.1 Explanatory Function Identification

We now introduce a criteria for a learning machine to successfully infer a function.

Definition 3 [16, 5, 11] Let $a \in N \cup \{*\}$.

(a) $\mathbf{M}$ $\mathbf{Ex}^a$-*identifies* f (written: $f \in \mathbf{Ex}^a(\mathbf{M})$) just in case $(\exists i \mid \varphi_i =^a f)[\mathbf{M}(f)\downarrow = i]$.

(b) $\mathbf{Ex}^a = \{\mathcal{S} \mid (\exists \mathbf{M})[\mathcal{S} \subseteq \mathbf{Ex}^a(\mathbf{M})]\}$.

We sometimes write $\mathbf{Ex}$ for $\mathbf{Ex}^0$ including in the names of those learning classes introduced in later sections where '$\mathbf{Ex}^0$' is a proper substring of those names.

The notion of $\mathbf{Ex}^*$ identification is due to Blum and Blum [5]. For a given f and $\mathbf{M}$, we refer to each instance of the case, $? \neq \mathbf{M}(f[n]) \neq \mathbf{M}(f[n+1])$ as a *mind change* by $\mathbf{M}$ on f. Case and Smith [11] (see also [4]) introduce a refinement of the above notion of $\mathbf{Ex}$-identification by bounding the number of times a learning machine is allowed to change its mind before converging to a correct program for the function being learned. Definition 4 below describes this notion.

Definition 4 [11] Suppose $a, b \in N \cup \{*\}$.

(a) $\mathbf{M}$ $\mathbf{Ex}^a_b$-*identifies* f (written: $f \in \mathbf{Ex}^a_b(\mathbf{M})$) just in case $[(\exists i \mid \varphi_i =^a f)$ $(\stackrel{\infty}{\forall} n)[\mathbf{M}(f[n]) = i]$ $\wedge$ $\mathrm{card}(\{n \mid ? \neq \mathbf{M}(f[n]) \neq \mathbf{M}(f[n+1])\}) \leq b]$.

(b) $\mathbf{Ex}^a_b = \{\mathcal{C} \mid (\exists \mathbf{M})[\mathcal{C} \subseteq \mathbf{Ex}^a_b(\mathbf{M})]\}$.

Clearly, $\mathbf{Ex}$-identification is the same as $\mathbf{Ex}^0_*$-identification. We sometimes write $\mathbf{Ex}^a$ for $\mathbf{Ex}^a_*$ including in the names of those learning classes introduced in later sections where '$\mathbf{Ex}^a_*$' is a proper substring of those names.

We now define a (partial) function mindchange.

Definition 5 $\mathrm{mindchange}(\mathbf{M}, f[n]) = \mathrm{card}(\{m < n \mid ? \neq \mathbf{M}(f[m]) \neq \mathbf{M}(f[m+1])\})$.
$\mathrm{mindchange}(\mathbf{M}, f) = \mathrm{card}(\{n \mid ? \neq \mathbf{M}(f[n]) \neq \mathbf{M}(f[n+1])\})$.

3.2 Behaviorally Correct Identification

Case and Smith [11] introduced another infinite hierarchy of identification criteria which we describe below. "**Bc**" stands for *behaviorally correct*. Barzdin [2] essentially introduced the notion $\mathbf{Bc}^0$.

Definition 6 [11] Let $a \in N \cup \{*\}$.

(a) $\mathbf{M}$ $\mathbf{Bc}^a$-*identifies* f (written: $f \in \mathbf{Bc}^a(\mathbf{M})$) just in case $(\stackrel{\infty}{\forall} n)[\varphi_{\mathbf{M}(f[n])} =^a f]$.

(b) $\mathbf{Bc}^a = \{\mathcal{S} \mid (\exists \mathbf{M})[\mathcal{S} \subseteq \mathbf{Bc}^a(\mathbf{M})]\}$.

We sometimes write **Bc** for $\mathbf{Bc}^0$ including in the names of those learning classes introduced in later sections where '$\mathbf{Bc}^0$' is a proper substring of those names.

Theorem 1 just below states some of the basic hierarchy results about the $\mathbf{Ex}_b^a$ and $\mathbf{Bc}^a$ classes.

Theorem 1 *For all m, n*

(a) $\mathbf{Ex}_0^{m+1} - \mathbf{Ex}^m \neq \emptyset$,

(b) $\mathbf{Ex}_{n+1}^0 - \mathbf{Ex}_n^* \neq \emptyset$,

(c) $\mathbf{Ex}_0^* - \bigcup_{m \in N} \mathbf{Ex}^m \neq \emptyset$,

(d) $\mathbf{Ex}_*^0 - \bigcup_{n \in N} \mathbf{Ex}_n^* \neq \emptyset$,

(e) $\mathbf{Ex}^* \subset \mathbf{Bc}$,

(f) $\mathbf{Bc}^m \subset \mathbf{Bc}^{m+1}$,

(g) $\bigcup_{m \in N} \mathbf{Bc}^m \subset \mathbf{Bc}^*$, *and*

(h) $\mathcal{R} \in \mathbf{Bc}^*$.

Parts (a), (b), (c), (d), (f) and (g) are due to Case and Smith [11]. John Steel first observed that $\mathbf{Ex}^* \subseteq \mathbf{Bc}$ and the diagonalization in part (e) is due to Harrington and Case [11]. Part (h) is due to Harrington [11]. Blum and Blum [5] first showed that $\mathbf{Ex} \subset \mathbf{Ex}^*$. Barzdin [2] first showed that $\mathbf{Ex} \subset \mathbf{Bc}$.

3.3 Popperian Function Identification

The following definitions are based on Popper's refutability principle: incorrect programs computing total functions are *always* refutable.

Definition 7 $\mathbf{M}$ is *Popperian on* f iff, for all n such that $\mathbf{M}(f[n]) \neq ?$, $\varphi_{\mathbf{M}(f[n])} \in \mathcal{R}$. $\mathbf{M}$ is *Popperian on* $\mathcal{C}$ just in case it is Popperian on each $f \in \mathcal{C}$. We say that $\mathbf{M}$ is *Popperian* just in case it is Popperian on $\mathcal{T}$.

Definition 8

(a) We say that $\mathbf{M}$ $\mathbf{P}_{\mathcal{C}}\mathbf{Ex}^a_b$-*identifies* f (written $f \in \mathbf{P}_{\mathcal{C}}\mathbf{Ex}^a_b(\mathbf{M})$) just in case $\mathbf{M}$ is Popperian on $\mathcal{C}$ and $\mathbf{M}$ $\mathbf{Ex}^a_b$-identifies f.

(b) $\mathbf{P}_{\mathcal{C}}\mathbf{Ex}^a_b = \{\mathcal{S} \mid (\exists \mathbf{M})[\mathcal{S} \subseteq \mathbf{P}_{\mathcal{C}}\mathbf{Ex}^a_b(\mathbf{M})]\}$.

Definition 9

(a) We say that $\mathbf{M}$ $\mathbf{P}_{\mathcal{C}}\mathbf{Bc}^a$-*identifies* f (written $f \in \mathbf{P}_{\mathcal{C}}\mathbf{Bc}^a(\mathbf{M})$) just in case $\mathbf{M}$ is Popperian on $\mathcal{C}$ and $\mathbf{M}$ $\mathbf{Bc}^a$-identifies f.

(b) $\mathbf{P}_{\mathcal{C}}\mathbf{Bc}^a = \{\mathcal{S} \mid (\exists \mathbf{M})[\mathcal{S} \subseteq \mathbf{P}_{\mathcal{C}}\mathbf{Bc}^a(\mathbf{M})]\}$.

For $\mathbf{I} \in \{\mathbf{Ex}^a_b, \mathbf{Bc}^a\}$, we sometimes write $\mathbf{PI}$ for $\mathbf{P}_{\mathcal{T}}\mathbf{I}$. Besides its relation to Popper's refutability principle, $\mathbf{PEx}$ is a mathematically natural class with many characterizations and closure properties. The reader will see some of these characterizations in Section 4.

$\mathbf{PEx}$ above adheres closest to Popper's refutability principle. We consider some less restricted variations below.

Definition 10 $\mathbf{M}$ is $*$-*Popperian on* f iff, for all but finitely many n such that $\mathbf{M}(f[n]) \neq ?$, $\varphi_{\mathbf{M}(f[n])} \in \mathcal{R}$. $\mathbf{M}$ is $*$-*Popperian on* $\mathcal{C}$ just in case it is $*$-Popperian on each $f \in \mathcal{C}$. We say that $\mathbf{M}$ is $*$-Popperian just in case it is $*$-Popperian on $\mathcal{T}$.

Definition 11

(a) We say that $\mathbf{M}$ $\mathbf{P}^*_{\mathcal{C}}\mathbf{Ex}^a_b$-*identifies* f (written $f \in \mathbf{P}^*_{\mathcal{C}}\mathbf{Ex}^a_b(\mathbf{M})$) just in case $\mathbf{M}$ is $*$-Popperian on $\mathcal{C}$ and $\mathbf{M}$ $\mathbf{Ex}^a_b$-identifies f.

(b) $\mathbf{P}^*_{\mathcal{C}}\mathbf{Ex}^a_b = \{\mathcal{S} \mid (\exists \mathbf{M})[\mathcal{S} \subseteq \mathbf{P}^*_{\mathcal{C}}\mathbf{Ex}^a_b(\mathbf{M})]\}$.

Definition 12

(a) We say that $\mathbf{M}$ $\mathbf{P}^*_{\mathcal{C}}\mathbf{Bc}^a$-*identifies* f (written $f \in \mathbf{P}^*_{\mathcal{C}}\mathbf{Bc}^a(\mathbf{M})$) just in case $\mathbf{M}$ is $*$-Popperian on $\mathcal{C}$ and $\mathbf{M}$ $\mathbf{Bc}^a$-identifies f.

(b) $\mathbf{P}^*_{\mathcal{C}}\mathbf{Bc}^a = \{\mathcal{S} \mid (\exists \mathbf{M})[\mathcal{S} \subseteq \mathbf{P}^*_{\mathcal{C}}\mathbf{Bc}^a(\mathbf{M})\}]$.

In this paper we will be interested in the above definitions for $\mathcal{C} \in \{\mathcal{R}, \mathcal{T}\}$.

In the following definitions we relax the Popper refutability restriction in another direction. In these definitions we require a machine to output programs for total functions only on the graphs of functions which it identifies.

Definition 13

(a) We say that $\mathbf{M}$ $\mathbf{TEx}^a_b$-*identifies* f (written: $f \in \mathbf{TEx}^a_b(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{Ex}^a_b$-identifies f and $(\forall n)[\mathbf{M}(f[n]) =?\ \vee\ \varphi_{\mathbf{M}(f[n])} \in \mathcal{R}]$.

(b) $\mathbf{TEx}^a_b = \{\mathcal{C} \mid (\exists \mathbf{M})[\mathcal{C} \subseteq \mathbf{TEx}^a_b(\mathbf{M})]\}$.

Definition 14

(a) We say that $\mathbf{M}$ $\mathbf{TBc}^a$-*identifies* f (written: $f \in \mathbf{TBc}^a(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{Bc}^a$-identifies f and $(\forall n)[\mathbf{M}(f[n]) =?\ \vee\ \varphi_{\mathbf{M}(f[n])} \in \mathcal{R}]$.

(b) $\mathbf{TBc}^a = \{\mathcal{C} \mid (\exists \mathbf{M})[\mathcal{C} \subseteq \mathbf{TBc}^a(\mathbf{M})]\}$.

The following proposition is immediate from the definitions.

Proposition 1

(a) $\mathbf{PEx}^a_b \subseteq \mathbf{TEx}^a_b \subseteq \mathbf{Ex}^a_b$.

(b) $\mathbf{PEx}^a_b \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}^a_b \subseteq \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}^a_b \subseteq \mathbf{Ex}^a_b$.

(c) $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex} \subseteq \mathbf{P}^*_{\mathcal{R}}\mathbf{Bc}$.

(d) $\mathbf{P}^*_{\mathcal{T}}\mathbf{Ex} \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}$.

(e) $\mathbf{TEx}^a \subseteq \mathbf{TBc}^a$.

Which of the subset relations in Proposition 1 are proper is completely spelled out in Section 4.

4 Relating Different Popperian Classes

Following Theorem gives a characterization of **PEx**.

