Quasi-Isometric Reductions Between Infinite Strings

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Abstract

This paper studies the recursion- and automata-theoretic aspects of large-scale geometries of infinite strings, a subject initiated by Khoussainov and Takisaka (2017). We first investigate several notions of quasi-isometric reductions between recursive infinite strings and prove various results on the equivalence classes of such reductions. The main result is the construction of two infinite recursive strings α and β such that α is strictly quasi-isometrically reducible to β, but the reduction cannot be made recursive. This answers an open problem posed by Khoussainov and Takisaka. Furthermore, we also study automatic quasi-isometric reductions between automatic structures, and show that automatic quasi-isometry may be separable from general quasi-isometry depending on the growth of the automatic domain.

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1 Introduction

Quasi-isometry is an important concept in geometric group theory that has been used to solve problems in group theory. Loosely speaking, two metric spaces are said to be quasi-isometric iff there is a mapping (called a *quasi-isometry*) from one metric space to the other that preserves the distance between any two points in the first metric space up to some multiplicative and additive constants. Thus, for example, while the Euclidean plane is not isometric to $\mathbb{R}^2$ equipped with the taxicab distance, the two spaces are quasi-isometric to each other since the Euclidean distance between any two points does not differ from the taxicab distance between them up to a multiplicative factor of $\sqrt{2}$. The study of group properties—where groups are represented by their Cayley graphs—that are invariant under quasi-isometries is quite a prominent theme in geometric group theory; examples of such group properties include hyperbolicity and growth rate [4].

Khoussainov and Takisaka [10] introduced a refined notion of quasi-isometry for infinite strings called *colour-preserving quasi-isometry*, thus enabling the study of global patterns on strings and linking the study of large-scale geometries with automata theory, recursion theory and model theory. Here, an infinite string $\alpha \in \Sigma^\omega$ is represented by the metric space $\mathbb{N}$ where each number i has a specific colour $\sigma \in \Sigma$ depending on the i-th symbol in the string α. This paper builds on the results from Khoussainov and Takisaka [10] further, by studying, in particular, *recursive* quasi-isometry between *recursive* infinite strings.

In addition to Khoussainov and Takisaka's paper [10] which studied quasi-isometry between infinite strings specifically, quasi-isometries between hyperbolic metric spaces in general—an example of which is an infinite string when viewed as a coloured metric space—are well-studied in geometric group theory. Isometries between computable metric spaces have also been studied by Melnikov [17].

Among the various questions investigated by Khoussainov and Takisaka was the computational complexity of the *quasi-isometry problem*: given any two infinite strings α and β, is there a quasi-isometry from α to β? They found that for any two quasi-isometric strings, a quasi-isometry that is recursive in the halting problem relative to α and β always exists between them, and that the quasi-isometry problem between any two recursive strings is Σ^0_2-complete [11][1]. In comparison, the corresponding problem for isometry with respect to recursive strings is Π^0_1-complete [17]. Khoussainov and Takisaka also had the following open problem which was mentioned in many talks and discussions: *if a quasi-isometric reduction from α to β exists, does there always exist a recursive quasi-isometric reduction?* This is a very natural question for computer science, specifically for computability theory, since it seeks to understand how complex such a reduction is. We answered this question in the negative, that is, there are cases where the reduction exists but cannot be made recursive. The fourth author's bachelor thesis [16] which contains this result was cited by Khoussainov and Takisaka in the journal version [11] of their paper [10].

To complete the picture, the present work examines, in more detail, the recursion-theoretic aspects of quasi-isometries between infinite strings. We study various natural restrictions on quasi-isometric reductions between strings: first, *many-one reductions*, where the quasi-isometric reduction is required to be *recursive* and *many-one*; second, *one-one reductions*, which are injective many-one reductions; third, *permutation quasi-isometric reductions*, which are surjective one-one reductions.

The main subjects of this work are the structural properties of the equivalence classes induced by the different types of reductions and the relationships between these reductions. In accordance with recursion-theoretic terminology, we call an equivalence class induced by a reduction type a *degree* of that reduction type. We show, for example, that within each many-one quasi-isometry degree, any pair of strings has a common upper bound as well as a common lower bound with respect to one-one reductions. Furthermore, there are two strings for which their many-one quasi-isometry degrees have a unique least common upper bound. The main result is the separation of quasi-isometry from *recursive* quasi-isometry, that is, we construct two recursive strings such that one is quasi-isometrically reducible to the other but

[1] Note that [11] is the journal version of [10], containing some corrections from the earlier paper.

no recursive many-one quasi-isometry exists between them. This main result answers the above-mentioned open problem posed by Khoussainov and Takisaka.

In addition, we also investigate the automata-theoretic aspects of quasi-isometries. In this case, the question is whether automatic quasi-isometry can be separated from general quasi-isometry. Our results show that the answer depends on the growth of the domain. In linear domains, quasi-isometry is equivalent to automatic quasi-isometry. On the other hand, in superlinear but polynomial domains, one can always find automatic colourings α and β such that α is quasi-isometrically reducible to β but not automatically. While it is known that quasi-isometry can be separated from its automatic counterparts in some exponential domains, it is unknown whether such separation is always possible in all domains with exponential growth.

2 Notation

Any unexplained recursion-theoretic notation may be found in [18, 21, 23]. The set of positive integers will be denoted by $\mathbb{N}$; $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. The finite set Σ will denote the alphabet used. We assume knowledge of elementary computability theory over different size alphabets [2]. An infinite string $\alpha \in \Sigma^\omega$ can also be viewed as a Σ-valued function defined on $\mathbb{N}$. The length of an interval I is denoted by $|I|$. For $\alpha_i \in \Sigma^*$ and $i \in \mathbb{N}$, we write $(\alpha_i)_{i=1}^\infty$ to denote $\alpha_1\alpha_2\cdots$, a possibly infinite string.

3 Coloured Metric Spaces and Infinite Strings

▶ **Definition 1** (Coloured Metric Spaces, [10]). *A* coloured metric space $(M; d_M, Cl)$ *consists of the* underlying metric space $(M; d_M)$ *with metric d_M and the colour function $Cl : M \to \Sigma$, where Σ is a finite set of colours called an* alphabet. *We say that $m \in M$* has colour $\sigma \in \Sigma$ *if $\sigma = Cl(m)$.*

▶ **Definition 2** (Quasi-isometries Between Coloured Metric Spaces, [10]). *For any $A \geq 1$ and $B \geq 0$, an* (A, B)-quasi-isometry *from a metric space $\mathcal{M}_1 = (M_1; d_1)$ to a metric space $\mathcal{M}_2 = (M_2; d_2)$ is a function $f : M_1 \to M_2$ such that for all $x, y \in M_1$, $\frac{1}{A} \cdot d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A \cdot d_1(x, y) + B$, and for all $y \in M_2$, there exists an $x \in M_1$ such that $d_2(f(x), y) \leq A$.*

Given two coloured metric spaces $\mathcal{M}_1 = (M_1; d_1, Cl_1)$ and $\mathcal{M}_2 = (M_2; d_2, Cl_2)$, a function $f : M_1 \to M_2$ is a quasi-isometric reduction *from $\mathcal{M}_1$ to $\mathcal{M}_2$ iff for some $A \geq 1$ and $B \geq 0$, f is an (A, B)-quasi-isometry from $(M_1; d_1)$ to $(M_2; d_2)$ and f is* colour-preserving*, that is, for all $x \in M_1$, $Cl_1(x) = Cl_2(f(x))$.*

An infinite string α can then be seen as a coloured metric space $(\mathbb{N}; d, \alpha)$, where d is the metric on $\mathbb{N}$ defined by $d(i, j) = |i - j|$ and $\alpha : \mathbb{N} \to \Sigma$ is the colour function. For any two infinite strings α and β, we write $\alpha \leq_{qi} \beta$ to mean that there is a quasi-isometric reduction from α to β. The relation $\leq_{qi}$ is a preorder on Σ^ω. For any pair of distinct letters $a_1, a_2 \in \Sigma$, a_1^ω and a_2^ω are incomparable with respect to $\leq_{qi}$, so this relation is not total.

The following proposition gives a useful simplification of the definition of quasi-isometry in the context of infinite strings.

▶ **Proposition 3.** *Given two infinite strings α and β, let $f : \mathbb{N} \to \mathbb{N}$ be a colour-preserving function. Then, f is a quasi-isometric reduction from α to β iff there exists a constant $C \geq 1$ such that for all x, y in the domain of α, the following conditions hold:*

(a) $d(f(x), f(x + 1)) \leq C$;

(b) $x + C < y \Rightarrow f(x) < f(y)$.

Proof. First, suppose that $f : \mathbb{N} \to \mathbb{N}$ is a colour-preserving quasi-isometric reduction from α to β. We show that there exists a constant $C \geq 1$ for which Conditions (a) and (b) hold for any $x, y \in \mathbb{N}$. By the definition of a quasi-isometric reduction, there exist constants $A \geq 1$ and $B \geq 0$ such that

$$\frac{1}{A} \cdot d(x, y) - B \leq d(f(x), f(y)) \leq A \cdot d(x, y) + B. \tag{1}$$

We first derive, for each of the two conditions, a choice of C satisfying it.

(i) Plugging $y = x + 1$ into the upper bound in (1) yields $d(f(x), f(x+1)) \leq A + B$.

(ii) Assume for the sake of a contradiction that for all $C \geq 1$, there are $x \in \mathbb{N}$ and $C' > C$ such that $f(x + C') \leq f(x)$. We show that if C is chosen so that $A + B \leq \frac{1}{A} \cdot C - B$, then the existence of some $C' > C$ with $f(x + C') \leq f(x)$ would lead to a contradiction. Fix such a C, and suppose there were indeed some C' with $C' > C \geq 1$ and

$$f(x + C') \leq f(x)\,. \tag{2}$$

Then,

$$\begin{aligned} f(x + C' + 1) - f(x + C') &\leq d(f(x + C' + 1), f(x + C')) \\ &\leq A + B \quad \text{(by statement (i))} \\ &\leq \frac{1}{A} \cdot C - B \quad \text{(by the choice of } C\text{)} \\ &< \frac{1}{A} \cdot C' - B \quad \text{(since } C' > C\text{)} \\ &\leq f(x) - f(x + C') \quad \text{(by (1) and (2))}\,, \end{aligned}$$

giving $f(x + C' + 1) < f(x)$. One can repeat the preceding argument inductively, yielding the inequality $f(x + C' + k + 1) - f(x + C' + k) < f(x) - f(x + C' + k)$, or equivalently $f(x + C' + k + 1) < f(x)$, for each $k \geq 0$. But this is impossible since $f(x)$ is finite and $d(f(x + C' + k + 1), f(x + C' + k' + 1)) > 0$ whenever $|k - k'|$ is sufficiently large.

It follows from (i) and (ii) that Conditions (a) and (b) are satisfied for $C = A \cdot (A + 2B)$.

For a proof of the converse direction, fix a C satisfying Conditions (a) and (b). Suppose $x \in \mathbb{N}$. Then, by Condition (a), $d(f(x), f(x+1)) \leq C$. Inductively, assume that $d(f(x), f(x + n)) \leq n \cdot C$. Then, by the inductive hypothesis and Condition (a), $d(f(x), f(x + n + 1)) \leq d(f(x), f(x+n)) + d(f(x+n), f(x+n+1)) \leq n \cdot C + C = (n+1) \cdot C$ where the first inequality follows from the triangle inequality. Consequently, for all $x, y \in \mathbb{N}$, $d(f(x), f(y)) \leq d(x, y) \cdot C$.

Next, we establish a lower bound for $d(f(x), f(y))$. Without loss of generality, assume $x < y$. Write $y = x + i(C+1) + j$ for some $i \in \mathbb{N}_0$ and $0 \leq j \leq C$. By a simple induction, one can show that $f(x + i(C+1)) \geq f(x) + i$ and thus $d(f(x), f(x + i(C+1)) \geq i$. Furthermore, $d(f(x + i(C+1)), f(y)) \leq C^2$. Thus, $d(f(x), f(y)) \geq i - C^2$ and $i \geq d(x, y)/(C+1) - 1$. It follows that $d(f(x), f(y)) \geq d(x, y)/(C+1) - 1 - C^2$.

Thus, one can select $A = C + 1$ and $B = C^2 + 1$ to establish the required bounds for the quasi-isometric mapping. ◀

By Proposition 3, we can now redefine quasi-isometric reduction in terms of one constant C, instead of two constants A and B as in Definition 2, reducing the number of constants by 1.