Theorem 2 [3, 11] $\mathcal{C} \in \mathbf{PEx}$ *iff there exists an r.e. class* $\mathcal{C}' \subseteq \mathcal{R}$, *such that* $\mathcal{C} \subseteq \mathcal{C}'$.

PROOF. Suppose $\mathcal{C} \in \mathbf{PEx}$. Let $\mathbf{M}$ be a Popperian machine which **Ex**-identifies each function in $\mathcal{C}$. Let $\mathcal{C}' = \{\varphi_{\mathbf{M}(\sigma)} \mid \mathbf{M}(\sigma) \neq ?\}$. It is easy to see that $\mathcal{R} \supseteq \mathcal{C}' \supseteq \mathcal{C}$ and $\mathcal{C}'$ is an r.e. class.

Suppose $\mathcal{C}' \neq \emptyset$ is an r.e. class. We construct a Popperian machine $\mathbf{M}$ such that $\mathcal{C}' \subseteq \mathbf{Ex}(\mathbf{M})$. Suppose $g \in \mathcal{R}$ is such that $\{\varphi_{g(i)} \mid i \in N\} = \mathcal{C}'$. Define $\mathbf{M}$ as follows.

$$\mathbf{M}(f[n]) = \min(\{n\} \cup \{i < n \mid (\forall x < n)[f(x) = \varphi_{g(i)}(x)]\})$$

It is easy to see that $\mathbf{M}$ is Popperian and that $\mathcal{C}' \subseteq \mathbf{Ex}(\mathbf{M})$. ∎

Using the above characterization of **PEx** we obtain the following closure property of **PEx**.

Theorem 3 $\mathbf{P}_{\mathcal{R}}\mathbf{Bc}^* = \mathbf{PEx}$.

PROOF. Clearly any machine which is Popperian on all the recursive functions is Popperian. Thus it suffices to show that $\mathbf{PBc}^* \subseteq \mathbf{PEx}$. Suppose $\mathbf{M}$ is Popperian and $\mathbf{M}$ $\mathbf{Bc}^*$-identifies $\mathcal{C}$. Let $\mathcal{C}' = \{\varphi_{\mathbf{M}(\sigma)} \mid \mathbf{M}(\sigma) \neq ?\}$. Clearly, $\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{R}$ and $\mathcal{C}'$ is an r.e. class. Thus, by Theorem 2, $\mathcal{C} \in \mathbf{PEx}$. ∎

The following characterization of **PEx** was first proved for **NV** (see Definition 19 in Section 5; it is proved in Section 6 that $\mathbf{NV} = \mathbf{PEx}$) by Barzdin and Freivalds [3] and independently by Adelman [5].

Theorem 4 $\mathbf{PEx} = \{\mathcal{C} \mid (\exists h \in \mathcal{R})(\forall f \in \mathcal{C})(\exists i)[\varphi_i = f \;\wedge\; (\stackrel{\infty}{\forall} x)[\Phi_i(x) \leq h(x)]]\}$.

Let $K = \{x \mid \varphi_x(x)\downarrow\}$. A class of functions $\mathcal{C}$ is said to be *r.e. in K class* iff either $\mathcal{C}$ is empty or there exists a function g, which can be computed relative to an oracle for K, such that $\mathcal{C} = \{\varphi_{g(i)} \mid i \in N\}$.

Theorem 5 [11] $\mathbf{PEx} = \{\mathcal{C} \mid \mathcal{C}$ *is contained in some r.e. in K class of recursive functions* $\}$.

The following theorem gives a closure property of **TEx**. The proof was obtained in collaboration with K. J. Chen.

Theorem 6 $\mathbf{TBc}^* = \mathbf{TEx}$.

PROOF. It suffices to show that $\mathbf{TBc}^* \subseteq \mathbf{TEx}$. Suppose $\mathbf{M}$ is given. We construct an $\mathbf{M}'$ such that $\mathbf{TBc}^*(\mathbf{M}) \subseteq \mathbf{TEx}(\mathbf{M}')$. Let $\mathbf{M}'$ be defined as follows. Let p_0 be a program for $\lambda x.0$.

$$\mathbf{M}'(f[n]) = \begin{cases} ?, & \text{if } \mathbf{M}(f[n]) =?; \\ \mathbf{M}(f[m]), & \text{if } (\exists j < n \mid \mathbf{M}(f[j]) \neq ?) \\ & [\text{card}(\{x < n \mid \Phi_{\mathbf{M}(f[j])}(x) < n \ \wedge \ \varphi_{\mathbf{M}(f[j])}(x) \neq f(x)\}) = 0], \text{ and} \\ & m = \min(\{j \mid \mathbf{M}(f[j]) \neq ? \ \wedge \\ & \text{card}(\{x < n \mid \Phi_{\mathbf{M}(f[j])}(x) < n \ \wedge \ \varphi_{\mathbf{M}(f[j])}(x) \neq f(x)\}) = 0\}); \\ p_0, & \text{otherwise.} \end{cases}$$

It is easy to verify that $\mathbf{M}'$ $\mathbf{TEx}$-identifies any function $\mathbf{TBc}^*$-identified by $\mathbf{M}$. ∎

The following theorem gives the closure properties for $\mathbf{P}^*_{\mathcal{C}}\mathbf{Ex}$, for $\mathcal{C} \in \{\mathcal{R}, \mathcal{T}\}$.

Theorem 7

(a) $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}^* = \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}$.

(b) $\mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}^* = \mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}$.

PROOF. Suppose $\mathbf{M}$ is given. Let $\text{converr}(f[n]) = \{x < n \mid \Phi_{\mathbf{M}(f[n])} < n \ \wedge \ \varphi_{\mathbf{M}(f[n])} \neq f(x)\}$. Let patch be a recursive function such that, for all i, x and finite set of pairs, D,

$$\varphi_{\text{patch}(i,D)}(x) = \begin{cases} y_m, & \text{if } (\exists y)[(x,y) \in D] \text{ and } y_m = \min(\{y \mid (x,y) \in D\}); \\ \varphi_i(x), & \text{otherwise.} \end{cases}$$

Define $\mathbf{M}'$ as follows.

If $\mathbf{M}(f[n]) =?$, then $\mathbf{M}'(f[n]) =?$; otherwise $\mathbf{M}'(f[n]) = \text{patch}(\mathbf{M}(f[n]), \{(x,y) \mid f(x) = y \ \wedge \ x \in \text{converr}(f[n])\})$. It is easy to see that if $\mathbf{M}$ is $*$-Popperian on $\mathcal{C}$, then so is $\mathbf{M}'$. Also, if for an f, $\varphi_{\mathbf{M}(f)} =^* f$ and $\varphi_{\mathbf{M}(f)} \in \mathcal{R}$, then $\varphi_{\mathbf{M}'(f)} = f$. ∎

We now proceed to determine the exact relationship between the different learning classes, defined in this paper, based on Popper's refutability principle.

Theorem 8 *For all n, $\mathbf{PEx}_{n+1} - \mathbf{Ex}^*_n \neq \emptyset$.*

PROOF. Let $\mathcal{C}_n = \{f \mid \text{card}(\{x \mid f(x) \neq f(x+1)\}) \leq n+1\}$. It is easy to see that $\mathcal{C}_n \in \mathbf{PEx}_{n+1}$. It was shown in [11] that $\mathcal{C}_n \notin \mathbf{Ex}^*_n$. ∎

The following theorem witnesses an interesting tradeoff. For Popperian classes, it is possible to completely tradeoff anomalies for mind change bounds. Surprising similar tradeoff results have also been found for team learning [29, 21] (see the discussion after Corollary 2 below), with teamsize in the role of mind changes.

Theorem 9 *Suppose m, n, m' are given. Let $n' = (1+n)\cdot(1+\lfloor\frac{m}{m'+1}\rfloor) - 1$. Then, for $\mathbf{I} \in \{\mathbf{PEx}, \mathbf{TEx}, \mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}, \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}\}$, $\mathbf{I}^m_n \subseteq \mathbf{I}^{m'}_{n'}$.*

PROOF. Suppose $\mathbf{M}$ $\mathbf{I}^m_n$-identifies $\mathcal{C}$. We give an $\mathbf{M}'$ which $\mathbf{I}^{m'}_{n'}$-identifies $\mathcal{C}$. If $\mathbf{M}(f[n]) =?$, then let $\mathrm{converr}(f[n]) = \emptyset$; otherwise let $\mathrm{converr}(f[n]) = \{x < n \mid \Phi_{\mathbf{M}(f[n])} < n \ \wedge\ \varphi_{\mathbf{M}(f[n])} \neq f(x)\}$. Let patch be a recursive function such that for all i, f, n,

$$\varphi_{\mathrm{patch}(i,f[n])}(x) = \begin{cases} f(x), & \text{if } x \in \mathrm{converr}(f[n]); \\ \varphi_i(x), & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}'$ be defined as follows.

$$\mathbf{M}'(f[n]) = \begin{cases} ?, & \text{if } \mathbf{M}(f[n]) =?; \\ \mathbf{M}(f[n]), & \text{if } n = 0 \text{ or } \mathbf{M}(f[n]) \neq \mathbf{M}(f[n-1]); \\ \mathrm{patch}(\mathbf{M}(f[n]), f[n]), & \text{if } n > 0 \text{ and } \mathbf{M}(f[n]) = \mathbf{M}(f[n-1]) \neq ? \text{ and} \\ & (\exists i \mid 0 < i \leq \lfloor\frac{m}{m'+1}\rfloor)[\mathrm{card}(\mathrm{converr}(f[n-1])) \\ & \quad < i\cdot(m'+1) \leq \mathrm{card}(\mathrm{converr}(f[n]))]; \\ \mathbf{M}'(f[n-1]), & \text{otherwise.} \end{cases}$$

It is easy to see that,

(a) if $\varphi_{\mathbf{M}(f[n])}$ is total, then so is $\varphi_{\mathbf{M}'(f[n])}$,
(b) if the number of mind changes by $\mathbf{M}$ on f is bounded by n, then the number of mind changes by $\mathbf{M}'$ on f is bounded by $(n+1)\cdot(1+\lfloor\frac{m}{m'+1}\rfloor) - 1$, and
(c) if $\varphi_{\mathbf{M}(f)} \in \mathcal{R}$ and $\varphi_{\mathbf{M}(f)} =^m f$, then $\varphi_{\mathbf{M}(f)} =^{m'} f$.

This completes the proof of the theorem. ∎

Definition 15 *f is a* STEP^m *function iff there exist i, $n_0, n_1, \ldots, n_i$, $S_0, S_1, \ldots, S_i$ such that the following conditions are satisfied.*

(a) $0 = n_0 \leq n_1 \leq n_2 \leq \ldots \leq n_i$.

(b) $n_j < \min(S_j), \max(S_j) < n_{j+1}$, *for $j \leq i$ (where n_{i+1} is taken to be ∞).*

(c) $\mathrm{card}(S_j) \leq m$, *for $j \leq i$.*

(d) $(\forall x \in \bigcup_{j\leq i} S_j)[f(x) = 0]$.

(e) $(\forall j \leq i)(\forall x \mid n_j \leq x < n_{j+1} \ \wedge\ x \notin S_j)[f(x) = j+1]$.

Definition 16 $\mathrm{STEP}^m(n) = \{f \mid f$ *is a* STEP^m *function and* $\max(\mathrm{range}(f)) \leq n\}$.

Definition 17 *For* $n > 0$, *let* $\text{STEPINIT}^m(n) = \{f[x] \mid x \in N \ \wedge\ f \in \text{STEP}^m(n) \ \wedge\ f(x) = n\}$. *Let* $\text{STEPINIT}^m(0) = \{\emptyset\}$.

Theorem 10 *Suppose* m, n, m' *are given. Then for* $n' = (1+n)\cdot(1+\lfloor\frac{m}{m'+1}\rfloor) - 2$, $\mathbf{PEx}_n^m - \mathbf{Ex}_{n'}^{m'} \neq \emptyset$.

PROOF. Clearly, $\text{STEP}^m(n+1) \in \mathbf{PEx}_n^m$. Suppose that $\mathbf{M}$ $\mathbf{Ex}^{m'}$-identifies $\text{STEP}^m(n+1)$.

Claim 1 *For all* $l > 0$, *for all* $\sigma \in \text{STEPINIT}^m(l)$, *there exists a* $\tau \in \text{STEPINIT}^m(l+1)$ *such that* $\sigma \subseteq \tau$ *and* $\text{mindchange}(\mathbf{M}, \tau) \geq (1+\lfloor\frac{m}{m'+1}\rfloor) + \text{mindchange}(\mathbf{M}, \sigma)$.
There exists a $\tau \in \text{STEPINIT}^m(1)$ *such that* $\text{mindchange}(\mathbf{M}, \tau) \geq (1+\lfloor\frac{m}{m'+1}\rfloor) - 1$.

The above claim follows immediately from the following easy to prove claim.

Claim 2 *Suppose* $l > 0$, $\sigma \in \text{STEPINIT}^m(l)$ *such that, there exists an* $i < \lfloor\frac{m}{m'+1}\rfloor$, $\text{card}(\{x \mid (x,0) \in \tau \ \wedge\ x > \min(\{y \mid (y,l) \in \sigma\})\}) = i\cdot(m'+1)$. *Then there exists a* $\tau \in \text{STEPINIT}^m(l)$ *such that,* $\sigma \subseteq \tau$, $\varphi_{\mathbf{M}(\tau)} =^* l$, $\mathbf{M}(\sigma) \neq \mathbf{M}(\tau)$ *and* $\text{card}(\{x \mid (x,0) \in \tau \ \wedge\ x > \min(\{y \mid (y,l) \in \sigma\})\}) = (i+1)\cdot(m'+1)$.

Now the existence of an $f \in \text{STEP}^m(n+1)$ such that $\text{mindchange}(\mathbf{M}, f) \geq (1+n)(1+\lfloor\frac{m}{m'+1}\rfloor) - 1$ follows from Claim 1. Thus $\text{STEP}^m(n+1) \notin \mathbf{Ex}_{n'}^{m'}$. ∎

We have the following corollary to the proof of Theorem 10.

Corollary 1 $\mathbf{PEx}_0^* - \bigcup_m \bigcup_n \mathbf{Ex}_n^m \neq \emptyset$.

As a corollary to Theorems 9 and 10 we have

Corollary 2 *Suppose* $[\mathbf{I} = \mathbf{PEx}$ *and* $\mathbf{J} \in \{\mathbf{PEx}, \mathbf{TEx}, \mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}, \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}, \mathbf{Ex}\}]$ *or* $[\mathbf{I} = \mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}$ *and* $\mathbf{J} \in \{\mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}, \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}, \mathbf{Ex}\}]$ *or* $[\mathbf{I} \in \{\mathbf{TEx}, \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}\}$ *and* $\mathbf{J} \in \{\mathbf{I}, \mathbf{Ex}\}]$. *Then* $\mathbf{I}_b^a \subseteq \mathbf{J}_{b'}^{a'}$ *iff the following two conditions are satisfied.*

(a) $b \leq b'$.

(b) $a' = * \ \vee\ b' = * \ \vee\ [a \in N \ \wedge\ \frac{b'+1}{b+1} \geq \lceil\frac{a+1}{a'+1}\rceil]$.

The inequality with fractions in part (b) of Corollary 2 is just an approximate *product tradeoff* between anomalies and mind changes. Anomalies can be traded for mind changes, but part (a) blocks tradeoffs in the other direction. We surprisingly see the exact same tradeoff formula in team learning [29, 21], an apparently quite different context. Somewhat similar tradeoff formulas for **Bc**-identification were observed in [12, 13].

Proposition 2 $\mathbf{TEx}_0^0 = \mathbf{Ex}_0^0$.

PROOF. Follows from the Definitions of the classes. ∎

Theorem 11 $\mathbf{TEx}_0 - \bigcup_m \mathbf{P}^*_{\mathcal{R}}\mathbf{Bc}^m \neq \emptyset$.

PROOF. Let $\mathcal{C} = \{f \mid \varphi_{f(0)} = f\}$. Clearly, $\mathcal{C} \in \mathbf{TEx}_0$. Suppose $\mathbf{M}$ and m are given. We show below that either $\mathbf{M}$ is not $*$-Popperian on $\mathcal{R}$ or there exists an $f \in \mathcal{C}$ which is not $\mathbf{Bc}^m$-identified by $\mathbf{M}$. Without loss of generality we assume that, for all σ, $\mathbf{M}(\sigma) \neq ?$. By the implicit use of the Kleene recursion theorem ([27, Page 214]), there exists an e such that the (partial) function φ_e may be described as follows.

Let x_s denote the least x such that $\varphi_e(x)$ has not been defined before stage s. Let $\varphi_e(0) = e$. Go to stage 0.

Begin stage s

1. Search for a $\sigma \in \mathrm{INIT}$, such that $\varphi_e[x_s] \subseteq \sigma$ and, for all $x \leq m$, $\varphi_{\mathbf{M}(\sigma)}(|\sigma| + x)\downarrow$.
2. If and when such a σ is found,

   for x such that $x_s \leq x < |\sigma|$, let $\varphi_e(x) = y$, where $(x, y) \in \sigma$ and
   for x such that $|\sigma| \leq x \leq |\sigma| + m$, let $\varphi_e(x) = \varphi_{\mathbf{M}(\sigma)}(x) + 1$.
3. Go to stage $s + 1$.

End stage s.

Now we consider the following cases.

Case 1: All stages terminate.

In this case clearly, $\varphi_e \in \mathcal{C}$. However, by the diagonalization at step 2, for infinitely many l, $\varphi_{\mathbf{M}(\varphi_e[l])} \neq^m \varphi_e$. Thus $\mathbf{M}$ does not $\mathbf{Bc}^m$-identify $\varphi_e \in \mathcal{C}$.

Case 2: Some stage s does not terminate.

In this case $\mathbf{M}$ does not output a program for a total function, on any extension of $\varphi_e[x_s]$. Thus $\mathbf{M}$ is not $*$-Popperian on $\mathcal{R}$.

From the above cases we have that $\mathcal{C} \notin \mathbf{P}^*_{\mathcal{R}}\mathbf{Bc}^m$. ∎

Corollary 3 $\mathbf{TEx}_0 - \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}^* \neq \emptyset$.

Theorem 12 $\mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}^0_0 - \mathbf{PEx} \neq \emptyset$.

PROOF. Let $\mathcal{C} = \{f \mid \varphi_{f(0)} = f \ \wedge \ (\forall x)[\Phi_{f(0)}(x) \leq f(x+1)]\}$. The concept behind this $\mathcal{C}$ is from [5]. It is easy to see that $\mathcal{C} \in \mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}^0_0$. Suppose by way of contradiction $\mathbf{M}$, a Popperian machine, is given, such that $\mathcal{C} \subseteq \mathbf{Ex}(\mathbf{M})$. Without loss of generality we assume that $\mathbf{M}$ is such that, for all σ, $\mathbf{M}(\sigma) \neq ?$. By the implicit use of Kleene's recursion theorem there exists an e such that the (partial) function φ_e may be described as follows.

$$\varphi_e(x) = \begin{cases} e, & \text{if } x = 0; \\ \Phi_e(x-1) + \varphi_{\mathbf{M}(\varphi_e[x])}(x) + 1, & \text{otherwise.} \end{cases}$$

Clearly, if $\mathbf{M}$ is Popperian, then $\varphi_e \in \mathcal{C}$. Moreover, it is easy to see that $\varphi_e \notin \mathbf{PEx}(\mathbf{M})$. ∎

Theorem 13 $\mathbf{Ex}^m_n \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$.

PROOF. Suppose $\mathbf{M}$ $\mathbf{Ex}^m_n$-identifies $\mathcal{C}$. Let p_0 be a program such that, for all x, $\varphi_{p_0}(x) = 0$. Let prog be a recursive function such that, for all i, f, l and x

$$\varphi_{\mathrm{prog}(i,f[l])}(x) = \begin{cases} f(x), & \text{if } x < l; \\ 0, & \text{if } \mathrm{card}(\{y < l \mid \Phi_i(y) > x\}) > m; \\ \varphi_i(x), & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}'$ be defined as follows.

$$\mathbf{M}'(f[l]) = \begin{cases} ?, & \text{if } \mathbf{M}(f[l]) = ?; \\ p_0, & \text{if } \mathrm{mindchange}(\mathbf{M}, f[l]) > n; \\ \mathrm{prog}(\mathbf{M}(f[l]), f[l]), & \text{otherwise.} \end{cases}$$

Clearly, $\mathbf{Ex}^m_n(\mathbf{M}) \subseteq \mathbf{Bc}^*(\mathbf{M}')$. Now we show that $\mathbf{M}'$ is $*$-Popperian on $\mathcal{T}$. Suppose f is a total function. It is easy to see that $\mathbf{M}'$ is $*$-Popperian on f by considering each of the following three cases.

(1) $\mathrm{mindchange}(\mathbf{M}, f) > n$,
(2) $\mathrm{mindchange}(\mathbf{M}, f) \leq n$ and $\mathrm{card}(\{x \mid \varphi_{\mathbf{M}(f)}(x)\uparrow\}) > m$,
(3) $\mathrm{mindchange}(\mathbf{M}, f) \leq n$ and $\mathrm{card}(\{x \mid \varphi_{\mathbf{M}(f)}(x)\uparrow\}) \leq m$.

Thus $\mathbf{Ex}^m_n \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$. ∎

Theorem 14 $\mathbf{TEx} \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$.

PROOF. This proof is very similar to the proof of Theorem 13. Suppose $\mathbf{M}$ $\mathbf{TEx}$-identifies $\mathcal{C}$. Without loss of generality assume that $\mathbf{M}$ is such that, for all σ, $\mathbf{M}(\sigma) \neq ?$. Define $\mathbf{M}'$ as follows. Let prog be a recursive function such that, for all f, l and x

$$\varphi_{\mathrm{prog}(f[l])}(x) = \begin{cases} f(x), & \text{if } x < l; \\ 0, & \text{if } (\exists l' < l)[\mathrm{card}(\{y < l \mid \Phi_{\mathbf{M}(f[l'])}(y) > x\}) > 0]; \\ \varphi_{\mathbf{M}(f[l])}(x), & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}'$ be defined as follows. $\mathbf{M}'(f[l]) = \text{prog}(f[l])$. Clearly, $\mathbf{TEx}(\mathbf{M}) \subseteq \mathbf{Bc}^*(\mathbf{M}')$. Now we show that $\mathbf{M}'$ is $*$-Popperian on $\mathcal{T}$. Suppose f is a total function. Now it is easy to see that $\mathbf{M}'$ is $*$-Popperian on f by considering the following two cases.

(1) $(\forall l)[\varphi_{\mathbf{M}(f[l])} \in \mathcal{R}]$.
(2) $\neg(\forall l)[\varphi_{\mathbf{M}(f[l])} \in \mathcal{R}]$.

Thus $\mathbf{TEx} \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$. ∎

Combining the ideas of the proofs of Theorems 13 and 14 it can be shown that

Theorem 15 $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}^*_n \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$.

Theorem 16 $\mathbf{Ex}^*_0 - \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^* \neq \emptyset$.

Proof. Let $\mathcal{C} = \{f \mid \varphi_{f(0)} =^* f\}$. Clearly, $\mathcal{C} \in \mathbf{Ex}^*_0$. Suppose by way of contradiction that $\mathbf{M}$ is $*$-Popperian on $\mathcal{T}$ and $\mathcal{C} \subseteq \mathbf{Bc}^*(\mathbf{M})$. Without loss of generality we assume that $\mathbf{M}$ is such that, for all σ, $\mathbf{M}(\sigma) \neq ?$. Then by the implicit use of Kleene's recursion theorem, there exists an e, such that the partial function φ_e may be described as follows.

We assume without loss of generality that Φ is such that, for all i, x, $\Phi_i(x) \geq x$. Let $\varphi_e(0) = e$. Let φ_e^s denote the part of φ_e defined before stage s. For each s, let $x_s^0 = 0$ and, for $i > 0$, $x_s^i = i$-th member of complement(domain(φ_e^s)) in increasing order. Let g_s denote the 0-extension of φ_e^s. Go to stage 0.

Begin stage s

1. **for** $i = \langle 0, 0\rangle$ **to** ∞ **do**
   Let j, k be such that $i = \langle j, k\rangle$.
   if $(\forall w \mid x_s^j \leq w \leq x_s^{j+1})[\Phi_{\mathbf{M}(g_s[w])}(x_s^{j+1}) \leq k]$ **then**
   Go to step 2.
   endif
   endfor
2. Let $\varphi_e(x_s^{j+1}) = 1 + \max(\{\varphi_{\mathbf{M}(g_s[w])(x_s^{j+1})} \mid x_s^j \leq w \leq x_s^{j+1}\})$.
   Go to stage $s+1$.