▶ **Definition 4.** *Suppose $C \geq 1$. Given infinite strings α and β, a C-*quasi-isometry* from α to β is a colour-preserving function $f : \mathbb{N} \to \mathbb{N}$ such that for all x, y in the domain of α,*

(a) *$f(1) \leq C$ and $f(x) - C \leq f(x+1) \leq f(x) + C$;*
(b) *$x + C < y \Rightarrow f(x) < f(y)$.*
For the rest of the paper, we shall use "Condition (a)" and "Condition (b)" to refer to the above conditions respectively, without necessarily mentioning the definition number.

A useful property of a C-quasi-isometry f from α to β is that any position of β has at most $C + 1$ pre-images under f.

▶ **Lemma 5** ([10, Corollary II.4]). *Given two infinite strings α and β, suppose that f is a C-quasi-isometry from α to β. Then, for all $y \in \mathbb{N}$, $|f^{-1}(y)| \leq C + 1$.*

Proof. If $y \in \mathbb{N}$ is not in the range of f, then $|f^{-1}(y)| = 0$. So assume that $f^{-1}(y)$ is not empty. Set $x_0 = \min\{x : f(x) = y\}$. Now for all $x \in f^{-1}(y)$, we must have $x_0 + C \geq x \geq x_0$. The first inequality follows from Condition (b) since $f(x_0) = y = f(x)$. The second inequality follows from the choice of x_0. Hence, $f^{-1}(y) \subseteq \{x_0, \ldots, x_0 + C\}$ and so $|f^{-1}(y)| \leq C + 1$. ◀

It was proven earlier that for any infinite strings α, β and any C-quasi-isometry f from α to β, there is a constant D such that each position of β is at most D positions away from some image of f. The next lemma states that each position of β in the range of f is at most C positions away from a different image of f.

▶ **Lemma 6.** *Let α and β be infinite strings and let f be a C-quasi-isometry from α to β. Then, $\min\{f(x) : x \in \mathbb{N}\} \leq C$ and for each $y \in \mathbb{N}$, $\min\{f(x) > f(y) : x \in \mathbb{N}\} \leq f(y) + C$. Hence, for each $z \in \mathbb{N}$, there is some $x \in \mathbb{N}$ such that $d(f(x), z) \leq C$.*

Proof. By definition, $\min\{f(x) : x \in \mathbb{N}\} \leq f(1) \leq C$. For each $y \in \mathbb{N}$, take the smallest y' such that $f(y') > f(y)$. Such y' must exist by Condition (b). If $y' = 1$, then $f(y') \leq C \leq f(y) + C$. Otherwise, $f(y' - 1) \leq f(y)$. By Condition (a), $f(y') \leq f(y' - 1) + C \leq f(y) + C$. Hence, $\min\{f(x) > f(y) : x \in \mathbb{N}\} \leq f(y') \leq f(y) + C$. ◀

▶ **Corollary 7.** *Let $\Sigma = \{a_1, \ldots, a_l\}$ and let α, β be two infinite strings. Let f be a C-quasi-isometry from α to β. Suppose that there is a positive integer K such that there is at least one occurrence of a_i in any interval of positions of α of length K. Then, there is at least one occurrence of a_i in any interval of positions of β of length KC.*

Proof. Consider an interval $[y+1, y+KC]$ of positions of β of length KC. By Condition (b), there is a least x such that $f(x) > y$ and $\alpha(x) = a_i$. We show that $f(x)$ lies in the interval $[y + 1, y + KC]$ and so is a position of some occurrence of a_i in β. Suppose that there is no occurrence of a_i before position x. Then, $x \leq K$, since there must be at least one occurrence of a_i in the first K positions of α. Hence, by Condition (a), $f(x) \leq KC \leq y + KC$. Otherwise, let x' be the position of the last occurrence of a_i before position x. By definition of x, $f(x') \leq y$, and by the assumption, $x - x' \leq K$. Then, by Condition (a), $f(x) \leq f(x') + KC \leq y + KC$. In both cases, we have $y + 1 \leq f(x) \leq y + KC$. ◀

A quasi-isometry f can fail to be order-preserving in that there are pairs $x, y \in \mathbb{N}$ with $x < y$ and $f(x) > f(y)$. Nonetheless, as Khoussainov and Takisaka noted [10, Lemma II.2], every quasi-isometry enforces a uniform upper bound on the size of a *cross-over*—the difference $f(x) - f(y)$ for such a pair $x, y \in \mathbb{N}$. This may be proven using the alternative definition of a quasi-isometry given in Definition 4.

▶ **Lemma 8** (Small Cross-Over Lemma, [10, Lemma II.2]). *Given two infinite strings α and β, suppose that f is a C-quasi-isometry from α to β. Then, for all $n, m \in \mathbb{N}$ with $n < m$, we have $f(n) - f(m) \leq C^2$.*

Proof. Consider any $n, m \in \mathbb{N}$ with $n < m$. If $n + C < m$, then by Condition (b), $f(n) - f(m) < 0$. If $n + C \geq m$, then $d(m, n) = m - n \leq C$, so by applying Condition (a) $m - n$ times, one has $f(n) - f(m) \leq d(f(m), f(n)) \leq (m - n) \cdot C \leq C^2$. Thus, $f(n) - f(m) \leq C^2$ holds for any choices of $m, n \in \mathbb{N}$ with $n < m$. ◀

4 Recursive Quasi-Isometric Reductions

Khoussainov and Takisaka [10] investigated the structure of the partial-order Σ^ω_{qi} of the quasi-isometry degrees over an alphabet $\Sigma = \{a_1, \dots, a_l\}$. They proved that Σ^ω_{qi} has a greatest element, namely the degree of $(a_1 \cdots a_n)^\omega$, and that Σ^ω_{qi} contains uncountably many minimal elements. Furthermore, they showed that Σ^ω_{qi} includes a chain of the type of the integers, and that it includes an antichain. In connection with computability theory, in particular with the arithmetical hierarchy, they established that the quasi-isometry relation on recursive infinite strings is Σ^0_2-complete [11]. In this section, we continue research into the recursion-theoretic aspects of quasi-isometries on infinite strings. We consider the notions of many-one and one-one recursive reducibilities first introduced by Post [19] as relations between recursive functions, and apply them to quasi-isometric reductions. We also define a third type of quasi-isometric reducibility—permutation reducibility—which is bijective. We then prove a variety of results on the degrees of such reductions.

▶ **Definition 9** (Many-One Reducibility). *A string α is* many-one reducible*, or* mqi-reducible*, to a string β iff there exists a quasi-isometric reduction f from α to β such that f is* recursive*. We call such an f a* many-one quasi-isometry (or mqi-reduction)*, and write $\alpha \leq_{mqi} \beta$ to mean that α is many-one reducible to β; if, in addition, f is a C-quasi-isometry, then we call f a* C-many-one quasi-isometry *(or* C-mqi-reduction*). We write $\alpha <_{mqi} \beta$ to mean that $\alpha \leq_{mqi} \beta$ and $\beta \not\leq_{mqi} \alpha$.*

▶ **Definition 10** (One-One Reducibility). *A string α is* one-one reducible*, or* 1qi-reducible*, to a string β iff there exists a many-one quasi-isometry f from α to β such that f is one-one. We call such an f a* one-one quasi-isometry (or 1qi-reduction)*, and write $\alpha \leq_{1qi} \beta$ to mean that α is one-one reducible to β; if, in addition, f is a C-quasi-isometry, then we call f a* C-one-one quasi-isometry *(or* C-1qi-reduction*). We write $\alpha <_{1qi} \beta$ to mean that $\alpha \leq_{1qi} \beta$ and $\beta \not\leq_{1qi} \alpha$.*

▶ **Definition 11** (Permutation Reducibility). *A string α is* permutation reducible*, or* pqi-reducible*, to a string β iff there exists a one-one quasi-isometry f from α to β such that f is surjective. We call such an f a* permutation quasi-isometry (or pqi-reduction)*, and write $\alpha \leq_{pqi} \beta$ to mean that α is permutation reducible to β; if, in addition, f is a C-quasi-isometry, then we call f a* C-permutation quasi-isometry *(or* C-pqi-reduction*). We write $\alpha <_{pqi} \beta$ to mean that $\alpha \leq_{pqi} \beta$ and $\beta \not\leq_{pqi} \alpha$.*

Given an alphabet Σ, the relations $\leq_{mqi}$, $\leq_{1qi}$, $\leq_{pqi}$ and $\leq_{qi}$ are preorders on the class of infinite strings over Σ. Let $\equiv_{mqi}$ be the relation on Σ^ω such that $\alpha \equiv_{mqi} \beta$ iff $\alpha \leq_{mqi} \beta$ and $\beta \leq_{mqi} \alpha$. Then, $\equiv_{mqi}$ is an equivalence relation on Σ^ω. We call an equivalence class on Σ^ω induced by $\equiv_{mqi}$ a *many-one quasi-isometry degree* (or *mqi-degree*), and denote the mqi-degree of an infinite string α by $[\alpha]_{mqi}$. Analogous definitions apply to $\equiv_{1qi}$, $[\alpha]_{1qi}$, $\equiv_{pqi}$, $[\alpha]_{pqi}$, $\equiv_{qi}$ and $[\alpha]_{qi}$.

We denote the partial orders induced by $\leq_{pqi}, \leq_{1qi}, \leq_{mqi}$ and $\leq_{qi}$ on the pqi-degrees, 1qi-degrees, mqi-degrees and qi-degrees by $\Sigma^\omega_{pqi}, \Sigma^\omega_{1qi}, \Sigma^\omega_{mqi}$ and Σ^ω_{qi} respectively.

By definition, Σ^ω_{pqi} is a refinement of Σ^ω_{1qi} in the sense that for all infinite strings α and β, $[\alpha]_{pqi} \leq_{pqi} [\beta]_{pqi} \Rightarrow [\alpha]_{1qi} \leq_{1qi} [\beta]_{1qi}$. In a similar manner, Σ^ω_{1qi} is a refinement of Σ^ω_{mqi}, which is in turn a refinement of Σ^ω_{qi}. The first subsection deals with the mqi-degrees, starting with the inner structure of each mqi-degree.

4.1 Structure of the mqi-Degrees

Fix any two distinct infinite strings β and γ belonging to $[\alpha]_{mqi}$. It can be shown that β and γ have a common upper bound as well as a common lower bound in $[\alpha]_{mqi}$ such that these bounds are witnessed by 1qi-reductions.

▶ **Proposition 12.** *For any two distinct infinite strings $\beta, \gamma \in [\alpha]_{mqi}$, there exists a $\delta \in [\alpha]_{mqi}$ such that $\beta \leq_{1qi} \delta$ and $\gamma \leq_{1qi} \delta$.*

Proof. Let f be a C-mqi-reduction from β to γ. Let δ be the infinite string obtained from γ by repeating $C+1$ times each letter of γ. Then, $\gamma \leq_{1qi} \delta$ via a $(C+1)$-1qi-reduction g defined by $g(n) = (n-1)\cdot(C+1)+1$ for each $n \in \mathbb{N}$. Furthermore, $\delta \leq_{mqi} \gamma$ via a C-mqi-reduction g' defined by $g'(n) = \lceil \frac{n}{C+1} \rceil$. Thus, $\delta \in [\alpha]_{mqi}$.

Next, one constructs a (C^2+2C)-1qi-reduction f' from β to δ using the function f. For each y in the range of f, map the pre-image of y under f, which by Lemma 5 has at most $C+1$ elements, to the set of positions of δ corresponding to the $C+1$ copies of the letter at position y. Formally, define

$$f'(n) = \begin{cases} g(f(n))\,, & \text{if } f(n) \neq f(n') \text{ for all } n' < n; \\ g(f(n)) + C'\,, & \text{otherwise; where } 1 \leq C' < C+1 \text{ is minimum such that} \\ & g(f(n)) + C' \neq f'(n') \text{ for all } n' < n. \end{cases}$$

We verify that f' is an injective (C^2+2C)-quasi-isometry. Injectiveness follows from the definition of f': in the first case, the injectiveness of g ensures that $f'(x) \neq f'(x')$ for all $x' < x$; in the second case, it is directly enforced that $f'(x) \neq f(x')$ for all $x' < x$. Since f is a C-reduction, $x + C < y \Rightarrow f(x) < f(y) \Rightarrow g(f(x)) < g(f(y)) \Rightarrow f'(x) < f'(y)$, and so f' satisfies Condition (b) with constant C. Now we show that f' satisfies Condition (a) with constant C^2+2C. By Condition (a), $d(f(x), f(x+1)) \leq C$. Without loss of generality, assume that $f(x) \leq f(x+1)$. By the definition of f', $f'(x) \geq g(f(x))$ and $f'(x+1) \leq g(f(x+1)) + C$. Since $f(x) \leq f(x+1)$, it follows that $f'(x) \leq f'(x+1)$ and so

$$\begin{aligned} d(f'(x+1), f'(x)) &\leq g(f(x+1)) + C - g(f(x)) \\ &= (C+1)\cdot(f(x+1)-1) + 1 + C - (C+1)\cdot(f(x)-1) - 1 \\ &= (C+1)\cdot(f(x+1) - f(x)) + C \\ &\leq C\cdot(C+1) + C \\ &= C^2 + 2C\,. \end{aligned}$$

This completes the proof. ◀

Next, we prove a lower bound counterpart of Proposition 12.