End stage s

Since $\mathbf{M}$ is $*$-Popperian on $\mathcal{T}$, all stages must terminate.

Let $x^i = \lim_{s\to\infty} x_s^i$. Now consider the following cases.

Case 1: For all but finitely many i, $x^i\uparrow$.

In this case clearly φ_e converges on all but finitely many inputs. Let $g =$ the 0-extension of φ_e. Clearly, $g \in \mathcal{C}$. Let i_l be the largest i such that $x^i\downarrow$. Let s be such that $x_s^{i_l} = x^{i_l}$. Now for all $s' > s$, $(\forall w \mid x^{i_l} \leq w \leq x_{s'}^{i_l+1})[\varphi_e(x_{s'}^{i_l+1})\downarrow \neq \varphi_{\mathbf{M}(g[w])}(x_{s'}^{i_l+1})\downarrow]$ (by the diagonalization at step 2 in some stage $s'' > s'$; this diagonalization must happen since $x^{i_l+1}\uparrow$).

Case 2: For each i, $x^i\downarrow$.

In this case it is easy to see that $\mathbf{M}$ cannot be $*$-Popperian on $\mathcal{T}$.

From the above cases it follows that $\mathcal{C} \notin \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$. ∎

Theorem 17 $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex} - \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^* \neq \emptyset$.

Proof. Suppose q is such that W_q is a simple set [27]. Let

$\mathcal{C} = \{f \in \mathcal{R} \mid$ The following two conditions are satisfied.

1. $\{x \mid f(3x) = 0\} \subseteq \text{complement}(W_q)$;
   (* *Note that this implies* $\{x \mid f(3x) = 0\}$ *is finite.* *)
2. For $y = \max(\{x \mid f(3x) = 0\})$ and $e = f(3y+1)$,
   $$\varphi_e = f \text{ and } (\forall z > y)[\max(\{\Phi_e(x) \mid x \leq 3z+1\}) \leq f(3z+2)].$$

$\}$

It is easy to see that $\mathcal{C} \in \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}$. Suppose $\mathbf{M}$ is given. Then we construct an f such that one of the following two properties is satisfied.

(a) $\{x \mid f(3x) = 0\}$ is infinite and $(\forall x)(\exists x' > x)[\varphi_{\mathbf{M}(f[x'])} \notin \mathcal{R}]$.
(b) $f \in \mathcal{C}$ and $f \notin \mathbf{Bc}^*(\mathbf{M})$.

It will immediately follow that $\mathcal{C} \notin \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$.

Without loss of generality we assume that $\mathbf{M}$ is such that, for all σ, $\mathbf{M}(\sigma) \neq ?$.

Now we proceed to construct an f as claimed above. We construct f in stages. x_s denotes the least x, if any, such that $f(x)$ has not been defined in any stages $< s$. x_s, if it exists, will be a multiple of 3. Note that f constructed may not be recursive. Execute stages $0, 1, 2, \ldots$.

Begin stage s

If f gets totally defined by stages $< s$, then this stage does not do anything; otherwise the following steps are executed.

0. By the implicit use of Kleene's recursion theorem, there exists an e_s, such that the (partial) function φ_{e_s} is defined as follows.

1. For $x < x_s$, let $\varphi_{e_s}(x) = f(x)$.

2. Let $f(x_s) = \varphi_{e_s}(x_s) = 0$, $f(x_s+1) = \varphi_{e_s}(x_s+1) = e_s$, $f(x_s+2) = \varphi_{e_s}(x_s+2) = \max(\{\Phi_{e_s}(x) \mid x \leq x_s+1\})+1$.

3. **for** $y = 1 + (x_s/3)$ **to** ∞ **do**

 if $(\forall w \mid x_s < w \leq 3y)[\varphi_{\mathbf{M}(\varphi_{e_s}[w])}(3y)\downarrow] \ \vee \ y \in W_q$ **then**

  if $\max(\{\Phi_{\mathbf{M}(\varphi_{e_s}[w])}(3y) \mid x_s < w \leq 3y\}) < \Phi_q(y)$ **then**

  3.1. Let $f(3y) = \varphi_{e_s}(3y) = 1 + \max(\{\varphi_{\mathbf{M}(\varphi_{e_s}[w])}(3y) \mid x_s < w \leq 3y\})$.

  else

   Let $f(3y) = \varphi_{e_s}(3y) = 1$.

  endif

  Let $f(3y+1) = \varphi_{e_s}(3y+1) = 0$.

  Let $f(3y+2) = \varphi_{e_s}(3y+2) = 1 + \max(\{\Phi_{e_s}(x) \mid x < 3y+2\})$.

 else φ_{e_s} does not get defined anymore.

 endif

endfor

End stage s

We now consider the following cases.

Case 1: Infinitely many stages are needed to define f.

In this case clearly, $\{x \mid f(3x) = 0\}$ is infinite and $(\forall x)(\exists x' > x)[\varphi_{\mathbf{M}(f[x'])} \notin \mathcal{R}]$.

Case 2: For some stage s, f gets completely defined in stage s.

Let s be such that, for all but finitely many x, $f(x)$ gets defined in stage s. Now by construction $\varphi_{e_s} = f \in \mathcal{C}$. But by the diagonalization in step 3.1 (which must happen infinitely often, since complement(W_q) is infinite), for all $w > x_s$, $\varphi_{\mathbf{M}(f[w])} \neq^* f$.

From the above cases it follows that $\mathcal{C} \not\subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$. ∎

As a Corollary we obtain

Corollary 4 *$\mathcal{R} \notin \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$.*

As a Corollary to Theorems 25 and 27 below, we have

Corollary 5 *$\mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}^0_1 - \mathbf{TEx} \neq \emptyset$.*

A slight modification to the proof of Theorem 9 can be used to show that

Theorem 18 $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}^m_0 \subseteq \mathbf{TEx}^{m'}_{n'}$, *for* $n' \geq \lfloor \frac{m}{m'+1} \rfloor$.

Theorem 19 *For all* n, $\mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^{n+1} - \mathbf{Bc}^n \neq \emptyset$.

PROOF. Fix n. Let $\mathbf{M}_0, \mathbf{M}_1, \ldots$ denote a recursive sequence of inductive inference machines such that $\mathbf{Bc}^n = \{\mathcal{C} \mid (\exists i)[\mathcal{C} \subseteq \mathbf{Bc}^n(\mathbf{M}_i)]\}$ (clearly, such recursive sequence of machines exist [20]). In [8, 9] a recursive function p is constructed such that, for each i, the following three conditions are satisfied.

(A) $(\exists j)[\varphi_{p(\langle i,j\rangle)}$ is total $]$.

(B) $(\forall j,k \mid j \leq k)[\mathrm{domain}(\varphi_{p(\langle i,k\rangle)}) = \emptyset \ \vee \ \mathrm{domain}(\varphi_{p(\langle i,j\rangle)}) \subseteq \mathrm{domain}(\varphi_{p(\langle i,k\rangle)})]$.

(C) $\{f \in \mathcal{R} \mid f(0) = i \ \wedge \ (\forall j)[\mathrm{card}(\{x \mid \varphi_{p(\langle i,j\rangle)}(x)\downarrow \neq f(x)\}) \leq n+1]\} \not\subseteq \mathbf{Bc}^n(\mathbf{M}_i)$.

Let $\mathcal{C}_i = \{f \in \mathcal{R} \mid f(0) = i \ \wedge \ (\forall j)[\mathrm{card}(\{x \mid \varphi_{p(\langle i,j\rangle)}(x)\downarrow \neq f(x)\}) \leq n+1]\}$.

Note that conditions (A) and (B) imply that, for each i,

$[(\overset{\infty}{\forall} k)[\mathrm{domain}(\varphi_{p(\langle i,k\rangle)}) = \emptyset \ \vee \ \varphi_{p(\langle i,k\rangle)}$ is total $] \ \wedge$
$[\mathrm{card}(\{k \mid \mathrm{domain}(\varphi_{p(\langle i,k\rangle)}) \neq \emptyset\}) < \infty \Rightarrow \varphi_{p(\langle i, \max(\{k \mid \mathrm{domain}(\varphi_{p(i,k)}) \neq \emptyset\})\rangle)}$ is total $]]$.

Thus, for all f,

$$f \in \mathcal{C}_i \Rightarrow (\overset{\infty}{\forall} m)[\varphi_{p(i, \max(\{k<m \mid \Phi_{p(\langle i,k\rangle)}(0)<m\}))} =^{n+1} f].$$

We take $\mathcal{C} = \bigcup_{i \in N} \mathcal{C}_i$. By condition (C) above $\mathcal{C} \notin \mathbf{Bc}^n$. Now define $\mathbf{M}$ as follows.

$\mathbf{M}(f[0]) = 0$. $\mathbf{M}(f[l]+1) = p(\langle f(0), \max(\{j < l \mid \Phi_{p(\langle f(0),j\rangle)}(0) \leq l\})\rangle)$. It is easy to verify that $\mathbf{M}$ is $*$-Popperian on $\mathcal{T}$ and $\mathcal{C} \in \mathbf{Bc}^{n+1}(\mathbf{M})$. ∎

A proof similar to that of Theorem 19 can be used to show that

Theorem 20 $\mathbf{P}^*_{\mathcal{T}}\mathbf{Bc} - \mathbf{Ex}^* \neq \emptyset$.

The following theorem was obtained in collaboration with Chen.

Theorem 21 $\mathcal{R} \in \mathbf{P}^*_{\mathcal{R}}\mathbf{Bc}^*$.

PROOF. Define $\mathbf{M}$ as follows. $\mathbf{M}(f[n]) = i$ such that φ_i is defined as follows.

Begin φ_i on input y

Go to stage 0.

Begin stage s

Let $S = \{j < n \mid (\forall x < n)[\Phi_j(x) \leq y + s \ \wedge \ \varphi_j(x) = f(x)]\}$.

if $S = \emptyset$, **then** go to stage $s + 1$

else

if $\Phi_{\min(S)}(y) \leq s$, **then** output $\varphi_{\min(S)}(y)$

else go to stage $s + 1$.

endif

endif

End stage s

End

Now suppose $f \in \mathcal{R}$. Let $p = \min(\{j \mid \varphi_j = f\})$. Let $l > p$ be such that $(\forall j < p)(\exists x < l)[\varphi_j(x) \neq f(x)]$. Now consider $\varphi_{\mathbf{M}(f[t])}$, for $t > l$. It is easy to see that $\varphi_{\mathbf{M}(f[t])}$ is total. Let $t' = \max(\{\Phi_p(x) \mid x < t\})$. Now, for all $y > t'$, $\varphi_{\mathbf{M}(f[t])}(y) = f(y)$. Thus, $\mathbf{M}$ $\mathbf{P}^*_{\mathcal{R}}\mathbf{Bc}^*$-identifies $\mathcal{R}$. ∎

Theorem 22 *For each m, $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}^0_0 - \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^m \neq \emptyset$.*

Proof. Fix m. Let $\mathbf{M}_0$, $\mathbf{M}_1$, $\ldots$ be a recursive enumeration of IIMs such that $\mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^m = \{\mathcal{C} \mid (\exists i)[\mathcal{C} \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^m(\mathbf{M}_i)]\}$.

Without loss of generality, we assume that, Φ is such that, for each j, x, $\Phi_j(x) \geq 1$. Let $\text{cumcomp}(i, x) = \Sigma_{y \leq x}\Phi_i(y)$. Note that, given i, x, z, it can be effectively determined if $\text{cumcomp}(i, x) \geq z$.

We will give recursive functions p (of one argument) and g (of three arguments) such that, for each e, the following properties hold.