▶ **Proposition 13.** *For any two distinct infinite strings $\beta, \gamma \in [\alpha]_{mqi}$, there exists a $\delta \in [\alpha]_{mqi}$ such that $\delta \leq_{1qi} \beta$ and $\delta \leq_{1qi} \gamma$.*

Proof. Suppose $\beta = \beta_1\beta_2\ldots$, where $\beta_i \in \Sigma$. Let $f : \mathbb{N} \to \mathbb{N}$ be a C-mqi-reduction from β to γ. Now define $\delta = \beta_{i_1}\beta_{i_2}\ldots$, where i_k is the minimum index such that $i_k \neq i_l$ for all $l < k$ and for all $j < i_k$, $f(j) \neq f(i_k)$. By Condition (b), the range of f is infinite and thus each i_k is well-defined. We verify that $\delta \leq_{1qi} \beta$ and $\delta \leq_{1qi} \gamma$.

Define $f'(n) = i_n$ for all $n \in \mathbb{N}$. We show that f' is a 1qi-reduction from δ to β. By the choice of the i_n's, $f'(n) > f'(m)$ whenever $n > m$; in particular, f' is injective and Condition (b) holds for f'. Furthermore, given any n, by applying Condition (b) to f and all $n' \leq n$, it follows that $f'(n+1) \leq f'(n) + C + 1$. Hence, f' also satisfies Condition (a).

Next, define a 1qi-reduction f'' from δ to γ by $f''(n) = f(i_n)$. The injectiveness of f'' follows from the choice of the i_n's (though f'' is not necessarily strictly monotone increasing). Using the fact that $i_{n+1} \leq i_n + C + 1$, as well as applying Condition (a) $i_{n+1} - i_n$ times, $d(f''(n+1), f''(n)) = d(f(i_{n+1}), f(i_n)) \leq C \cdot d(i_{n+1}, i_n) \leq C \cdot (C+1)$. Hence, f'' satisfies Condition (a) with constant $C \cdot (C+1)$. Since the i_n's are strictly increasing, $m + C < n \Rightarrow i_m + C < i_n \Rightarrow f(i_m) < f(i_n)$. Thus, f'' is a $C \cdot (C+1)$-1qi-reduction.

Lastly, define a mqi-reduction g from β to δ by $g(n) = k$ where k is the minimum integer with $f(n) = f(i_k)$. As the i_n's cover the whole range of f, g is well-defined. For any given n, suppose $g(n) = k_1$ and $g(n+1) = k_2$, so that $f(n) = f(i_{k_1})$ and $f(n+1) = f(i_{k_2})$. By Condition (b), $d(n, i_{k_1}) \leq C$ and $d(n+1, i_{k_2}) \leq C$, and so

$$\begin{aligned} d(g(n), g(n+1)) &= d(k_1, k_2) \\ &\leq d(i_{k_1}, i_{k_2}) \\ &\leq d(n, i_{k_1}) + d(n, n+1) + d(n+1, i_{k_2}) \\ &\leq 2C + 1\,. \end{aligned}$$

Hence, g satisfies Condition (a) with constant $2C+1$. To verify that g satisfies Condition (b) for some constant, fix any n and apply Condition (b) $C \cdot (C+1)$ times to f, giving $f(n) + C \cdot (C+1) \leq f(n + C \cdot (C+1)^2)$. Suppose $g(n) = i_{k_1}$ and $g(n + C \cdot (C+1)^2) = i_{k_2}$, so that $f(n) = f(i_{k_1})$ and $f(n + C \cdot (C+1)^2) = f(i_{k_2})$. Then, $d(f(i_{k_1}), f(i_{k_2})) = d(f(n), f(n + C \cdot (C+1)^2)) \geq C \cdot (C+1)$. So by applying Condition (a) $d(i_{k_1}, i_{k_2})$ times to f, we have $C \cdot d(i_{k_1}, i_{k_2}) \geq d(f(i_{k_1}), f(i_{k_2})) \geq C \cdot (C+1)$. Dividing both sides of the inequality by C yields $d(i_{k_1}, i_{k_2}) \geq C + 1$. Applying the contrapositive of Condition (b) to f then gives $f(i_{k_2}) \geq f(i_{k_1}) \Rightarrow i_{k_2} + C \geq i_{k_1}$. Since $d(i_{k_1}, i_{k_2}) \geq C+1$, this implies that $g(n + C \cdot (C+1)^2)$ $= i_{k_2} > i_{k_1} = g(n)$. Thus, g satisfies Condition (b) with constant $C \cdot (C+1)^2 - 1$. ◀

4.2 1qi-Degrees Within mqi-Degrees

We now investigate the structural properties of 1qi-degrees within individual mqi-degrees. As will be seen shortly, these properties can vary quite a bit depending on the choice of the mqi-degree.

▶ **Proposition 14.** *There exists an infinite string α such that $[\alpha]_{mqi}$ is the union of an infinite ascending chain of 1qi-degrees.*

Proof. Let $\Sigma = \{0, 1\}$ and let $\alpha = 10^\omega$. Then, $[\alpha]_{mqi}$ consists of all infinite strings with a finite, positive number of occurrences of 1. Given any infinite string β with $k \geq 1$ occurrences of 1, β is 1qi-equivalent to a string γ in $[\alpha]_{mqi}$ iff γ has exactly k occurrences of 1. If $1 \leq k < k'$, then each string $\beta \in [\alpha]_{mqi}$ with exactly k occurrences of 1 is 1qi-reducible to any string $\beta' \in [\alpha]_{mqi}$ with exactly k' occurrences of 1. Thus, $[\alpha]_{mqi}$ is the union of an ascending chain $[\alpha]_{1qi} < [110^\omega]_{1qi} < [1110^\omega]_{1qi} < \ldots$, where the i-th term of this chain is 1^i0^ω. ◀

▶ **Proposition 15.** *There exists an infinite string α such that the poset of 1qi-degrees within $[\alpha]_{mqi}$ is isomorphic to $\mathbb{N}^2$ with the componentwise ordering. That is, $[\alpha]_{mqi}$ is the union of infinitely many disjoint infinite ascending chains of 1qi-degrees such that every pair of these ascending chains has incomparable elements. Also, $[\alpha]_{mqi}$ does not contain infinite anti-chains of 1qi-degrees.*

Proof. Let $\Sigma = \{0, 1, 2\}$ and let $\alpha = 120^\omega$. Then, $[\alpha]_{mqi}$ consists of all infinite strings with a finite, positive number of 1's and a finite, positive number of 2's. Furthermore, $[\alpha]_{1qi}$ consists of all infinite strings with exactly one occurrence of 1 and exactly one occurrence of 2.

Based on the proof of Proposition 14, $[\alpha]_{mqi}$ is the union, over all $k \geq 1$, of chains of the form $[12^k0^\omega]_{1qi} < [1^22^k0^\omega]_{1qi} < \ldots$, where the i-th term of each chain is $[1^i2^k0^\omega]_{1qi}$. Given any two chains $\Gamma_j = \{[1^i2^j0^\omega]_{1qi} : i \in \mathbb{N}\}$ and $\Gamma_k = \{[1^i2^k0^\omega]_{1qi} : i \in \mathbb{N}\}$, where $j < k$, the classes $[1^22^j0^\omega]_{1qi} \in \Gamma_j$ and $[12^k0^\omega]_{1qi} \in \Gamma_k$ are incomparable with respect to $\leq_{1qi}$.

It remains to show that any anti-chain of 1qi-degrees contained in $[\alpha]_{mqi}$ must be finite. Consider any anti-chain of 1qi-degrees containing the class $[1^i2^j0^\omega]_{1qi} \subsetneq [\alpha]_{mqi}$. Every element of this anti-chain that is different from $[1^i2^j0^\omega]_{1qi}$ is of the form $[1^{i'}2^{j'}0^\omega]_{1qi}$, where either $i < i'$ and $j > j'$, or $i > i'$ and $j < j'$. Thus, if the anti-chain were infinite, then it would contain at least 2 1qi-degrees, $[\beta]_{1qi}$ and $[\gamma]_{1qi}$, such that either β has the same number of occurrences of 1 as γ, or β has the same number of occurrences of 2 as γ. This is a contradiction as it would imply that either $\beta \leq_{1qi} \gamma$ or $\gamma \leq_{1qi} \beta$. ◀

4.3 pqi-Reductions

We now discuss pqi-reductions, which are the most stringent kind of quasi-isometric reductions considered in the present work. Pqi-reductions are 1qi-reductions that are surjective; an example of such a reduction is the mapping $2m - 1 \mapsto 2m$, $2m \mapsto 2m - 1$ from $(01)^\omega$ to $(10)^\omega$. We record a few elementary properties of pqi-reductions.

▶ **Lemma 16.** *If f is a pqi-reduction and if $x + D = f(x)$ for some $D \geq 1$ and some $x \in \mathbb{N}$, then there are at least D positions $y > x$ such that $f(y) < f(x)$.*

Proof. If $x + D = f(x)$ for some $D \geq 1$, then $\{1, \ldots, x + D - 1\} \setminus \{f(1), \ldots, f(x - 1)\}$ must contain at least D elements as the former set contains D more elements than the latter. Thus, for f to be a bijection, there must exist at least D positions $y > x$ that are mapped by f into $\{1, \ldots, x + D - 1\} \setminus \{f(1), \ldots, f(x - 1)\}$. ◀

We next observe that for any pqi-reduction f, there is a uniform upper bound on the difference $x - f(x)$.

▶ **Proposition 17.** *If f is a C-pqi-reduction, then for all $x \in \mathbb{N}$, $x - f(x) < 2C^2 + 1$.*

Proof. Assume, by way of contradiction, that there is some $x \in \mathbb{N}$ such that $x - f(x) \geq 2C^2 + 1$. First, suppose that there are at least $C^2 + 1$ numbers z such that $z > x$ and $f(z) \in \{f(x) + 1, f(x) + 2, \ldots, x - 1\}$. Then, there are at least $C^2 + 1$ numbers z' such that $z' < x$ and $f(z') > x > f(x)$, among which there is at least one z'_0 with $f(z'_0) \geq x + C^2 + 1$. This would contradict the fact that by the Small Cross-Over Lemma (Lemma 8), $z'_0 < x \Rightarrow f(z'_0) \leq f(x) + C^2 < x + C^2$.

Second, suppose that f maps at most C^2 numbers greater than x into $\{f(x) + 1, f(x) + 2, \ldots, x - 1\}$. Then, there are at least $C^2 + 1$ numbers less than x that are mapped into $\{f(x) + 1, f(x) + 2, \ldots, x - 1\}$ and in particular, there is at least one number $y < x$ such that $f(y) \geq f(x) + C^2 + 1$, contradicting the Small Cross-over Lemma. Thus, for all $x \in \mathbb{N}$, $x - f(x) < 2C^2 + 1$. ◀

Lemma 16 and Proposition 17 together give a uniform upper bound on the absolute difference between any position number and its image under a C-pqi-reduction.

▶ **Corollary 18.** *If f is a C-pqi-reduction, then for all $x \in \mathbb{N}$, $|x - f(x)| < 2C^2 + 1$.*

Proof. By Condition (b), there cannot be more than C numbers y such that $y > x$ and $f(y) < f(x)$. Lemma 16 thus implies that there cannot exist any $D > C$ such that $x + D = f(x)$, and so $f(x) - x \leq C$. Combining the latter inequality with that in Proposition 17 yields $|x - f(x)| < \max\{C+1, 2C^2+1\} = 2C^2 + 1$. ◀

Given any infinite string α, it was observed earlier that by the definitions of pqi, 1qi and mqi-reductions, $[\alpha]_{pqi} \subseteq [\alpha]_{1qi} \subseteq [\alpha]_{mqi}$. In the following example, we give instances of strings α where each of the two subset relations is proper or can be replaced with the equals relation.