(A) $\varphi_{p(e)}(e) = 1$ and, for each $x < e$, $\varphi_{p(e)}(x) = 0$.
(B) g is a monotone increasing function in its third argument.
(C) $(\forall j)(\forall^{j+1} x > \max(\{e, 2j\}))[(\exists y \leq x)[\varphi_j(y)\downarrow \neq \varphi_{p(e)}(y)\downarrow] \ \vee \ [\Phi_{p(e)}(x) \leq g(e, x, \text{cumcomp}(i, x))]]$.
(D) $\varphi_{p(e)}$ is either total or $\text{card}(\{x \mid \varphi_{p(e)}(x)\uparrow\}) = \infty$.
(E) $\mathbf{M}_e$ is $*$-Popperian on $\mathcal{T} \Rightarrow [\varphi_{p(e)} \in \mathcal{R} \ \wedge \ \varphi_{p(e)} \notin \mathbf{Bc}^m(\mathbf{M}_e)]$.

Let $\mathcal{C} = \{\varphi_{p(e)} \mid \varphi_{p(e)} \in \mathcal{R}\}$.

We will construct g and p as above later. First we show that $\mathcal{C} \in \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}_0$.

Define $\mathbf{M}$ as follows. Let p_0 be such that $(\forall x)[\varphi_{p_0}(x) = 0]$.

$$\mathbf{M}(f[n]) = \begin{cases} ?, & \text{if } (\forall x < n)[f(x) = 0]; \\ p(e), & \text{if } e < n \ \wedge \ f(e) = 1 \ \wedge \ (\forall x < e)[f(x) = 0] \ \wedge \\ & \neg(\exists x < n)[\Phi_{p(e)}(x) < n \ \wedge \ \varphi_{p(e)}(x) \neq f(x)] \ \wedge \\ & \neg(\exists j \leq n)(\exists x_0, \ldots, x_{j+1} \mid \max(\{e, 2j\}) < x_0 < x_1 < \ldots < x_{j+1} < n) \\ & [(\forall y \leq x_{j+1})[\Phi_j(y) \leq n \ \wedge \ \varphi_j(y) = f(y)] \ \wedge \\ & (\forall r \leq j+1)[\Phi_{p(e)}(x_r) > g(e, x_r, \text{cumcomp}(j, x_r))]]; \\ p_0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathbf{M}$ is $*$-Popperian on $\mathcal{R}$ (using properties of p and g discussed above) and $\mathbf{Ex}_0$-identifies each $f \in \mathcal{C}$. Moreover by clause (E), in the properties of p and g mentioned above, $\mathcal{C} \notin \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^m$.

We now proceed to construct p and g as claimed above. p and g are constructed by implicit use of operator recursion theorem [7]. For a given e, we describe below $\varphi_{p(e)}$ and $g(e, \cdot, \cdot)$ in stages. Let $\varphi_{p(e)}(x) = 0$, for $x < e$. Let $\varphi_{p(e)}(e) = 1$. Let $\varphi^s_{p(e)}$ denote the finite portion of $\varphi_{p(e)}$ which is defined before stage s. Let cancel $= \emptyset$. Go to stage 0.

Begin stage s

1. For $t < s$, let $g(e, s, t) = t$.
2. For $x \leq s$, such that $\varphi_{p(e)}(x)$ has been defined till now, let $g(e, x, s) = g(e, x, s-1) + 1$ (if $s = 0$, then let $g(e, 0, 0) = 0$).
3. **for** $x = 0$ **to** s **do**

    3.1. **if** $x \notin \text{domain}(\varphi^s_{p(e)})$ **then**

        3.2. Let $X = \{j \mid [j \notin \text{cancel}] \ \wedge \ x > \max(\{e, 2j\}) \ \wedge \ \text{cumcomp}(j, x) \leq s\}$.

        3.3. If $X \neq \emptyset$ **then**

            3.4. Let $j = \min(X)$.

            3.5. Let $\varphi_{p(e)}(x) = 1 \dot{-} \varphi_j(x)$.

            3.6. Let cancel $=$ cancel $\cup \{j\}$.

            3.7. Let $g(e, x, s) = \Phi_{p(e)}(x) + 1$.

            endif

        endif

    endfor

4. Let f be the 0-extension of $\varphi_{p(e)}$ defined till now.
5. **if** there exist l, $x_0, x_1, \ldots, x_m$ such that the following three conditions are satisfied.

    (a) $l = x_0 < x_1 < x_2 < \ldots < x_m < s$.

    (b) For $r \leq m$, $\varphi_{p(e)}(x_r)$ has not been defined till now.

(c) $\Phi_{\mathbf{M}(f[l])}(x_r) \leq s$, for $r \leq m$.

then

5.1. For $x < l$ such that $\varphi_{p(e)}$ has not been defined till now

let $\varphi_{p(e)}(x) = 0$, and

$g(e, x, s) = \Phi_{p(e)}(x) + s$.

5.2. For $r \leq m$

let $\varphi_{p(e)}(x_r) = 1 \dot{-} \varphi_{\mathbf{M}(f[l])}(x_r)$, and

$g(e, x_r, s) = \Phi_{p(e)}(x_r) + s$.

endif

For each $x \leq s$, such that $\varphi_{p(e)}$ has not been defined till now, let $g(e, x, s) = s$.

Go to stage $s + 1$

End stage s

We now show that g and p satisfy the properties (A)-(E) claimed above. Properties (A) and (B) are immediate from the construction. For property (C) note that, for each j and $x > \max(\{e, 2j\})$, if cumcomp$(j, x) \leq s$ and $\neg(\exists y < x)[\varphi_{p(e)}(y)\downarrow \neq \varphi_j(y)\downarrow]$, then $j \in X -$cancel at step 3.2 in the for loop iteration (at step 3), for x, in stage s. But j can be in $X -$ cancel at step 3.2 for at most $j + 1$ of the x's. Thus (C) holds. If $\mathbf{M}_e$ is $*$-Popperian on $\mathcal{T}$, then the if clause at step 5, succeeds for infinitely many l. Thus $\varphi_{p(e)}$ is total. Thus (E) holds. If $\varphi_{p(e)}$ is not total, then the if clause at step 5 succeeds for only finitely many l. But then $\varphi_{p(e)}$ must diverge on infinitely many inputs since $\varphi_{p(e)}$ can be defined on at most j inputs $\leq \max(\{e, 2j\})$ due to step 3. Thus (D) holds. ∎

5 Prediction Paradigms

Definition 18 A *prediction machine* is an algorithmic device which computes a partial mapping from INIT into N.

We let $\mathcal{M}$, with or without decorations, range over prediction machines.

Definition 19 [1, 5]

(a) $\mathcal{M}$ **NV**-*identifies* a function f (written: $f \in \mathbf{NV}(\mathcal{M})$) iff $[(\forall\sigma)[\mathcal{M}(\sigma)\downarrow] \wedge (\stackrel{\infty}{\forall} n)[\mathcal{M}(f[n]) = f(n)]]$.

(b) $\mathbf{NV} = \{\mathcal{C} \mid (\exists\mathcal{M})[\mathcal{C} \subseteq \mathbf{NV}(\mathcal{M})]\}$.

Definition 20 [3]

(a) $\mathcal{M}$ $\mathbf{NV}'$-*identifies* a function f (written: $f \in \mathbf{NV}'(\mathcal{M})$) iff $[(\forall n)[\mathcal{M}(f[n])\downarrow] \wedge (\overset{\infty}{\forall} n)[\mathcal{M}(f[n]) = f(n)]]$.

(b) $\mathbf{NV}' = \{\mathcal{C} \mid (\exists \mathcal{M})[\mathcal{C} \subseteq \mathbf{NV}'(\mathcal{M})]\}$.

Definition 21 [22]

(a) $\mathcal{M}$ $\mathbf{NV}''$-*identifies* a function f (written: $f \in \mathbf{NV}''(\mathcal{M})$) iff $(\overset{\infty}{\forall} n)[\mathcal{M}(f[n]) = f(n)]$.

(b) $\mathbf{NV}'' = \{\mathcal{C} \mid (\exists \mathcal{M})[\mathcal{C} \subseteq \mathbf{NV}''(\mathcal{M})]\}$.

6 Relating Prediction Classes with Identification Classes

The Theorem below was first pointed out to us by Jan van Leeuwen. It has been independently noted by Barzdin.

Theorem 23 $\mathbf{NV} = \mathbf{PEx}$.

PROOF. Suppose $\mathcal{C} \in \mathbf{PEx}$ and $\mathcal{C} \neq \emptyset$. Let g be a recursive function such that $\mathcal{C} \subseteq \{\varphi_{g(i)} \mid i \in N\} \subseteq \mathcal{R}$ (by Theorem 2 such a g exists). Define a predicting machine $\mathcal{M}$ as follows. Let $\mathrm{pmin}(f[l]) = \min(\{l\} \cup \{i < l \mid (\forall x < l)[\varphi_{g(i)}(x) = f(x)]\})$. Let $\mathcal{M}(f[l]) = \varphi_{g(\mathrm{pmin}(f[l]))}(l)$. It is easy to see that $\mathcal{M}$ $\mathbf{NV}$-identifies $\mathcal{C}$. Thus $\mathbf{PEx} \subseteq \mathbf{NV}$.

Suppose $\mathcal{M}$ $\mathbf{NV}$-identifies $\mathcal{C}$. Let g be a recursive function such that, for all f, l and x,

$$\varphi_{g(f[l])}(x) = \begin{cases} f(x), & \text{if } x < l; \\ \mathcal{M}(\varphi_{g(f[l])}[x]), & \text{otherwise.} \end{cases}$$

Now suppose $\mathcal{M}$ $\mathbf{NV}$-identifies f. Let l be such that, for all $x \geq l$, $\mathcal{M}(f[x]) = f(x)$. Then $\varphi_{g(f[l])} = f$. Thus $\mathbf{NV}(\mathcal{M}) \subseteq \{\varphi_{g(f[l])} \mid f \in \mathcal{R} \wedge l \in N\}$. It follows by Theorem 2 that $\mathbf{NV}(\mathcal{M}) \in \mathbf{PEx}$. ∎

Theorem 24 [22] $\mathbf{NV}'' = \mathbf{Bc}$.

PROOF. Suppose $\mathbf{M}$ is given. Without loss generality assume that $\mathbf{M}$ is such that, for all σ, $\mathbf{M}(\sigma) \neq ?$. Define $\mathcal{M}$ as follows. $\mathcal{M}(f[l]) = \varphi_{\mathbf{M}(f[l])}(l)$. It is easy to see that $\mathbf{Bc}(\mathbf{M}) \subseteq \mathbf{NV}''(\mathcal{M})$.

Suppose $\mathcal{M}$ is given. Let g be a recursive function such that for all f, l and x,

$$\varphi_{g(f[l])}(x) = \begin{cases} f(x), & \text{if } x < l; \\ \mathcal{M}(\varphi_{g(f[l])}[x]), & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}(f[l]) = g(f[l])$. It is easy to see that $\mathbf{NV}''(\mathcal{M}) \subseteq \mathbf{Bc}(\mathbf{M})$. ∎

Theorem 25 $\mathbf{TEx} \subseteq \mathbf{NV}'$.

Proof. Suppose $\mathbf{M}$ is given. Without loss generality assume that $\mathbf{M}$ is such that, for all σ, $\mathbf{M}(\sigma) \neq ?$. Define $\mathcal{M}$ as follows. $\mathcal{M}(f[l]) = \varphi_{\mathbf{M}(f[l])}(l)$. It is easy to see that $\mathbf{TEx}(\mathbf{M}) \subseteq \mathbf{NV}'(\mathcal{M})$. ∎

Theorem 26 [22] $\mathbf{NV}' \subseteq \mathbf{Ex}$.

Proof. Suppose $\mathcal{M}$ is given. We give a machine $\mathbf{M}$ which may diverge on some inputs. However, for all $f \in \mathbf{NV}'(\mathcal{M})$, there exists i such that $\varphi_i = f$ and $(\overset{\infty}{\forall} l)[\mathbf{M}(f[l]) = i]$. It is easy to extend [20] $\mathbf{M}$ to a total machine $\mathbf{M}'$ which $\mathbf{Ex}$-identifies each function $\mathbf{NV}'$-identified by $\mathcal{M}$. Let g be a recursive function such that for all f, l and x,

$$\varphi_{g(f[l])}(x) = \begin{cases} f(x), & \text{if } x < l; \\ \mathcal{M}(\varphi_{g(f[l])}[x]), & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}(f[n])$ be defined as follows. If $(\exists l < n)[\mathcal{M}(f[l])\uparrow]$, then $\mathbf{M}(f[n])$ diverges; otherwise, let $m = \min(\{x \mid (\forall y \mid x \leq y < n)[\mathcal{M}(f[y]) = f(y)]\})$, and then let $\mathbf{M}(f[n]) = g(f[m])$. It is easy to verify that $\mathbf{M}$ satisfies the properties claimed. ∎

Traces of the trick used by Jun Tarui to prove Corollary 8 below are used in the proof of the following theorem.