▶ **Example 19.** **(a)** $[\alpha]_{pqi} = [\alpha]_{1qi} = [\alpha]_{mqi}$. Set $\alpha = 0^\omega$. For any infinite string γ such that $\gamma \leq_{mqi} 0^\omega$, γ can only contain occurrences of 0, and therefore $[0^\omega]_{pqi} = [0^\omega]_{1qi} = [0^\omega]_{mqi} = \{0^\omega\}$.

(b) $[\alpha]_{1qi} = [\alpha]_{mqi}$ and $[\alpha]_{pqi} \subset [\alpha]_{1qi}$. Set $\alpha = (01)^\omega$. First, $(001)^\omega \leq_{1qi} (01)^\omega$, as witnessed by the 1qi-reduction $3n-2 \mapsto 4n-3, 3n-1 \mapsto 4n-1, 3n \mapsto 4n$ for $n \in \mathbb{N}$. We also have $(01)^\omega \leq_{1qi} (001)^\omega$ via the 1qi-reduction $2n-1 \mapsto 3n-2, 2n \mapsto 3n$ for $n \in \mathbb{N}$. However, $(001)^\omega \notin [(01)^\omega]_{pqi}$ because the density of 0's and 1's in the two strings are different, making it impossible to construct a permutation reduction between them. More formally, if there were a pqi-reduction from $(001)^\omega$ to $(01)^\omega$, then by Corollary 18, there would be a constant D such that for each n, the first $3n$ positions of $(001)^\omega$ are mapped into the first $3n + D$ positions of $(01)^\omega$. But the first $3n$ positions of $(001)^\omega$ contain $2n$ occurrences of 0 while the first $3n + D$ positions of $(01)^\omega$ contain at most $\lceil 1.5n + \frac{D}{2} \rceil$ occurrences of 0, and for large enough n, one has $2n > \lceil 1.5n + \frac{D}{2} \rceil$. Hence, no pqi-reduction from $(001)^\omega$ to $(01)^\omega$ can exist, and so $[\alpha]_{pqi} \subset [\alpha]_{1qi}$.
To see that $[(01)^\omega]_{mqi} \subseteq [(01)^\omega]_{1qi}$, we first note that any string that is mqi-reducible to $(01)^\omega$ (or to any other recursive string) must be recursive. Thus, if $\beta \leq_{mqi} (01)^\omega$, then a 1qi-reduction from β to $(01)^\omega$ can be constructed by mapping the n-th position of β to the position of the matching letter in the n-th occurrence of 01 in $(01)^\omega$. Next, suppose that f is a C-mqi-reduction from $(01)^\omega$ to β. By Corollary 7, f maps the positions of $(01)^\omega$ to a sequence of positions of β that contains 0 and 1 every $2C$ positions. Thus, a 1qi-reduction can be constructed from $(01)^\omega$ to β by mapping, for each n, the $(2n-1)$-st and $(2n)$-th positions of $(01)^\omega$ to the positions of the first occurrence of 0 and first occurrence of 1 respectively in the interval $[2C(n-1)+1, 2Cn]$ of positions of β. Therefore, $\beta \in [(01)^\omega]_{1qi}$.

(c) $[\alpha]_{1qi} \subset [\alpha]_{mqi}$ and $[\alpha]_{pqi} = [\alpha]_{1qi}$. Set $\alpha = 10^\omega$. We recall from the proof of Proposition 14 that $[10^\omega]_{pqi}$ and $[10^\omega]_{1qi}$ consist of all binary strings with a single occurrence of 1, while $[10^\omega]_{mqi}$ consists of all binary strings with a finite, positive number of occurrences of 1. Thus, $[10^\omega]_{pqi} = [10^\omega]_{1qi}$ and $[10^\omega]_{pqi} \neq [10^\omega]_{mqi}$.

(d) $[\alpha]_{pqi} \subset [\alpha]_{1qi} \subset [\alpha]_{mqi}$. Set $\alpha = (0^n 1)_{n=1}^\infty$, the concatenation of all strings $0^n 1$ where $n \in \mathbb{N}$. Then, $\beta = (0^n 11)_{n=1}^\infty \in [\alpha]_{mqi}$; however, $\beta \notin [\alpha]_{1qi}$ as each pair of adjacent positions of 1's in β must be mapped to distinct positions of 1's in α, but the distance between the n-th and $(n+1)$-st occurrences of 1 in α increases linearly with n, meaning that Condition (a) cannot be satisfied.
To construct an mqi-reduction from β to α, map the positions of the substring $0^n 11$ of β to the positions of the substring $0^n 1$ of α as follows: for $k \in \{1, \ldots, n\}$, the position of the k-th occurrence of 0 in $0^n 11$ is mapped to that of the k-th occurrence of 0 in $0^n 1$, while the two positions of 1's in $0^n 11$ are mapped to the position of the single 1 in $0^n 1$. For an mqi-reduction from α to β, for each substring $0^n 1$ of α and each substring $0^n 11$ of β, the positions of 0^n in $0^n 1$ are mapped to the corresponding positions of 0^n in $0^n 11$,

while the position of 1 in 0^n1 is mapped to the position of the first occurrence of 1 in 0^n11. Thus, $\beta \in [\alpha]_{mqi}$.
Furthermore, $\gamma = 1(0^n1)_{n=1}^{\infty} \in [\alpha]_{1qi}$ but $\gamma \notin [\alpha]_{pqi}$. The reason for γ not being pqi-reducible to α is similar to that given in Example (b). If such a pqi-reduction did exist, then by Corollary 18, there would exist a constant D such that for all n, the first $1 + \sum_{k=1}^{n}(k+1) = 1 + \frac{n(n+3)}{2}$ positions of γ are mapped into the first $1 + \frac{n(n+3)}{2} + D$ positions of α. But the first $1 + \frac{n(n+3)}{2}$ positions of γ contain $n+1$ occurrences of 1 and for large enough n, the first $1 + \frac{n(n+3)}{2} + D$ positions of α contain at most n occurrences of 1. Hence, no pqi-reduction from γ to α is possible.
For a 1qi-reduction from γ to α, map the starting position of γ, where the letter 1 occurs, to the first occurrence of 1 in α. For subsequent positions of γ, for each $n \geq 1$, the set of positions of γ where the substring 0^n1 occurs can be mapped in a one-to-one fashion into the set of positions of α where the substring $0^{n+1}1$ occurs. To see that α is 1qi-reducible to γ, it suffices to observe that α is a suffix of γ, so one can map the positions of α in a one-to-one fashion to the positions of the suffix of γ corresponding to α.

Proposition 21 extends the first example in Example 19 by characterising all recursive strings whose pqi, 1qi and mqi-degrees all coincide. In fact, there are only $|\Sigma|$ many such strings: those of the form a_i^ω, where $a_i \in \Sigma$. We call the pqi, 1qi and mqi-degrees of such strings *trivial*.

▶ **Definition 20.** *The pqi, 1qi and mqi-degrees of each string a_i^ω, where $a_i \in \Sigma$, will be called* trivial *pqi, 1qi and mqi-degrees respectively.*

▶ **Proposition 21.** *If, for some recursive string α, $[\alpha]_{pqi} = [\alpha]_{1qi} = [\alpha]_{mqi}$, then all three degree classes are trivial.*

Proof. Suppose $\Sigma = \{a_1, \ldots, a_l\}$, where without loss of generality it may be assumed that $l \geq 2$. Fix any string α whose pqi, 1qi and mqi-degrees all coincide, and assume, by way of contradiction, that at least two distinct letters occur in α. Now we consider the following case distinction.

Case (i): *There is some a_i that occurs a finite, positive number of times in α.* Let $\beta = a_i\alpha$. Then, $\beta \notin [\alpha]_{pqi}$ as β contains exactly one more occurrence of a_i than α, so a colour-preserving bijection between α and β cannot exist. On the other hand, $\beta \in [\alpha]_{mqi}$. First, $\beta \leq_{mqi} \alpha$ via a reduction that maps the first position of β to the position of the first occurrence of a_i in α; the positions of the suffix α of β are then mapped in a one-to-one fashion to the corresponding positions of α. Second, $\alpha \leq_{1qi} \beta$ as α is a suffix of β.

Case (ii): *Each letter in α occurs infinitely often.*

Subcase (a): *There is some letter a_i occurring in α such that there is no uniform upper bound on the distance between successive occurrences of a_i.* Consider the string $\beta = a_i\alpha$. The same argument as in Case (i) shows that $\beta \in [\alpha]_{mqi}$. We show that $\beta \not\leq_{pqi} \alpha$. If there were a pqi-reduction from β to α, then by Corollary 18, there would exist a constant D such that for all n, the first n positions of β are mapped into the first $n + D$ positions of α. Now pick k large enough so that the distance between the k-th and $(k+1)$-st occurrences of a_i in α is greater than $D + 1$. Let n be the position number of the $(k+1)$-st occurrence of a_i in β; then $n - 1$ is the position number of the k-th occurrence of a_i in α. By Corollary 18, the positions of the first $k + 1$ occurrences of a_i in β must be mapped into the first $n + D$ positions of α, but since the $(k+1)$-st occurrence of a_i in α is at a position greater than $n - 1 + D + 1 = n + D$, it follows that such a mapping cannot be one-to-one. Hence, $\beta \notin [\alpha]_{pqi}$.

Subcase (b): *There is some $D > 0$ such that for every letter a_i occurring in α, the distance between successive occurrences of a_i is at most D.* Let D be the largest distance between successive occurrences of the same letter in α. As α is recursive, one can construct a 1qi-reduction from α to $(a_1 \cdots a_l)^\omega$ by mapping the n-th position of α to the position in the n-th occurrence of $a_1 \cdots a_l$ where the corresponding letters match. Furthermore, a 1qi-reduction from $(a_1 \cdots a_l)^\omega$ to α may be constructed as follows. Let m be the first position of α such that every letter in α occurs at least once before position $m+1$. By the choice of D, any interval of D positions of α starting after position m contains every letter. We define a mapping from $(a_1 \cdots a_l)^\omega$ to α such that the position of the first occurrence of each letter in $(a_1 \cdots a_l)^\omega$ is mapped to the position of the first occurrence of the corresponding letter in α, and for $k \geq 2$, for the k-th occurrence of $a_1 \cdots a_l$, the position of each letter is mapped to the position of the first occurrence of the corresponding letter in the interval $[m + (k-1)D + 1, m + kD]$ of positions of α. This mapping is one-to-one and also satisfies Conditions (a) and (b) for the constant $\max\{m + D, 2D\}$, so it is indeed a 1qi-reduction. Therefore, $[(a_1 \cdots a_l)^\omega]_{1qi} = [\alpha]_{1qi} = [\alpha]_{mqi}$.
Now consider $\beta = (a_1 a_1 a_2 \cdots a_l)^\omega$. A 1qi-reduction from β to $(a_1 \cdots a_l)^\omega$ can be constructed by mapping, for the k-th occurrence of $a_1 a_1 a_2 \cdots a_l$, the position of the first occurrence of a_1 to the position of a_1 in the $(2k-1)$-st occurrence of $a_1 \cdots a_l$, and the positions of subsequent letters to the positions of matching letters in the $(2k)$-th occurrence of $a_1 \cdots a_l$. A 1qi-reduction from $(a_1 \cdots a_l)^\omega$ to β can also be constructed by mapping the positions of the k-th occurrence of $(a_1 \cdots a_l)$ to positions of matching letters in the k-th occurrence of $(a_1 a_1 a_2 \cdots a_l)$. Thus, $\beta \in [\alpha]_{1qi}$. On the other hand, a proof entirely similar to that in part (b) of Example 19 shows that $\beta \notin [\alpha]_{pqi}$. We conclude that $[\alpha]_{pqi} \neq [\alpha]_{1qi}$. ◀

We observe next that every non-trivial pqi degree must be infinite.

▶ **Proposition 22.** *All non-trivial pqi-degrees are infinite.*

Proof. Suppose that at least two distinct letters occur in α. Fix a letter, say a_1, that occurs infinitely often in α. Let a_2 be a letter different from a_1 that occurs in α. For each $n \in \mathbb{N}$, let $\beta_n = a_1^n a_2 \alpha^{(n+1)}$, where $\alpha^{(n+1)}$ is obtained from α by removing the first occurrence of a_2 as well as the first n occurrences of a_1. Since β_n is built from α by permuting the letters occurring at a finite set of positions of α, $\beta_n \in [\alpha]_{pqi}$. As the β_n's are all distinct, it follows that $[\alpha]_{pqi}$ is indeed infinite. ◀

We close this subsection by illustrating an application of Proposition 22, showing that if the mqi-degree of α contains at least two distinct strings such that one is 1qi-reducible to the other, then the first string is 1qi-reducible to infinitely many strings in $[\alpha]_{mqi}$.