Theorem 27 $\mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}_1 - \mathbf{NV}' \neq \emptyset$.

Proof. Define Dips as follows [11]. $\text{Dips}(f) = \{x + 1 \mid f(x) > f(x+1)\}$. Let

$\mathcal{C} = \{f \mid$

$[\text{Dips}(f) = \emptyset \ \wedge \ \varphi_{f(0)} = f \ \wedge \ (\forall x)[\Phi_{f(0)}(x) \leq f(x+1)]] \ \vee$

$[\text{Dips}(f) = \{n\} \ \wedge \ (\forall x < n-1)[\Phi_{f(0)}(x) \leq f(x+1)] \ \wedge \ \varphi_{f(n)} = f \ \wedge \ (\forall x \geq n)[\Phi_{f(n)}(x) \leq f(x+1)]]$

$\}$

It easy to see that $\mathcal{C} \in \mathbf{P}^*_{\mathcal{T}}\mathbf{Ex}^0_1$. We show that $\mathcal{C} \notin \mathbf{NV}'$. Suppose $\mathcal{M}$ is given. We construct an f such that $f \notin \mathbf{NV}'(\mathcal{M})$. By implicit use of Kleene's recursion theorem there exists an e such that the (partial) function φ_e may be described as follows.

$$\varphi_e(x) = \begin{cases} e, & \text{if } x = 0; \\ \mathcal{M}(\varphi_e[x-1])(x) + \varphi_e(x-1) + \Phi_e(x-1) + 1, & \text{otherwise.} \end{cases}$$

Now consider the following cases.

Case 1: φ_e is total.

In this case clearly, $\varphi_e \in \mathcal{C}$ and $\mathcal{M}$ does not $\mathbf{NV}'$-identify φ_e.

Case 2: φ_e is not total.

Let x_u be the least x such that $\varphi_e(x)$ is not defined. By implicit use of recursion theorem there exists an e' such that the following holds.

$$\varphi_{e'}(x) = \begin{cases} \varphi_e(x), & \text{if } x < x_u; \\ \Phi_e(x_u - 1) + e' + \varphi_e(x_u - 1) + 1, & \text{if } x = x_u; \\ e', & \text{if } x = x_u + 1; \\ 1 + \Phi_{e'}(x-1) + \varphi_{e'}(x-1), & \text{otherwise.} \end{cases}$$

It is easy to see that $\varphi_{e'} \in \mathcal{C}$. Also, since $\mathcal{M}(\varphi_{e'}[x_u])\uparrow$, $\varphi_{e'} \notin \mathbf{NV}'(\mathcal{M})$.

From the above cases it follows that $\mathcal{C} \notin \mathbf{NV}'$. ∎

Since $\mathbf{TEx} \subseteq \mathbf{NV}'$, from Theorems 10 and 11 we have the following corollaries.

Corollary 6 $\mathbf{NV}' - \mathbf{Ex}^*_n \neq \emptyset$.

Corollary 7 $\mathbf{NV}' - \bigcup_m \mathbf{P}^*_{\mathcal{R}}\mathbf{Bc}^m \neq \emptyset$.

Theorem 28 $\mathbf{NV}' - \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^* \neq \emptyset$.

PROOF. Suppose q is such that W_q is a simple set [27]. Let

$\mathcal{C} = \{f \in \mathcal{R} \mid$ The following two conditions are satisfied.

1. $\{x \mid f(3x) = 0\} \subseteq \text{complement}(W_q)$;

   (Note that this implies* $\{x \mid f(3x) = 0\}$ *is finite. *)*

2. For $y = \max(\{x \mid f(3x) = 0\})$ and $e = f(3y+1)$,

   $\varphi_e = f \wedge$

   $(\forall w < y \mid f(3w) = 0)[\text{domain}(\varphi_{f(3w+1)}) \supseteq \{x \mid x \leq \min(\{3w'+2 \mid w' > w \wedge f(3w') = 0\})\}]$.

$\}$

It is easy to see that $\mathcal{C} \in \mathbf{NV}'$. The diagonalization just below is a modification of the diagonalization in the proof of Theorem 17. Suppose $\mathbf{M}$ is given. Then we construct an f such that one of the following two properties is satisfied.

(a) $\{x \mid f(3x) = 0\}$ is infinite and $(\forall x)(\exists x' > x)[\varphi_{\mathbf{M}(f[x'])} \notin \mathcal{R}]$.

(b) $f \in \mathcal{C}$ and $f \notin \mathbf{Bc}^*(\mathbf{M})$.

It will immediately follow that $\mathcal{C} \notin \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$.

Without loss of generality we assume that $\mathbf{M}$ is such that, for all σ, $\mathbf{M}(\sigma) \neq ?$.

Now we proceed to construct an f as claimed above. We construct f in stages. x_s denotes the least x, if any, such that $f(x)$ has not been defined in stages $< s$. x_s, if it exists, will be a multiple of 3. Note that f constructed may not be recursive. Execute stages 0, 1, 2, … .

Begin stage s

If f gets totally defined by stages $< s$, then this stage does not do anything; otherwise following steps are executed.

0. By implicit use of Kleene's recursion theorem there exists an e_s such that the (partial) function φ_{e_s} may be defined as follows.
1. For $x < x_s$, let $\varphi_{e_s}(x) = f(x)$.
2. Let $f(x_s) = \varphi_{e_s}(x_s) = 0$, $f(x_s+1) = \varphi_{e_s}(x_s+1) = e_s$, $f(x_s+2) = \varphi_{e_s}(x_s+2) = 1 + \max(\{\Phi_{e_s}(x) \mid x \leq x_s+1\})$.

   Let $\varphi_{e_s}(x_s+3) = 1$, $\varphi_{e_s}(x_s+4) = 0$, $\varphi_{e_s}(x_s+5) = 1 + \max(\{\Phi_{e_s}(x) \mid x \leq x_s+4\})$.
3. **for** $y = 1 + (x_s/3)$ **to** ∞ **do**

   if $(\forall w \mid x_s < w \leq 3y)[\varphi_{\mathbf{M}(\varphi_{e_s}[w])}(3y+3)\downarrow] \ \vee \ y \in W_q$ **then**

   if $\max(\{\Phi_{\mathbf{M}(\varphi_{e_s}[w])}(3y+3) \mid x_s < w \leq 3y\}) < \Phi_q(y)$ **then**

   3.1. Let $\varphi_{e_s}(3y+3) = 1 + \max(\{\varphi_{\mathbf{M}(\varphi_{e_s}[w])}(3y+3) \mid x_s < w \leq 3y\})$.

   else

   Let $\varphi_{e_s}(3y+3) = 1$.

   endif

   Let $\varphi_{e_s}(3y+4) = 0$.

   Let $\varphi_{e_s}(3y+5) = 1 + \max(\{\Phi_{e_s}(x) \mid x < 3y+5\})$.

   Let $f(3y+x) = \varphi_{e_s}(3y+x)$, for $x \leq 2$.

   else φ_{e_s} does not get defined anymore.

   endif

   endfor

End stage s

We now consider the following cases.

Case 1: Infinitely many stages are needed to define f.

In this case clearly, $\{x \mid f(3x) = 0\}$ is infinite and $(\forall x)(\exists x' > x)[\varphi_{\mathbf{M}(f[x'])} \notin \mathcal{R}]$.

Case 2: For some stage s, f gets completely defined in stage s.

Let s be such that, for all but finitely many x, $f(x)$ gets defined in stage s. Now by construction $\varphi_{e_s} = f \in \mathcal{C}$. But by the diagonalization in step 3.1 (which must happen infinitely often, since complement(W_q) is infinite), for all $w > x_s$, $\varphi_{\mathbf{M}(f[w])} \neq^* f$.

From the above cases it follows that $\mathcal{C} \not\subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$. ∎

Corollary 8 (Jun Tarui) $\mathbf{NV}' - \mathbf{TEx} \neq \emptyset$.

7 Reliable Function Identification

Definition 22 [5, 19] An IIM is $\mathbf{Ex}^a$-*reliable* on $\mathcal{C}$ just in case $(\forall f \in \mathcal{C})[\mathbf{M}(f)\downarrow \Rightarrow \varphi_{\mathbf{M}(f)} =^a f]$. We say that $\mathbf{M}$ is reliable on $\mathcal{C}$ if it is $\mathbf{Ex}$-reliable on $\mathcal{C}$.

Definition 23

(a) We say that $\mathbf{M}$ $\mathbf{R}_{\mathcal{C}}\mathbf{Ex}^a_b$-*identifies* f (written $f \in \mathbf{R}_{\mathcal{C}}\mathbf{Ex}^a_b(\mathbf{M})$) just in case $\mathbf{M}$ is $\mathbf{Ex}^a$-reliable on $\mathcal{C}$ and $\mathbf{M}$ $\mathbf{Ex}^a_b$-identifies f.

(b) $\mathbf{R}_{\mathcal{C}}\mathbf{Ex}^a_b = \{\mathcal{S} \mid (\exists \mathbf{M})[\mathcal{S} \subseteq \mathbf{R}_{\mathcal{C}}\mathbf{Ex}^a_b(\mathbf{M})]\}$.

In this paper, regarding reliable identification, we will be interested only in the classes $\mathbf{R}_{\mathcal{R}}\mathbf{Ex}^a_b$ and $\mathbf{R}_{\mathcal{T}}\mathbf{Ex}^a_b$.

Intuitively a machine is reliable if it doesn't converge on functions it fails to identify. [5] gives a characterization of $\mathbf{R}_{\mathcal{R}}\mathbf{Ex}$-identification. In the proofs of Theorem 22 above and Theorem 41 below we employ some the ideas behind this characterization from [5].

Proposition 3 $\mathbf{R}_{\mathcal{T}}\mathbf{Ex}^a_b \subseteq \mathbf{R}_{\mathcal{R}}\mathbf{Ex}^a_b \subseteq \mathbf{Ex}^a_b$.

8 Relating Popperian Classes with Reliable Classes

In [14, 15] it was shown that $\mathbf{R}_{\mathcal{R}}\mathbf{Ex}^{n+1} - \mathbf{Ex}^n \neq \emptyset$. We extend the theorem to the following.

Theorem 29 *For all n,* $\mathbf{R}_{\mathcal{T}}\mathbf{Ex}^{n+1}_0 - \mathbf{Ex}^n \neq \emptyset$.

Proof. In [9, 8] a recursive function p was constructed such that, for each i, $\varphi_{p(i)}(0) = i$, card$(\{x \mid \varphi_{p(i)}(x)\uparrow\}) \leq n+1$, and $\{f \mid (\exists i)[\varphi_{p(i)} \subseteq f]\} \notin \mathbf{Ex}^n$. It is easy to see that, for p as claimed above, $\{f \mid (\exists i)[\varphi_{p(i)} \subseteq f]\} \in \mathbf{R}_{\mathcal{T}}\mathbf{Ex}^{n+1}_0$. ∎

Theorem 30 *For all n,* $\mathbf{R}_{\mathcal{T}}\mathbf{Ex}_{n+1} - \mathbf{Ex}^*_n \neq \emptyset$.

PROOF. Let $\mathcal{C}_n = \{f \mid \mathrm{card}(\{x \mid f(x) \neq f(x+1)\}) \leq n+1\}$. It is easy to see that $\mathcal{C}_n \in \mathbf{R}_{\mathcal{T}}\mathbf{Ex}_{n+1}$. It was shown in [11] that $\mathcal{C}_n \notin \mathbf{Ex}_n^*$. ∎

Theorem 31 $\mathbf{P}_{\mathcal{C}}^*\mathbf{Ex}_0^m \subseteq \mathbf{R}_{\mathcal{C}}\mathbf{Ex}_n^{m'}$, *where* $n \geq \lfloor \frac{m}{m'+1} \rfloor$.

PROOF. Proof of Theorem 9 can be easily adapted to prove the above. We leave the details to the reader. ∎

Theorem 32 $\mathbf{PEx}_0^* - \mathbf{R}_{\mathcal{R}}\mathbf{Ex}_n^* \neq \emptyset$.

PROOF. Let $\mathcal{C} = \{f \mid (\overset{\infty}{\forall} x)[f(x) = 0]\}$. Clearly, $f \in \mathbf{PEx}_0^*$. Suppose $\mathbf{M}$ is given, which is $\mathbf{Ex}^*$-reliable on $\mathcal{R}$ and $\mathbf{Ex}^*$-identifies $\mathcal{C}$. Since $\mathcal{R} \notin \mathbf{Ex}$, there exists a σ such that $\mathrm{mindchange}(\mathbf{M}, \sigma) > n$. Let f be an extension of σ such that $(\overset{\infty}{\forall} x)[f(x) = 0]$. Clearly, then $f \notin \mathbf{R}_{\mathcal{R}}\mathbf{Ex}_n^*(\mathbf{M})$. It follows that $\mathcal{C} \notin \mathbf{R}_{\mathcal{R}}\mathbf{Ex}_n^*$. ∎

Theorem 33 $\mathbf{P}_{\mathcal{C}}^*\mathbf{Ex} \subseteq \mathbf{R}_{\mathcal{C}}\mathbf{Ex}$.

PROOF SKETCH. Suppose $\mathbf{M}$ is given, which is $*$-Popperian on $\mathcal{C}$. Define, informally an $\mathbf{M}'$, which, on any input function f, outputs a program, which patches the convergent errors, if any, of the last program output by $\mathbf{M}$ on f. It is easy to see $\mathbf{M}'$ as described above will be reliable on $\mathcal{C}$ and $\mathbf{Ex}$-identifies each function $\mathbf{Ex}$ identified by $\mathbf{M}$. ∎

Theorem 34 $\mathbf{PEx}_1^0 - \mathbf{R}_{\mathcal{R}}\mathbf{Ex}_n^* \neq \emptyset$.

PROOF. Fix n. Let $\mathcal{C} = \{f \in \mathrm{STEP}^0(n+2) \mid \max(\mathrm{range}(f)) = n+2\} \cup \{\lambda x.1\}$. Clearly, $\mathcal{C} \in \mathbf{PEx}_1^0$. Suppose $\mathbf{M}$, a $\mathbf{Ex}^*$-reliable machine on $\mathcal{R}$, is given such that $\mathcal{C} \subseteq \mathbf{Ex}^*(\mathbf{M})$. Let $\gamma \subset \lambda x.1$ be such that $\mathbf{M}(\gamma) \neq ?$. Let $\mathrm{INITSTEP}(k) = \{f[m] \mid f \in \mathrm{STEP}^0(k)\}$. The theorem now follows using the following claim.

Claim 3 $(\forall k > 0, \sigma \in \mathrm{INITSTEP}(k) \mid \gamma \subseteq \sigma)(\exists \tau \in \mathrm{INITSTEP}(k) \mid \sigma \subseteq \tau)[\varphi_{\mathbf{M}(\tau)} =^* \lambda x.k]$.

PROOF. Suppose σ, k is given. Let f be such that, for all $x > |\sigma|, f(x) = k$. Now there exists an $l > |\sigma|$, such that $\varphi_{\mathbf{M}(f[l])} =^* \lambda x.k$ (otherwise $\mathbf{M}$ is not $\mathbf{Ex}^*$-reliable on $\mathcal{R}$). Since $f[l] \in \mathrm{INITSTEP}(k)$, the claim follows. ∎ (Claim 3 and Theorem 34)

Theorem 35 $\mathbf{R}_{\mathcal{C}}\mathbf{Ex}_n^0 \subseteq \mathbf{P}_{\mathcal{C}}^*\mathbf{Ex}_n^0$.

PROOF. Suppose $\mathbf{M}$ is given. Define $\mathbf{M}'$ as follows. Let p_0 be a program such that $(\forall x)[\varphi_{p_0}(x) = 0]$.

$$\mathbf{M}'(f[l]) = \begin{cases} \mathbf{M}(f[l]), & \text{if mindchange}(\mathbf{M}, f[l]) \leq n; \\ p_0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathbf{Ex}_n^0(\mathbf{M}) \subseteq \mathbf{Ex}_n^0(\mathbf{M}')$. Moreover if $\mathbf{M}$ is reliable on f, then $\mathbf{M}'$ is $*$-Popperian on f. ▮

Theorem 36 $\mathbf{TEx}_0^0 - \mathbf{R}_{\mathcal{R}}\mathbf{Ex}^* \neq \emptyset$.

PROOF. First we prove the following claim.

Claim 4 *If $\mathbf{M}$ is $\mathbf{Ex}^*$-reliable on $\mathcal{R}$, then for all σ such that $\mathbf{M}(\sigma) \neq ?$, there exists a $\tau \supseteq \sigma$ such that $\mathbf{M}(\sigma) \neq \mathbf{M}(\tau)$.*

PROOF. Suppose otherwise. Then for all f such that $\sigma \subseteq f$, $\mathbf{M}(f) = \mathbf{M}(\sigma)$. This along with $\mathbf{Ex}^*$-reliability of $\mathbf{M}$ on $\mathcal{R}$ implies that $(\forall f \in \mathcal{R} \mid \sigma \subseteq f)[f =^* \varphi_{\mathbf{M}(\sigma)}]$. This is clearly false. A contradiction. ▮ (Claim 4)

Now let $\mathcal{C} = \{f \mid \varphi_{f(0)} = f\}$. Clearly, $\mathcal{C} \in \mathbf{TEx}_0^0$. Suppose by way of contradiction that $\mathbf{M}$ $\mathbf{R}_{\mathcal{R}}\mathbf{Ex}^*$-identifies $\mathcal{C}$. Then by the implicit use of Kleene's recursion theorem there exists an e such that the (partial) function φ_e may be described as follows. Let $\varphi_e(0) = e$. Let x_s denote the least x such that $\varphi_e(x)$ has not been defined before stage s. Go to stage s.

Begin stage s

  if $\mathbf{M}(\varphi_e[x_s]) = ?$

  then

    let $\varphi_e(x_s) = 0$. Go to stage $s+1$.

  endif

  Search for an extension τ of $\varphi_e[x_s]$ such that $\mathbf{M}(\varphi_e[x_s]) \neq \mathbf{M}(\tau)$.

  If and when such a τ is found,

    for x such that $x_s \leq x < |\tau|$, let $\varphi_e(x) = y$ such that $(x, y) \in \tau$.

  Go to stage $s+1$.

End stage s

Now consider the following cases.

Case 1: All stages terminate.

In this case clearly φ_e is total and a member of $\mathcal{C}$. Moreover, $\mathbf{M}(\varphi_e)\uparrow$.

Case 2: Some stage s does not terminate.

In this case, by Claim 4, $\mathbf{M}$ is not $\mathbf{Ex}^*$-reliable on $\mathcal{R}$.

From the above cases we have that $\mathcal{C} \notin \mathbf{R}_{\mathcal{R}}\mathbf{Ex}^*$. ∎

Theorem 37 $\mathbf{R}_{\mathcal{T}}\mathbf{Ex}_0^0 - \mathbf{PEx} \neq \emptyset$.

Proof. Let $\mathcal{C} = \{f \mid \varphi_{f(0)} = f \ \wedge \ (\forall x)[\Phi_{f(0)}(x) \leq f(x+1)]\}$. Clearly, $\mathcal{C} \in \mathbf{R}_{\mathcal{T}}\mathbf{Ex}_0^0$. Suppose $\mathbf{M}$ is given. Without loss of generality assume that, for all σ, $\mathbf{M}(\sigma) \neq ?$. By the implicit use of Kleene's recursion theorem there exists an e such that the (partial) function φ_e may be defined as follows.

$$\varphi_e(x) = \begin{cases} e, & \text{if } x = 0; \\ 1 + \Phi_e(x-1) + \varphi_{\mathbf{M}(\varphi_e[x])}(x), & \text{otherwise.} \end{cases}$$

Now if $\mathbf{M}$ is Popperian, then φ_e is (a) total, (b) a member of $\mathcal{C}$, and (c) not $\mathbf{Ex}$-identified by $\mathbf{M}$. ∎

Theorem 38 $\mathbf{R}_{\mathcal{T}}\mathbf{Ex}_1^0 - \mathbf{NV}' \neq \emptyset$.

Proof. Let $\mathcal{C}$ be as defined in the proof of Theorem 27. It is easy to see that $\mathcal{C} \in \mathbf{R}_{\mathcal{T}}\mathbf{Ex}_1^0$. ∎

Theorem 39 *For all n, $\mathbf{R}_{\mathcal{C}}\mathbf{Ex}_n^* \subseteq \mathbf{P}_{\mathcal{C}}^*\mathbf{Bc}$.*

Proof. Fix n, $\mathcal{C}$. Suppose $\mathbf{M}$, $\mathbf{Ex}^*$-reliable on $\mathcal{C}$, is given. Define $\mathbf{M}'$ as follows. Let patch be a recursive function such that, for all i, f, l and x,

$$\varphi_{\mathrm{patch}(i,f[l])}(x) = \begin{cases} f(x), & \text{if } x < l; \\ \varphi_i(x), & \text{otherwise} \end{cases}$$

Let p_0 be such that $\varphi_{p_0}(x) = 0$, for all x.

$$\mathbf{M}'(f[l]) = \begin{cases} ?, & \text{if } \mathbf{M}(f[l]) = ?; \\ p_0, & \text{if } \mathrm{mindchange}(\mathbf{M}, f[l]) > n; \\ \mathrm{patch}(\mathbf{M}(f[l]), f[l]), & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathbf{M}'$ is $*$-Popperian on $\mathcal{C}$ and $\mathbf{Ex}_n^*(\mathbf{M}) \subseteq \mathbf{Bc}(\mathbf{M}')$. ∎

Theorem 40 $\mathbf{R}_{\mathcal{R}}\mathbf{Ex}_0^* - \mathbf{P}_{\mathcal{T}}^*\mathbf{Bc}^* \neq \emptyset$.

Proof. Let q be such that W_q is a simple set. Let $\mathbf{M}_0, \mathbf{M}_1, \ldots$ be a recursive enumeration of the inductive inference machines such that

for all i, σ, $\mathbf{M}_i(\sigma) \neq ?$, and
$\mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^* = \{\mathcal{C} \mid (\exists i)[\mathcal{C} \subseteq \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*(\mathbf{M}_i)]\}$.

Clearly, such recursive enumerations exist.

The proof of this theorem extends the idea used in the proof of Theorem 16. We will describe a recursive sequence of (partial) recursive functions, $\varphi_{p(\cdot)}$. Then using these (partial) recursive functions, we will describe a sequence of total functions $h_0, h_1, h_2, \ldots$. h_i may or may not be recursive. However, if $\mathbf{M}_i$ is $*$-Popperian on $\mathcal{T}$, then h_i will be recursive. We will let $\mathcal{C} = \{h_i \mid h_i \in \mathcal{R}\}$ to witness the theorem.

We now proceed to describe the (partial) recursive function $\varphi_{p(e)}$ for arbitrary e.

We assume without loss of generality that Φ is such that, for all i and x, $\Phi_i(x) \geq x$. Let $\varphi_{p(e)}(0) = e$. Let $\varphi^s_{p(e)}$ denote the part of $\varphi_{p(e)}$ defined before stage s. For each s, let $x^0_s = 0$ and, for $i > 0$, x^i_s denote i-th member (in increasing order) of complement(domain(φ^s_e)). Let g_s denote the 0-extension of $\varphi^s_{p(e)}$. Let val^e_0 be the recursive function such that, for all x, $\mathrm{val}^e_0(x) = x$. Let $t = 1$. Intuitively t denotes a time variable. The use of this time variable on outputs of $\varphi_{p(e)}$ will allow us to determine, effectively in e, x and y, whether $\varphi_{p(e)}(x) = y + 1$. Go to stage 0.

Begin stage s

1. **for** $i = \langle 0, 0\rangle$ **to** ∞ **do**

   Let j, k be such that $i = \langle j, k\rangle$.

   Let $t = t + 1$.

   if $(\forall w \mid x^j_s \leq w \leq x^{j+1}_s)[\Phi_{\mathbf{M}_e(g_s[w])}(x^{j+1}_s) \leq k]$ **then**

   Go to step 2.

   else if $\mathrm{val}^e_s(j+1) \in W_{q,k}$ **then**

   Go to step 3.

   endif

   endfor

2. Let $\varphi_{p(e)}(x^{j+1}_s) = t + 1 + \max(\{\varphi_{\mathbf{M}(g_s[w])(x^{j+1}_s)} \mid x^j_s \leq w \leq x^{j+1}_s\})$.