▶ **Proposition 23.** *If there exist distinct $\beta \in [\alpha]_{mqi}$ and $\gamma \in [\alpha]_{mqi}$ such that $\beta \leq_{1qi} \gamma$, then β is 1qi-reducible to infinitely many strings in $[\alpha]_{mqi}$.*

Proof. Suppose that $\beta \leq_{1qi} \gamma$ and $\beta \neq \gamma$ for some $\beta \in [\alpha]_{mqi}$ and $\gamma \in [\alpha]_{mqi}$. Then, $[\alpha]_{mqi}$ is non-trivial, so by Proposition 22, $[\gamma]_{pqi}$ is infinite. Since $[\gamma]_{pqi} \subseteq [\gamma]_{1qi}$, $[\gamma]_{1qi}$ is also infinite. Thus, β is 1qi-reducible to each of the infinitely many strings in $[\gamma]_{1qi}$. ◀

4.4 The Partial Order of All mqi-Degrees

As discussed earlier, Khoussainov and Takisaka [10] observed that for any alphabet $\Sigma = \{a_1, \ldots, a_l\}$, the partial order Σ^ω_{qi} has a greatest element equal to $[(a_1 \cdots a_l)^\omega]_{qi}$. Their proof also extends to the partial order of all recursive mqi-degrees, showing that for each recursive string α, $[\alpha]_{mqi} \leq_{mqi} [(a_1 \cdots a_l)^\omega]_{mqi}$. We next prove that there is a pair of recursive mqi-degrees whose join is precisely the maximum recursive mqi-degree $[(a_1 \cdots a_l)^\omega]_{mqi}$.

▶ **Proposition 24.** *Suppose that $\Sigma = \{a_1, \ldots, a_l\}$. Then, there exist two distinct infinite strings α and β such that $[(a_1 \cdots a_l)^\omega]_{mqi}$ is the unique recursive common upper bound of $[\alpha]_{mqi}$ and of $[\beta]_{mqi}$ under $\leq_{mqi}$.*

Proof. Let $\alpha = (a_1)^\omega$ and $\beta = (a_2 a_3 \cdots a_l)^\omega$. Suppose that for some recursive string γ, $\alpha \leq_{mqi} \gamma$ via a C-mqi-reduction. Since a_1 is the only letter occurring in α, Condition (a) implies that there must be at least one occurrence of a_1 in γ every C positions. Similarly, if $\beta \leq_{mqi} \gamma$ via a C'-mqi-reduction, then for each a_i with $i \geq 2$, since a_i occurs every $l-1$ positions, it must also occur in γ every $C' \cdot (l-1)$ positions. Hence, there exists a constant C'' such that every substring of γ of length C'' contains at least one occurrence of a_i for every $i \in \{1, \ldots, l\}$, and therefore $(a_1 \cdots a_l)^\omega \leq_{mqi} \gamma$. Since $\gamma \leq_{mqi} (a_1 \cdots a_l)^\omega$ follows from the proof of [10, Proposition II.1], one has $\gamma \in [(a_1 \cdots a_l)^\omega]_{mqi}$, as required. ◀

Khoussainov and Takisaka [10] showed that the partial order Σ^ω_{qi} is not dense. In particular, given any distinct $a_i, a_j \in \Sigma$, there is no element $[\beta]_{qi}$ that is strictly between the minimal element $[(a_j)^\omega]_{qi}$ and the "atom" $[a_i(a_j)^\omega]_{qi}$ [10, Proposition II.1]. The next theorem shows similarly that the partial order Σ^ω_{mqi} is non-dense with respect to pairs of mqi-degrees.

▶ **Theorem 25.** *There exist two pairs (α, β) and (γ, δ) of recursive strings such that both α and β are mqi-reducible to γ as well as mqi-reducible to δ, but there is no string ξ such that $\alpha \leq_{mqi} \xi, \beta \leq_{mqi} \xi, \xi \leq_{mqi} \gamma$ and $\xi \leq_{mqi} \delta$.*

Proof. Let $\Sigma = \{0, 1\}$. Define the strings

$$\alpha = \sigma_1 \sigma_2 \ldots, \quad \text{where } \sigma_i = (01)^{2^{2^i}} 0^i 1^i\,;$$
$$\beta = \tau_1 \tau_2 \ldots, \quad \text{where } \tau_i = (01)^{2^{2^i}} 1^i 0^i\,;$$
$$\gamma = \mu_1 \mu_2 \ldots, \quad \text{where } \mu_i = (01)^{2^{2^i}} 0^i\,;$$
$$\delta = \nu_1 \nu_2 \ldots, \quad \text{where } \nu_i = (01)^{2^{2^i}} 1^i\,.$$

We first show that $\alpha \leq_{mqi} \gamma$. For each $i \in \mathbb{N}$, define the following intervals of positions.

$K_i = [k_i, k_i + 2^{2^i+1} - 1]$ is the interval of positions of the substring $(01)^{2^{2^i}}$ of σ_i in α.
$R_i = [r_i, r_i + 2i - 1]$ is the interval of positions of the substring $0^i 1^i$ of σ_i in α.
$L_i = [l_i, l_i + 2^{2^i+1} - 1]$ is the interval of positions of the substring $(01)^{2^{2^i}}$ of μ_i in γ.
$L'_i = [l_i + 2^{2^i+1}, l_i + 2^{2^i+1} + i - 1]$ is the interval of positions of the substring 0^i of μ_i in γ.

Define an mqi-reduction g from α to γ as follows. For $i \in \mathbb{N}$,

$$g(k_i + 4w + 2u + x) = l_i + 2(i-1) + 2w + x\,, \quad 0 \leq w \leq i-2, u, x \in \{0,1\}\,;$$
$$g(k_i + m) = l_i + m\,, \quad 4i - 4 \leq m \leq 2^{2^i+1} - 1\,;$$
$$g(r_i + m) = l_i + 2^{2^i+1} + m\,, \quad 0 \leq m \leq i-1\,;$$
$$g(r_i + i + m) = l_{i+1} + 2m + 1\,, \quad 0 \leq m \leq i-1.$$

In other words, g maps the positions of the substring $(01)^{2^{2^i}}$ of σ_i in α to the last $2^{2^i+1}-2(i-1)$ positions of the substring $(01)^{2^{2^i}}$ of μ_i in γ. The mapping is as follows. Each of the i-th to the $2(i-1)$-th pairs 01 of μ_i is an image of two consecutive pairs 01 of σ_i. Then, the last $2^{2^i+1}-4(i-1)$ positions of the substring $(01)^{2^{2^i}}$ of μ_i in γ are mapped from the last $2^{2^i+1}-4(i-1)$ positions of the substring $(01)^{2^{2^i}}$ of σ_i in α in a one-to-one fashion. Furthermore, g maps the positions of the substring 0^i of σ_i in α to the positions of the substring 0^i of μ_i in γ in a one-to-one fashion. Then, the positions of the substring 1^i of σ_i in α is mapped to the first i positions of 1 in μ_{i+1} in γ, ending at the $2i$-th position of the substring $(01)^{2^{2^{i+1}}}$ of μ_{i+1} in γ. Thus, g is a 4-mqi-reduction from α to γ.

A similar mqi-reduction can be constructed from β to γ. In this case, the mqi-reduction maps the interval K_i to the prefix of the interval L_i where the last $2i$ positions are cut off and each of the last i pairs of positions of the prefix is the image of two consecutive pairs of positions of K_i. By symmetrical constructions, one obtains mqi-reductions from α to δ as well as from β to δ.

Now assume, for the sake of contradiction, that there is a string ξ and there are mqi-reductions f_1 from α to ξ, f_2 from β to ξ, f_3 from ξ to γ and f_4 from ξ to δ with constants C_1, C_2, C_3 and C_4 respectively. Set $C = \max\{C_1, C_2, C_3, C_4\}$ and fix some $n > 2C^7 + 1$.

For $i \in \mathbb{N}$, let $K'_i = [k_i + C^2 + 2, k_i + 2^{2^i+1} - 3 - C^2]$ be the interval obtained from K_i by removing the first and last $C^2 + 2$ positions. We make the following observation.

▷ Claim 26. For all positions $m \in K'_n$, for $i \in \{1, 2\}$ and $j \in \{3, 4\}$, $f_j(f_i(m)) \in L_n$.

Proof. To simplify the subsequent argument, we assume, without loss of generality, that at least one position of γ lies in the intersection of $\bigcup_{i<n} L_i \cup L'_i$ and the range of $f_3 \circ f_1$. We first note that the map $f_3 \circ f_1$ is a C^2-mqi-reduction from α to γ. Since the length of the interval $[1, l_n - 1]$ is $\sum_{i=1}^{n-1}(i + 2^{2^i+1}) \le (n-1) \cdot (n - 1 + 2^{2^{n-1}+1}) \le 3n \cdot 2^{2^{n-1}}$, there are at most $3C^2 n \cdot 2^{2^{n-1}}$ positions of K_n in α that are mapped into the interval $[1, l_n - 1]$ of γ. Since $|K_n| = 2^{2^n+1} > 3C^2 n \cdot 2^{2^{n-1}}$ and there is, by assumption, at least one point in the range of $f_3 \circ f_1$ that lies in $\bigcup_{i<n} L_i \cup L'_i$, it follows from the fact that there cannot be gaps larger than C^2 in the range of $f_3 \circ f_1$ that at least one position of K_n, say m_0, must be mapped under $f_3 \circ f_1$ into the interval L_n of γ.

Thus, $f_3 \circ f_1$ cannot map any position of K_n into L_{n-1}. For, if there were a least such position $m_1 \in K_n$ and $m_1 < m_0$, then by Condition (a) and using the fact that $n - 1 > 2C^2$, $f_3 \circ f_1$ must map $m_1 + 1$ into either L_{n-1} or to one of the first C^2 positions of L'_{n-1}. In the former case, $f_3 \circ f_1$ must map $m_1 + 2$ into either L_{n-1} or to one of the first C^2 positions of L'_{n-1} and the same argument can be iterated. In the latter case, the letters of α at positions $m_1 + 1$ and $m_1 + 2$ must be 0 and 1 respectively, which implies that $f_3 \circ f_1$ must map position $m_1 + 2$ into L_{n-1}, and the same argument can again be iterated. Iterating the argument, it would then follow that $f_3 \circ f_1$ maps m_0 into L_{n-1} or to one of the first C^2 positions of L'_{n-1}, a contradiction. A similar argument applies in the case that $m_1 > m_0$.

Furthermore, if $f_3 \circ f_1$ maps some position m of K_n into L'_{n-1}, then m must be contained in the first $C^2 + 2$ positions of K_n. For, suppose that m occurs after the first $C^2 + 2$ positions of K_n, then each of the first two positions of K_n is at least $C^2 + 1$ positions away from m. So, by Condition (b), $f_3 \circ f_1$ must map the first two positions of K_n to some position of γ before $(f_3 \circ f_1)(m)$, which lies in L'_{n-1}; but this is impossible since the letter in the second position of K_n is 1 and L'_{n-1} contains only 0's and, as was shown earlier, no position of K_n is mapped into L_{n-1}. Therefore, $f_3 \circ f_1$ maps at most $C^2 + 2$ positions of K_n in α into the interval L'_{n-1} of γ, and these positions must occur within the first $C^2 + 2$ positions of K_n. Similarly, $f_3 \circ f_1$ maps at most $C^2 + 2$ positions of K_n in α into the interval L'_n of γ, and these positions must occur within the last $C^2 + 2$ positions of K_n. Summing up, for each position m in the interval $K'_n = [k_n + C^2 + 2, k_n + 2^{2^n+1} - 3 - C^2]$, $f_3(f_1(m)) \in L_n$. Similar arguments show that $f_j(f_i(m)) \in L_n$ for $(i,j) \in \{(1,4),(2,3),(2,4)\}$. ◁

Now define the sets $H_i = f_1(K'_i) \cup f_2(K'_i)$ for $i \in \mathbb{N}$. We show that the sets H_n and H_{n+1} are non-overlapping by proving $\max(H_n) < \min(H_{n+1})$. By Claim 26 above, for all $m \in H_n$ and $j \in \{3,4\}$ we have $f_j(m) \in L_n$. Then, we have $f(\min(H_{n+1})) - f(\max(H_n)) \geq \min(L_{n+1}) - \max(L_n) = n + 1 > 2C^7 + 2 > C^2$. So by the Small Cross-Over Lemma, $\min(H_{n+1}) > \max(H_n)$.