   Go to step 4.

3. Let $\varphi_{p(e)}(x^{j+1}_s) = t + 1$.

4. Let $\mathrm{val}^e_{s+1}(x) = \mathrm{val}^e_s(x)$, for $x \leq j$.

   Let $\mathrm{val}^e_{s+1}(x) = \mathrm{val}^e_s(x) + 1$, for $x > j$.

   Go to stage $s + 1$.

End stage s

Note that it can be determined, effectively in e, x and y, whether $\varphi_{p(e)}(x) = y + 1$.

We now proceed to define h_e.

Consider the following cases.

Case 1: For all but finitely many i, $x^i\uparrow$.

In this case clearly $\varphi_{p(e)}$ converges on all but finitely many inputs. Let h_e be the 0-extension of $\varphi_{p(e)}$. Let i_l be the largest i such that $x^i\downarrow$. Let s be such that $x^{i_l}_s = x^{i_l}$. Now for all $s' > s$ such that $\mathrm{val}^e_{s'}(i_l + 1) \notin W_q$, $(\forall w \mid x^{i_l} \leq w \leq x^{i_l+1}_{s'})[\varphi_{p(e)}(x^{i_l+1}_{s'})\downarrow \neq \varphi_{\mathbf{M}(g[w])}(x^{i_l+1}_{s'})\downarrow]$ (by the diagonalization at step 2 in some stage $s'' > s'$; this diagonalization must happen since $x^{i_l+1}\uparrow$). Since, for infinitely many s', $\mathrm{val}^e_{s'}(i_l + 1) \notin W_q$, we have that $\mathbf{M}_e$ does not $\mathbf{Bc}^*$-identify h_e.

Case 2: For each i, $x^i\downarrow$.

In this case, it is easy to see that $\mathbf{M}_e$ cannot be $*$-Popperian on $\mathcal{T}$. Let h_e be the 0-extension of $\varphi_{p(e)}$. We claim that h_e cannot be recursive. Suppose otherwise. Then let $0 < y_1 < y_2 < \ldots$ be all the inputs > 0 on which h_e is 0. Now consider the set $L_e = \{\mathrm{val}^e_s(j) \mid j > 0 \ \wedge \ x^s_j = y_j\}$. Clearly, L_e is recursively enumerable, infinite and $\subseteq$ complement(W_q). But this is not possible, since W_q is a simple set. Thus h_e cannot be recursive.

Now consider the class $\mathcal{C} = \{h_e \mid h_e \in \mathcal{R}\}$. By the analysis above $\mathcal{C} \notin \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^*$. We now informally describe the behavior of a machine $\mathbf{M}$, which is $\mathbf{Ex}^*$-reliable on $\mathcal{R}$, and $\mathbf{Ex}^*_0$-identifies $\mathcal{C}$. $\mathbf{M}$ on input function f first outputs $p(f(0))$. Then it searches for an x such that

(a) $\varphi_{p(f(0))}(x)\downarrow \neq f(x)$ or
(b) $f(x) > 0$ and $\varphi_{p(f(0))}(x) \neq f(x)$.

Note that by the comment at the end of construction above, (b) above can be effectively tested. If and when such an x is found $\mathbf{M}$, then, proceeds to diverge on the function f. It is easy to see that $\mathbf{M}$ is $\mathbf{Ex}^*$-reliable on $\mathcal{R}$ and $\mathbf{Ex}^*_0$-identifies $\mathcal{C}$. ∎

Lemma 1 [5] *Suppose* $\mathbf{M}$ *is reliable on* $\mathcal{T}$. *Then there exists a recursive* h *such that* $(\forall f \in \mathbf{Ex}(\mathbf{M}) \mid \mathrm{range}(f) \subseteq \{0,1\})(\exists j \mid \varphi_j = f)(\overset{\infty}{\forall} x)[\Phi_j(x) \leq h(x)]$.

Theorem 41 $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}_0 - \mathbf{R}_{\mathcal{T}}\mathbf{Ex} \neq \emptyset$.

Proof. The idea is to construct a class consisting of arbitrarily complex 0-1 functions which can be $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}$-identified, and then use Lemma 1 for the diagonalization.

We will simultaneously define recursive functions g and p such that, for each e, the following three conditions are satisfied.

(a) $\varphi_{p(e)}(e) = 1$ and, for $x < e$, $\varphi_{p(e)}(x) = 0$.

(b) domain($\varphi_{p(e)}$) is either N or an initial segment of N and range($\varphi_{p(e)}$) $\subseteq \{0, 1\}$.

(c) If φ_e is total, then

(c.1) $\varphi_{p(e)}$ is total,

(c.2) $(\forall j)(\forall^{j+1} x > \max(\{e, j\}))[(\exists y \leq x)[\varphi_j(y) \neq \varphi_{p(e)}(y)] \vee [\Phi_{p(e)}(x) \leq g(e, x, \Phi_j(x))]]$, and

(c.3) $(\forall j \mid \varphi_j = \varphi_{p(e)})(\overset{\infty}{\forall} x)[\Phi_j(x) > \varphi_e(x)]$.

Now let $\mathcal{C} = \{\varphi_{p(e)} \mid \varphi_e \text{ is total}\}$. We will construct g and p as above later. First we show that $\mathcal{C} \in \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}_0$.

Define $\mathbf{M}$ as follows. Let p_0 be such that $(\forall x)[\varphi_{p_0}(x) = 0]$.

$$\mathbf{M}(f[n]) = \begin{cases} ?, & \text{if } (\forall x < n)[f(x) = 0]; \\ p(e), & \text{if } e < n \ \wedge \ f(e) = 1 \ \wedge \ (\forall x < e)[f(x) = 0] \ \wedge \\ & \neg(\exists x < n)[\Phi_{p(e)}(x) < n \ \wedge \ \varphi_{p(e)}(x) \neq f(x)] \ \wedge \\ & \neg(\exists j \leq n)(\exists x_0, \ldots, x_{j+1} \mid \max(\{e, j\}) < x_0 < x_1 < \ldots < x_{j+1} < n) \\ & [(\forall y \leq x_{j+1})[\Phi_j(y) \leq n \ \wedge \ \varphi_j(y) = f(y)] \ \wedge \\ & \quad (\forall r \leq j+1)[\Phi_{p(e)}(x_r) > g(e, x_r, \Phi_j(x_r))]]; \\ p_0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathbf{M}$ is $*$-Popperian on $\mathcal{R}$ (using properties of p and g discussed above) and $\mathbf{Ex}_0$-identifies each $f \in \mathcal{C}$.

Also by Lemma 1 and clause (c.3) in the property of p and g given above we have that $\mathcal{C} \notin \mathbf{R}_{\mathcal{T}}\mathbf{Ex}$.

We now proceed to construct p and g as claimed above. p and g are constructed by implicit use of parametric recursion theorem. For given e and x, we describe in stages below $\varphi_{p(e)}(x)$ and $g(e, x, \cdot)$. (it will be easy to see that the definition is effective in e and x). The construction also maintains a cancellation list. cancel_x denotes the set of programs which have been diagonalized against on inputs $\leq x$. For $x < e$, let $\varphi_{p(e)}(x) = 0$. Let $\varphi_{p(e)}(e) = 1$. Let $\text{cancel}_e = \emptyset$. Go to stage 0.

$\varphi_{p(e)}(x)$ and $g(e, x, \cdot)$

Begin stage s

1. **if** $x \leq e$ or $s = 0$ or $\Phi_e(x) > s$ or $\Phi_{p(e)}(x-1) > s$ **then**

    Let $g(e, x, s) = s$.

    Go to stage $s + 1$.

   endif

2. *(* φ_e is defined on all inputs $\leq x$. *)*

Let $X = \{j \mid j \leq x \ \wedge \ \Phi_j(x) \leq \max(\{s, \varphi_e(x)\})\} - \mathrm{cancel}_{x-1}$.

3. **if** $\varphi_{p(e)}(x)$ has been defined before stage s **then** let $g(e,x,s) = g(e,x,s-1)+1$.
else if $X = \emptyset$ **then**
3.1 let $\varphi_{p(e)}(x) = 0$, and $g(e,x,s) = \Phi_{p(e)}(x)+1$.
else
3.2 let $\varphi_{p(e)}(x) = 1 \dot{-} \varphi_{\min(X)}(x)$, and $g(e,x,s) = \Phi_{p(e)}(x)+1$.
Let $\mathrm{cancel}_{x+1} = \mathrm{cancel}_x \cup \{\min(X)\}$.
endif

4. Go to stage $s+1$.

End stage s

End $\varphi_{p(e)}(x)$ and $g(e,x,\cdot)$

We now show that $\varphi_{p(e)}$ satisfies the properties claimed.

(a), (b) immediate from construction.

(c) Suppose φ_e is total. Clearly, $\varphi_{p(e)}$ is total, and thus (c.1) is satisfied. Also if j is such that $(\overset{\infty}{\exists}\, x)[\Phi_j(x) \leq \varphi_e(x)]$, then, by the diagonalization at step 3.2, $\varphi_j \neq \varphi_{p(e)}$. Thus (c.3) is satisfied. For (c.2), suppose that $x > e$ is such that $\varphi_{p(e)}(x)$ get defined at stage s. Then, for $j \leq x$, either $\Phi_j(x) > s$ (and thus $\Phi_{p(e)}(x) \leq g(e,x,\Phi_j(x))$), or $(\exists y < x)[\varphi_{p(e)}(y) \neq \varphi_j(x)]$ or $j \in X - \mathrm{cancel}_{x-1}$ at stage s. But j can be in $X - \mathrm{cancel}_{x-1}$ in some stage s, for at most $j+1$ x's greater than j. Thus (c.2) holds. ∎

Theorem 42 $\mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}_0 - \mathbf{R}_{\mathcal{T}}\mathbf{Ex}^* \neq \emptyset$.

Proof. For $f \in \mathcal{R}$ define f' as follows. $f'(\langle x, y\rangle) = f(y)$. Let $\mathcal{C}$ be as in the proof of Theorem 41. Let $\mathcal{C}' = \{f' \mid f \in \mathcal{C}\}$. Clearly, $\mathcal{C}' \in \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex}_0$. Moreover $\mathcal{C}' \in \mathbf{R}_{\mathcal{T}}\mathbf{Ex}^* \Leftrightarrow \mathcal{C} \in \mathbf{R}_{\mathcal{T}}\mathbf{Ex}$. But since $\mathcal{C} \notin \mathbf{R}_{\mathcal{T}}\mathbf{Ex}$, we have that $\mathcal{C}' \notin \mathbf{R}_{\mathcal{T}}\mathbf{Ex}^*$. ∎

Corollary 9 $\mathbf{R}_{\mathcal{R}}\mathbf{Ex}_0 - \mathbf{R}_{\mathcal{T}}\mathbf{Ex}^* \neq \emptyset$.

Theorem 43 $\mathbf{R}_{\mathcal{T}}\mathbf{Ex} - \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^* \neq \emptyset$.

Proof. Note that $\mathcal{C}$ as defined in Theorem 17 is a member of $\mathbf{R}_{\mathcal{T}}\mathbf{Ex}$. The theorem follows. ∎

As a corollary to Theorem 22 we have

Corollary 10 *For each m,* $\mathbf{R}_{\mathcal{R}}\mathbf{Ex}^0_0 - \mathbf{P}^*_{\mathcal{T}}\mathbf{Bc}^m \neq \emptyset$.

The results mentioned in this paper resolve all the questions regarding relationship between different learning classes introduced in this paper, except for the open problems mentioned below.

Open Question 1 *For* $\mathcal{C} \in \{\mathcal{R}, \mathcal{T}\}$*,* $\mathbf{R}_{\mathcal{C}}\mathbf{Ex} - \mathbf{P}^*_{\mathcal{R}}\mathbf{Ex} \neq \emptyset$*?*

Open Question 2 *For* $\mathcal{C} \in \{\mathcal{R}, \mathcal{T}\}$*, a and m,* $\mathbf{R}_{\mathcal{C}}\mathbf{Ex}^a - \mathbf{P}^*_{\mathcal{R}}\mathbf{Bc}^m \neq \emptyset$*?*

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