Consider the interval $[\min(H_n), \max(H_n)]$ in the domain of ξ. By Claim 26, f_3 (resp. f_4) maps each element of H_n into L_n. Fix any other position z in the interval. Then, f_3 cannot map z into L'_n, which is the set of positions in γ of the string 0^n. To see this, we note that if ℓ and $\ell + 1$ are the two largest values of K_n, then ℓ is at least $C^2 + 1$ more than the value x such that $f_i(x) = \max(H_n)$ for some $i \in \{1,2\}$, and so by Condition (b), $z + C < \max(H_n) + C < f_k(\ell + 1)$ for $k \in \{1,2\}$. Thus, $f_3(z) < f_3(f_k(\ell + 1))$ for $k \in \{1,2\}$. Furthermore, by applying Condition (a) repeatedly to f_3 and then to f_k, we have $d(f_3(f_k(\ell+1)), f_3(f_k(\max(K'_n)))) \leq C \cdot d(f_k(\ell+1), f_k(\max(K'_n))) \leq C^2 \cdot (C^2+2) = C^4 + 2C^2$. Since $f_3(f_k(\max(K'_n))) \in L_n$ and we fixed $n > 2C^7 + 1$, then $f_3(f_k(\ell + 1)) \notin L_{n+1}$. Furthermore, the letter at position $f_3(f_k(\ell + 1))$ of γ is 1. Thus, $f_3(z)$ cannot lie in L'_n as there is no occurrence of 1 in L'_n. A similar argument, using position ℓ rather than position $\ell + 1$, shows that $f_4(z)$ cannot lie in L'_n. One can also prove similarly that none of the positions in the interval $[\min(H_{n+1}), \max(H_{n+1})]$ is mapped by f_3 or f_4 into the interval L'_n.

Next, we consider the positions of ξ between $\max(H_n)$ and $\min(H_{n+1})$. Since none of the positions of ξ in the union $[\min(H_n), \max(H_n)] \cup [\min(H_{n+1}), \max(H_{n+1})]$ is mapped by f_3 into L'_n and L'_n is an interval of length $n > 2C^7$, Lemma 6 implies that there are at least $\lfloor \frac{n}{C_3} \rfloor$ positions of ξ between H_n and H_{n+1} which are mapped into L'_n.

▷ Claim 27. The string ξ contains a substring of 0's (resp. 1's) of length $\Omega(C^4)$ between H_n and H_{n+1} such that all positions of this substring are mapped by f_3 (resp. f_4) into L'_n.

Proof. Let $m_1, \ldots, m_\ell$ be all the positions of L'_n in the range of f_3, where $m_1 < m_2 < \ldots < m_\ell$. Since L'_n is an interval of length n, Lemma 6 implies that $\ell \geq \lfloor \frac{n}{C_3} \rfloor$. Let $P = f_3^{-1}(L'_n) \setminus f_3^{-1}(\{m_i : 1 \leq i \leq C_3\} \cup \{m_i : \ell - C_3 + 1 \leq i \leq \ell\})$ be the set of positions of ξ which are mapped into L'_n but not to any of the first C_3 or the last C_3 positions of $L'_n \cap \text{range}(f_3)$. By Lemma 5,

$$|f_3^{-1}(\{m_i : 1 \leq i \leq C_3\} \cup \{m_i : \ell - C_3 + 1 \leq i \leq \ell\})| \leq 2C^2\,,$$

and thus

$$|P| \geq \ell - 2C^2 \geq \left\lfloor \frac{n}{C_3} \right\rfloor - 2C^2 = \Omega(C^6),$$

where we have used the fact that $n > 2C^7$. The set P is split into at most $2C^2$ groups, each included in an interval not containing any position in $f_3^{-1}(\{m_i : 1 \leq i \leq C_3\} \cup \{m_i : \ell - C_3 + 1 \leq i \leq \ell\})$, and so by the pigeonhole principle there is an interval between H_n and H_{n+1} containing at least $\frac{\Omega(C^6)}{2C^2} = \Omega(C^4)$ positions that are mapped by f_3 into L'_n, but no position in this interval is mapped to any of the first C_3 or the last C_3 positions of L'_n. Let m' (resp. m'') be the minimum (resp. maximum) position in this interval which is mapped to a position in L'_n. If there were a least position $m''' \in [m', m'']$ such that $f_3(m''') \notin L'_n$, then by the choice of m' and m'', $f_3(m''') \in L_n \cup L_{n+1}$, but this is impossible as it would imply that $d(f_3(m'''), f_3(m'''-1)) > C_3$, contradicting Condition (a). Thus, $[m', m'']$ is an interval between H_n and H_{n+1} of length $\Omega(C^4)$ such that $f_3([m', m'']) \subseteq L'_n$. An analogous argument, replacing f_3 by f_4 (thereby considering the mapping from ξ to δ), shows that ξ contains a substring of 1's of length $\Omega(C^4)$ between H_n and H_{n+1} such that f_4 maps all positions of this substring into L'_n. ◁

It is shown next that between H_n and H_{n+1}, there cannot exist two $\Omega(C^4)$-long substrings of 0's (resp. 1's) such that an $\Omega(C^4)$-long substring of 1's (resp. 0's) lies between them.

▷ Claim 28. There cannot exist between H_n and H_{n+1} two substrings $\delta_1 \in \{0\}^*$ and $\delta_2 \in \{0\}^*$ of ξ with $|\delta_1| = \Omega(C^4)$ and $|\delta_2| = \Omega(C^4)$ such that some $\delta_3 \in \{1\}^*$ with $|\delta_3| = \Omega(C^4)$ is a substring of ξ between δ_1 and δ_2. The same statement holds when $\{1\}^*$ is interchanged with $\{0\}^*$.

Proof. Recall that for $i \in \mathbb{N}$, $[r_i, r_i + 2i - 1]$ is the interval of positions of α (resp. β) occupied by the substring $0^i 1^i$ (resp. $1^i 0^i$) of σ_i (resp. τ_i); denote this interval by R_i. Let I_n denote the interval of positions of ξ between (exclusive) H_n and H_{n+1}. We give a proof for the case where $\delta_1 \in \{0\}^*$, $\delta_2 \in \{0\}^*$ and $\delta_3 \in \{1\}^*$. Since range(f_1) cannot contain gaps of size more than C_1, there are $\frac{\Omega(C^4)}{C} = \Omega(C^3)$ positions of δ_1 (resp. δ_2, δ_3) that belong to range(f_1). We observe two facts: first, no position of α before K_n or after K_{n+1} is mapped by f_1 into I_n; second, f_1 maps at most $\mathcal{O}(C^2)$ positions in $(K_n \setminus K'_n) \cup (K_{n+1} \setminus K'_{n+1})$ into I_n. These two facts imply that f_1 maps $\Omega(C^3)$ positions of R_n into the interval occupied by δ_1 (resp. δ_2, δ_3). But then f_1 would have to map two consecutive positions of R_n occupied by 0's to positions in ξ that are at least $|\delta_3| = \Omega(C^4)$ positions apart, contradicting Condition (a). Hence, no such substrings δ_1, δ_2 and δ_3 can exist. ◁

Based on Claims 27 and 28, there are exactly two maximal intervals J_1 and J_2, each of length $\Omega(C^4)$, such that the substrings of ξ occupied by J_1 and J_2 belong to $\{0\}^*$ and $\{1\}^*$ respectively. Then, f_1 maps $\Omega(C^3)$ positions of $[r_n, r_n + n - 1]$ into J_1 and $\Omega(C^3)$ positions of $[r_n + n, r_n + 2n - 1]$ into J_2; further, there are two positions that are $\Omega(C^3)$ positions apart, one in $[r_n, r_n + n - 1]$ and the other in $[r_n + n, r_n + 2n - 1]$, such that f_1 maps the first position into J_1 and the second position into J_2. This implies that J_1 must precede J_2, for otherwise Condition (b) would be violated. Arguing similarly with f_2 in place of f_1 (that is, the mapping from β to ξ), it follows that J_2 must precede J_1, a contradiction. We conclude that the string ξ cannot exist. ◀

Example 19 established separations between various notions of recursive quasi-reducibility: pqi, 1qi and mqi-reducibilities. It remains to separate general quasi-isometry from its recursive counterpart. Due to space constraint, we only give a proof sketch of Theorem 29.

▶ **Theorem 29.** *There exist two recursive strings α and β such that $\alpha \leq_{qi} \beta$ but $\alpha \not\leq_{mqi} \beta$.*

Proof. We begin with an overview of the construction of α and β. To ensure that only non-recursive quasi-isometries between α and β exist, we use a tool from computability theory, which is a Kleene tree [12]—an infinite uniformly recursive binary tree with no infinite recursive branches (see, for example, [18, §V.5]). The idea of the proof is to encode a fixed Kleene tree into β, and construct α such that for any quasi-isometry f from α to β, an infinite branch of the encoded Kleene tree can be computed recursively from f. Hence, f cannot be recursive, as otherwise the chosen infinite branch of the Kleene tree must be recursive, contradicting the definition of a Kleene tree.

We now describe the construction of α and β based on some fixed Kleene tree $T \subseteq \{0,1\}^*$. The building blocks for α and β are called *blocks*, or more specifically, *n-blocks* for some

$n \in \mathbb{N}$. The construction will be done in stages, where at stage n, we concatenate some n-blocks to the existing prefixes of α and β. An *n-block* is defined to be a string of one of the following forms:

$$\begin{aligned} \lambda_{(n,0)} &= 0^n 1^n, \\ \lambda_{(n,i)} &= 0^{\lfloor \frac{n+1}{2} \rfloor} 1^i 0^{\lceil \frac{n+1}{2} \rceil} 1^n, \text{ for } 1 \le i \le n-1 \text{ or} \\ \lambda'_n &= (01)^n 1^n. \end{aligned}$$

The strings appended to α and β at stage n will be called θ_n and ζ_n respectively. Taking the limit as n grows to infinity, α and β have the following shapes:

$$\begin{aligned} \alpha &= \theta_1 \theta_2 \cdots = (\theta_n)_{n=1}^{\infty}, \\ \beta &= \zeta_1 \zeta_2 \cdots = (\zeta_n)_{n=1}^{\infty} \end{aligned}$$

where θ_n and ζ_n are made up of the same number of n-blocks and $|\zeta_n| \ge |\theta_n|$. We now define θ_n and ζ_n for $n \in \mathbb{N}$. Set

$$\theta_1 = \zeta_1 = \lambda_{(1,0)}.$$

For $n \ge 2$, each of the strings θ_n and ζ_n is composed of three main segments: a *scaling* segment, a *branching* segment and a *selection* segment. These segments are added to θ_n and ζ_n in the given order, each preceded by a *join* segment. Furthermore, a scaling segment is further made up of two *scaling parts* joined by a join segment. So, for $n \ge 2$, the structure of θ_n can be depicted as follows:

$$\theta_n = \underbrace{v_{n,1}}_{\text{Join}} \quad \underbrace{s_{n,1} \ v_{n,2} \ s_{n,2}}_{\text{Scaling}} \quad \underbrace{v_{n,3}}_{\text{Join}} \ \underbrace{t_n}_{\text{Branching}} \ \underbrace{v_{n,4}}_{\text{Join}} \ \underbrace{u_n}_{\text{Selection}}$$

and similarly,

$$\zeta_n = \underbrace{v'_{n,1}}_{\text{Join}} \quad \underbrace{s'_{n,1} \ v'_{n,2} \ s'_{n,2}}_{\text{Scaling}} \quad \underbrace{v'_{n,3}}_{\text{Join}} \ \underbrace{t'_n}_{\text{Branching}} \ \underbrace{v'_{n,4}}_{\text{Join}} \ \underbrace{u'_n}_{\text{Selection}}.$$

We can now define each segment of θ_n and ζ_n and briefly explain its function, and in the process give a high level overview of the proof.

Join segment. Each join segment serves as a connector between two different segments which aren't join segments. A join segment in θ_n or ζ_n is also called an *n-join segment*. For $i \in \{1, 2, 3, 4\}$, the n-join segments $v_{n,i}$ and $v'_{n,i}$ are defined as follows

$$v_{n,i} = v'_{n,i} = (\lambda_{(n,0)})^{3nB^n_{2i-1}}$$

where B^n_{2i-1} is the number of blocks in α before the start of $v_{n,i}$. We ensure that the corresponding segments of α and β have the same number of blocks. So, B^n_{2i-1} is also the number of blocks in β before the start of $v'_{n,i}$.

Given any fixed quasi-isometric reduction f from α to β, we define the *lead* ℓ of an n-join segment $v_{n,i}$ such that for all $nB^n_{2i-1} + 1 \le j \le 2nB^n_{2i-1}$, f maps the j-th $\lambda_{(n,0)}$ block of $v_{n,i}$ to the $(j+\ell)$-th $\lambda_{(n,0)}$ block of $v'_{(n,i)}$. We will show later that for large enough n, the lead ℓ is always defined and non-negative. To explain the functions of the other segments, we will describe how each segment affects the leads of join segments next to it.

Selection segment. The selection segment plays a key role in the encoding of the Kleene tree T into the string β. Before we define the selection segment, we first define

$$S_n = \left\{ \sum_{m=1}^{n-1} b_m 4^{n-1-m} : b_1 \cdots b_{n-1} \in T \cap \{0,1\}^{n-1} \right\}.$$

The set S_n encodes all the strings of length $n-1$ in the Kleene tree T, where each element $\sum_{m=1}^{n-1} b_m 4^{n-1-m} \in S_n$ is the number with base-4 representation $b_1 \cdots b_{n-1} \in T$ possibly with leading 0's. We can now define the *selection segments* u_n and u'_n. Define

$$u_n = \lambda_{(n,1)}(\lambda_{(n,0)})^{\max(S_n)}$$

and for $1 \leq i \leq \max(S_n)+1$, let the i-th block of u'_n be:

- $\lambda_{(n,0)}$ if $i-1 \notin S_n$ and
- $\lambda_{(n,1)}$ if $i-1 \in S_n$.

Then, the selection segment u'_n of β encodes the set S_n, which in turn encodes the set of all the strings in T of length $n-1$. So, all of the selection segments of β together encode the fixed Kleene tree T. Meanwhile, each selection segment of α has a single $\lambda_{(n,1)}$ block followed by $\lambda_{(n,0)}$ blocks. The single $\lambda_{(n,1)}$ block serves as a pointer which must be mapped by f to a $\lambda_{(n,1)}$ block in the respective selection segment of β. Hence, the selection segments of α and β ensure that for sufficiently large n, the lead of a quasi-isometric reduction from α to β in the n-join segment preceding a selection segment is a number in S_n. Moreover, the first $(n+1)$-join segment succeeds the n-th selection segment and has the same lead as the previous join segment.

The other segments ensure that the number in S_n is chosen appropriately such that an infinite branch of the Kleene tree can be computed from the leads of a quasi-isometric reduction in the join segments preceding the selection segments. More specifically, we need to make sure that for large enough n, the base-4 representation $b_1 \ldots b_{n-2}$ of the lead of the $(n-1)$-join segment preceding a selection segment and the base-4 representation $b'_1 \ldots b'_{n-1}$ of the lead of the n-join segment preceding a selection segment have the same first $n-c$ digits, where c is some constant independent of n. That is, $b_1 \ldots b_{n-c} = b'_1 \ldots b'_{n-c}$.

Scaling segment. To achieve the objective described above, each scaling segment helps by making sure that the lead of the join segment following the scaling segment is 4 times that of the previous join segment. To do this, a scaling segment is made up of two scaling parts joined together by a join segment, where each scaling part doubles the lead of the previous join segment. Then, the *scaling segments* of θ_n and ζ_n can be depicted as $s_{n,1}v_{n,2}s_{n,2}$ and $s'_{n,1}v'_{n,2}s'_{n,2}$ respectively, where $s_{n,1}$, $s_{n,2}$, $s'_{n,1}$ and $s'_{n,2}$ are *scaling parts* defined as follows:

$$s_{n,i} = s'_{n,i} = (\lambda_{(n,1)})^{nB_{2i}^n}(\lambda_{(n,2)})^{nB_{2i}^n} \ldots (\lambda_{(n,n-1)})^{nB_{2i}^n}(\lambda_{(n,0)})^{2nB_{2i}^n}$$

where $i \in \{1,2\}$ and B_{2i}^n is the number of blocks in α before the start of $s_{n,i}$. The doubling of the lead follows from the properties of the n-blocks chosen to make the scaling parts, which will be proven in the later parts of the proof.

Branching segment. Note that the join segment *after* a branching segment precedes a selection segment and so must have a lead which is in S_n. On the other hand, the join segment *before* a branching segment succeeds a scaling segment and may not be in S_n. So, the branching segment's purpose is to allow minor adjustments to the lead so that the lead after the branching segment is in S_n. Furthermore, this adjustment must be small enough so that for large enough n, the first $n-c$ digits of the base-4 representations $b_1 \ldots b_{n-2}$ and $b'_1 \ldots b'_{n-1}$ of the leads of join segments $v_{n-1,4}$ and $v_{n,4}$ match, where c is some constant independent of n. So, we define the *branching segments* t_n and t'_n as follows:

$$t_n = (\lambda_{(n,0)})^{2nB_6^n+1} \text{ and}$$
$$t'_n = (\lambda_{(n,0)})^{2nB_6^n}\lambda'_n$$

where B_6^n is the number of blocks in α before the start of t_n. The n-blocks chosen to make the branching segments t_n and t'_n ensure that given a C-quasi-isometric reduction f from α

to β and large enough n, the lead of the join segment after the branching segment is between $\ell - C$ and $\ell + 1$ inclusive, where ℓ is the lead of the previous join segment.

The above descriptions give a high-level overview of the proof while omitting the details of how certain properties are achieved. These details will be given in the rest of the proof.

Note also that a recursive formula for the number of blocks in each segment of θ_n or ζ_n may be determined in terms of n and $\max(S_n)$, although, as the present proof does not analyze time or space complexity issues, such a formula will not be explicitly stated.

We will now describe each segment in detail. In each segment, we will restate the definition of the segment and prove properties related to the segment. We will occasionally be informal and speak of mappings between two sequences of blocks; it is to be understood that in such a situation we are really referring to mappings between the sequences of positions of the block sequences in question.

Join segment. Recall that an *n-join segment* is a sequence of $3nB$ blocks $\lambda_{(n,0)}$, where B is the number of blocks in the prefix of α (resp. β) just before the start of the said sequence of $\lambda_{(n,0)}$ blocks. The sequence of the $(nB+1)$-st to the $(2nB)$-th blocks of an n-join segment will be called an *n-inner join segment.* An n-join segment and an n-inner join segment will be called a join segment and an inner join segment respectively when the choice of n is clear from the context.

The extension θ_n is defined so that for any quasi-isometric reduction f from α to β, if n is large enough and θ_n, ζ_n each contains at least K join segments, then f maps the sequence of positions of the K-th inner join segment in θ_n into a sequence of positions of the K-th join segment of ζ_n in a *monotonic and one-block-to-one-block* fashion, by which we mean that there is a constant t such that for $1 \leq i \leq nB$, f maps the i-th $\lambda_{(n,0)}$ block of the inner join segment in θ_n to the $(i+t)$-th $\lambda_{(n,0)}$ block of the join segment of ζ_n. Furthermore, suppose that the first $\lambda_{(n,0)}$ block of the K-th inner join segment in θ_n is the k_1-st block of $\theta_1 \cdots \theta_n$, and that the $(t+1)$-st $\lambda_{(n,0)}$ block of the K-th join segment in ζ_n is the k_2-nd block of $\zeta_1 \cdots \zeta_n$. Then, we call the quantity $k_2 - k_1$ the *lead* of f in the sequence of positions of the K-th join segment of θ_n. Thus, the lead of a quasi-isometry from α to β is defined with respect to the sequence of positions of a given n-join segment when n is large enough.

The extensions θ_n and ζ_n are chosen so that when n is large enough, the lead of f in each sequence of positions of a join segment of θ_n is nonnegative. Moreover, the extensions are chosen so that there is a constant C' (depending on f) such that when n is large enough, the lead ℓ of f in the sequence of positions of the last join segment of θ_n is contained in S_n and the string in $T \cap \{0,1\}^{n-1}$ corresponding to ℓ has a common prefix of length at least $n - C'$ with the string in $T \cap \{0,1\}^{n-2}$ corresponding to the analogously defined lead at the end of stage $n-1$. The idea is that by calculating successive values of the lead, one could then compute recursively in f an infinite branch of the Kleene tree.

Based on the preliminarily defined shapes of α and β, we now state a few useful properties of quasi-isometric reductions from α to β, in particular how they map between various types of blocks when the block lengths are large enough. Further details of the construction will be provided progressively.

For $n \geq 2$, define B_n to be the total number of blocks in $\zeta_1 \cdots \zeta_{n-1}$. The first observation is that for any C-quasi-isometric reduction, when n is sufficiently large and $i > nB_n$, the i-th occurrence of an n-block in α cannot be mapped to an m-block in β with $m < n$.

▷ Claim 30. Let f be any C-quasi-isometric reduction from α to β. Then, for all sufficiently large n and all $i > nB_n$, no position of the i-th occurrence of an n-block in α is mapped by f to the position of an m-block in β with $m < n$.

Proof. Fix any $n > 4C + 4$ and any $i > nB_n$, and let p be a position of the i-th n-block in α. There are at least $i - C - 1 > nB_n - C - 1$ n-blocks preceding position p in α such that all the positions of these n-blocks are more than C positions away from p. As $\lambda_{(n,0)}$ is the shortest n-block and has a length of $2n$, the total number of positions occupied by these blocks is more than $2n(nB_n - C - 1)$. By Condition (b), the images of the positions of these n-blocks under f precede $f(p)$. By Lemma 5, each position of β has at most $C + 1$ preimages, which means that there are at least $\frac{2n(nB_n-C-1)}{C+1}$ positions preceding $f(p)$. In other words, $f(p) > \frac{2n(nB_n-C-1)}{C+1}$. Since each k-block with $k < n$ has length at most $3(n-1)$ (a λ'_{n-1} block), the number of k-blocks with $k < n$ that can fit $\frac{2n(nB_n-C-1)}{C+1}$ positions must be at least $\frac{2n(nB_n-C-1)}{3(n-1)(C+1)}$. By the choice of n,

$$\begin{aligned}\frac{2n(nB_n - C - 1)}{3(n-1)(C+1)} &\geq \frac{2nB_n}{3(C+1)} - \frac{2n}{3(n-1)}\\ &> \frac{2(4C+4)B_n}{3(C+1)} - 2\\ &\geq 2B_n - 2\\ &\geq B_n.\end{aligned}$$

Since there are exactly B_n k-blocks with $k < n$, we conclude that $f(p)$ cannot occur in a k-block with $k < n$. ◁

The next observation gives a localised restriction on quasi-isometric mappings, in particular between θ_n and ζ_n.

▷ Claim 31. Let f be any C-quasi-isometric reduction from α to β. Then, for all sufficiently large n, no position of a block $\lambda_{(n,1)}$ is mapped by f to a position of a block $\lambda_{(n,0)}$ occurring in β.

Proof. Pick any $n > 2C + 2$. First, we observe that if at least one position in the sequence P of positions of $0^{\lfloor \frac{n+1}{2} \rfloor}$ is mapped to a position of some block $\lambda_{(n,0)}$, then f maps the whole sequence P into the sequence of the first n positions of $\lambda_{(n,0)}$. The reason is that if f maps some position of P to the position of another block, then, since the 0's occurring in the blocks adjacent to $\lambda_{(n,0)}$ are at least $n - 1 > 2C + 1$ positions away, Condition (a) would not be satisfied. The same observation applies to the sequence of positions of $0^{\lceil \frac{n+1}{2} \rceil}$.

Second, if the sequence of positions of $0^{\lfloor \frac{n+1}{2} \rfloor}$ is mapped into the sequence of positions of 0^n and the sequence of positions of $0^{\lceil \frac{n+1}{2} \rceil}$ is mapped into the sequence of positions of 0's in the block succeeding $\lambda_{(n,0)}$, then the position of the single 1 in $\lambda_{(n,1)}$ must be mapped into the sequence of positions of n 1's in $\lambda_{(n,0)}$. But the position of this 1 would then be at least $\frac{n}{2} > C + 1$ positions away from at least one of the images of the 0's adjacent to the single 1 occurring in $\lambda_{(n,1)}$, contradicting Condition (a).

Third, if the sequence of positions of $0^{\lfloor \frac{n+1}{2} \rfloor}$ and the sequence of positions of $0^{\lceil \frac{n+1}{2} \rceil}$ are both mapped into the sequence of positions of 0^n in $\lambda_{(n,0)}$, then the image of at least one position of $0^{\lceil \frac{n+1}{2} \rceil}$ that is at least $C + 1$ positions after the single 1 in $\lambda_{(n,1)}$ would precede the image of the single 1, contradicting Condition (b). For a similar reason, the sequence of positions of $0^{\lfloor \frac{n+1}{2} \rfloor}$ and the sequence of positions of $0^{\lceil \frac{n+1}{2} \rceil}$ cannot be both mapped into the sequence of positions of 0's in the succeeding block of $\lambda_{(n,0)}$. ◁

It is observed next that when n is large enough, every quasi-isometric mapping from an inner join segment of θ_n to a join segment of ζ_n is monotonic and one-block-to-one-block. We recall that B_n is the total number of blocks in $\zeta_1 \cdots \zeta_{n-1}$.

$\triangleright$ Claim 32. Let f be any quasi-isometric reduction from α to β. Then, for all sufficiently large n, if f maps a position of the K-th inner join segment of θ_n to a position of the K'-th join segment of ζ_n, then there is a constant t such that whenever $1 \leq i \leq nB_n$, f maps the sequence of positions of the i-th $\lambda_{(n,0)}$ block of the K-th inner join segment to the sequence of positions of the $(i+t)$-th $\lambda_{(n,0)}$ block of the K'-th join segment of ζ_n.

Proof. Suppose that f maps a position of the i-th $\lambda_{(n,0)}$ block of the K-th inner join segment of θ_n to a position of the j-th $\lambda_{(n,0)}$ block of the K'-th join segment of ζ_n. Using a similar argument as in the proof of Claim 31, if n is large enough, then all the positions of the i-th $\lambda_{(n,0)}$ block must be mapped into the sequence of positions of the j-th $\lambda_{(n,0)}$ block, for otherwise Condition (a) would fail. Inductively, assume that the $(i-k')$-th block is mapped to the $(j-k')$-th block and the $(i+k')$-th $\lambda_{(n,0)}$ block is mapped to the $(j+k')$-th block whenever $1 \leq k' \leq k$. By Condition (b), when n is large enough, f cannot map the $(i-k-1)$-st block to the $(j-k)$-th block or any subsequent block, and f also cannot map any block after the $(i-k)$-th block to the $(j-k-1)$-st block. If f does not map the $(i-k-1)$-st block to the $(j-k-1)$-st block, then the $(i-k-1)$-st block must be mapped to some sequence of positions wholly before the $(j-k-1)$-st block. But in this case, by Condition (b), no block before the $(i-k-1)$-st block can be mapped to the $(j-k-1)$-st block, and so the $(j-k-1)$-st $\lambda_{(n,0)}$ block of the K'-th join segment of ζ_n would have no preimage, which, in view of Lemma 6, is false for large enough values of n. Hence, the $(i-k-1)$-st block must be mapped to the $(j-k-1)$-st block. A similar argument shows that the $(i+k+1)$-st block is mapped to the $(j+k+1)$-st block (for large enough n). $\triangleleft$

The scaling, branching and selection segments of θ_n and ζ_n will now be described in detail. Along the way, we prove further properties of quasi-isometries from α to β.

Scaling segment. Set $B'_n = B_n + 3nB_n$, that is, B'_n is the number of blocks in $\zeta_1 \cdots \zeta_{n-1}$ plus the number of blocks in the first join segment of ζ_n. The scaling segments of θ_n and ζ_n are composed of two similar parts joined by an n-join segment. The first part of the scaling segment of θ_n and of ζ_n is

$$(\lambda_{(n,1)})^{nB'_n}(\lambda_{(n,2)})^{nB'_n} \cdots (\lambda_{(n,n-1)})^{nB'_n}(\lambda_{(n,0)})^{2nB'_n}.$$

We then append a join segment to the first part of the scaling segment of θ_n (resp. ζ_n). Let B''_n be the total number of blocks in the prefix of α built so far. The second part of the scaling segment of θ_n and of ζ_n is

$$(\lambda_{(n,1)})^{nB''_n}(\lambda_{(n,2)})^{nB''_n} \cdots (\lambda_{(n,n-1)})^{nB''_n}(\lambda_{(n,0)})^{2nB''_n}.$$

The structures of the scaling segments of θ_n and ζ_n imply that when n is large enough, the lead of a quasi-isometric reduction in the interval of positions of the first inner join segment of θ_n is nonnegative. This property will help to control the value of the lead in the last inner join segment of θ_n.

$\triangleright$ Claim 33. Let f be any C-quasi-isometric reduction from α to β. Then, for all sufficiently large n, f maps the sequence of positions of the first inner join segment of θ_n to the sequence of positions of the first join segment of ζ_n in a monotonic and one-block-to-one-block fashion.

Further, suppose that the first $\lambda_{(n,0)}$ block of the first inner join segment of θ_n is the $(i+1)$-st block of $\theta_1 \cdots \theta_n$. Then, there is a nonnegative constant ℓ, called the *lead* of f in the sequence of positions of the first inner join segment of θ_n, such that whenever $1 \leq k \leq nB_n$, f maps the sequence of positions of the $(i+k)$-th block of $\theta_1 \cdots \theta_n$ into the sequence of positions of the $(i+k+\ell)$-th block of $\zeta_1 \cdots \zeta_n$.

Proof. We first show that the first $\lambda_{(n,0)}$ block of the first inner join segment of θ_n is mapped to some $\lambda_{(n,0)}$ block of the first join segment of ζ_n. By Claim 30, $\lambda_{(n,0)}$ cannot be mapped to any k-block with $k < n$ for large enough n. By the definition of B_n, for large enough n, the positions of $\theta_1 \cdots \theta_{n-1}$ are mapped to at most nB_n blocks in the first join segment of ζ_n. By Claim 32, the first nB_n blocks $\lambda_{(n,0)}$ of the first join segment of θ_n are mapped to at most nB_n blocks in the first join segment of ζ_n. Thus, there are at least nB_n blocks $\lambda_{(n,0)}$ of the first join segment of ζ_n that have preimages after the first nB_n blocks $\lambda_{(n,0)}$ of the first join segment of θ_n. By Condition (b), the first $\lambda_{(n,0)}$ block of the first inner join segment of θ_n must therefore be mapped to some $\lambda_{(n,0)}$ block in the first join segment of ζ_n.

Now we show that the first $\lambda_{(n,0)}$ block of the first inner join segment of θ_n is mapped to a $\lambda_{(n,0)}$ block after the first nB_n blocks of the first join segment of ζ_n. Since, for each $i \leq n-1$, the number of blocks in ζ_i is equal to the number of blocks in θ_i, this would imply that the lead of f in the first inner join segment of θ_n is nonnegative. By Claim 32, for large enough n, if the first block of the first inner join segment of θ_n is mapped to one of the first nB_n blocks of the first join segment of ζ_n, then the sequence of positions of at least one $\lambda_{(n,0)}$ block of the first join segment of ζ_n has a preimage after the first join segment of θ_n. By Condition (b), if $n > C^2$, then this preimage is included in the sequence of positions of $(\lambda_{(n,1)})^{nB'_n}$, a prefix of the scaling segment. But by Claim 31, $\lambda_{(n,1)}$ cannot be mapped to $\lambda_{(n,0)}$. Hence, the first $\lambda_{(n,0)}$ block of the first inner join segment of θ_n is mapped to a $\lambda_{(n,0)}$ block after the first nB_n blocks of the first join segment of ζ_n. ◁

After the first part of the scaling segment, when n is large enough, the lead of f in the succeeding inner join segment is double the lead of f in the preceding inner join segment; after the second part, the lead is quadrupled. To see this, we first observe that within a scaling segment, when n is large enough and $i \leq n-2$, any quasi-isometric reduction must map a single $\lambda_{(n,i)}$ block to a single $\lambda_{(n,i)}$ block or to a single $\lambda_{(n,i+1)}$ block.

▷ Claim 34. Let f be any C-quasi-isometric reduction from α to β. Then, for large enough n and $i \leq n-2$, f maps the sequence of positions of each $\lambda_{(n,i)}$ block in a scaling segment of θ_n into either the sequence of positions of a $\lambda_{(n,i)}$ block in a scaling segment of ζ_n or the sequence of positions of a $\lambda_{(n,i+1)}$ block in a scaling segment of ζ_n.

Proof. According to Claim 33, when n is large enough, the lead of f in the preceding inner join segment is nonnegative. In this case, none of the blocks in the scaling segment of θ_n are mapped to blocks before the scaling segment of ζ_n. Furthermore, if $n > 4C+4$, then the positions of at most nB'_n blocks in the scaling segment of ζ_n have preimages before the scaling segment of θ_n. By Condition (b), only the first nB'_n blocks of the scaling segment can have preimages before the scaling segment of θ_n.

Now consider a $\lambda_{(n,i)}$ block for some $i \leq n-2$. When n is large and $j \leq n-2$, the positions of at most one $\lambda_{(n,i)}$ block can be mapped to a $\lambda_{(n,j)}$ block. To see this, we note the following subclaim.

▷ Subclaim 35. Suppose that n is large enough so that the $\lambda_{(n,0)}$ block just before the first part of the scaling segment of θ_n is mapped into the t-th block of $\zeta_1 \cdots \zeta_n$, where this t-th block is either the $\lambda_{(n,0)}$ block just before the first part of the scaling segment of ζ_n or a $\lambda_{(n,1)}$ block in the first part of the scaling segment of ζ_n. Then, for all s such that the s-th block of the first part of the scaling segment of θ_n is some $\lambda_{(n,i)}$ block with $i \leq n-2$, this s-th block is mapped into the $(t+s)$-th block of $\zeta_1 \cdots \zeta_n$.

Proof of Subclaim 35. By the choice of B'_n and for large enough n, the $(t+1)$-st block of $\zeta_1 \cdots \zeta_n$ is either a $\lambda_{(n,1)}$ block or a $\lambda_{(n,2)}$ block. Since a $\lambda_{(n,1)}$ block cannot be mapped into a $\lambda_{(n,0)}$ block, some parts of the first $\lambda_{(n,1)}$ block in the scaling segment must be mapped into the $(t+1)$-st block of $\zeta_1 \cdots \zeta_n$ in order to avoid causing large gaps in the range of f. However, the first $\lambda_{(n,1)}$ block in the scaling segment cannot be mapped across two adjacent blocks of the shape $\lambda_{(n,1)}\lambda_{(n,1)}$ or $\lambda_{(n,1)}\lambda_{(n,2)}$. For, if the single 1 were mapped into 1 in the first $\lambda_{(n,1)}$ block, then the succeeding substring $0^{\lceil \frac{n+1}{2} \rceil}1^n$ would have to be mapped into $0^{\lceil \frac{n+1}{2} \rceil}1^n$ in order to avoid causing large gaps in the range of f. For the same reason, the single 1 cannot be mapped to 1^n. Similarly, the single 1 cannot be mapped into 1 in the second $\lambda_{(n,1)}$ block or into 1^2 in the $\lambda_{(n,2)}$ block (for large enough n). The suffix 1^n of $\lambda_{(n,1)}$ would then have to be mapped into the suffix 1^n of $\lambda_{(n,1)}$. It follows that the first $\lambda_{(n,1)}$ block of the scaling segment must be wholly mapped into the $(t+1)$-st block of $\zeta_1 \cdots \zeta_n$.

Arguing as before, the next $\lambda_{(n,1)}$ or $\lambda_{(n,2)}$ block of θ_n cannot be mapped across two adjacent blocks of the shape $\lambda_{(n,1)}\lambda_{(n,1)}$, $\lambda_{(n,1)}\lambda_{(n,2)}$ or $\lambda_{(n,2)}\lambda_{(n,2)}$. By Condition (b), the next $\lambda_{(n,1)}$ or $\lambda_{(n,2)}$ block also cannot be entirely mapped into the $(t+1)$-st block of $\zeta_1 \cdots \zeta_n$. Thus, the next $\lambda_{(n,1)}$ or $\lambda_{(n,2)}$ block must be wholly mapped into the $(t+2)$-nd block of $\zeta_1 \cdots \zeta_n$. Applying the preceding arguments inductively to the next $\lambda_{(n,1)}$ or $\lambda_{(n,2)}$ block and then to subsequent $\lambda_{(n,i)}$ blocks, it follows that there cannot be two adjacent halves of $\lambda_{(n,i)}$ blocks or adjacent halves of a $\lambda_{(n,i)}$ and a $\lambda_{(n,i+1)}$ block that are mapped across a single $\lambda_{(n,j)}$ block for each $j \leq n-2$. Furthermore, for $i \leq n-2$, each s-th subsequent $\lambda_{(n,i)}$ block must be entirely mapped into the $(t+s)$-th block of $\zeta_1 \cdots \zeta_n$, which is either a $\lambda_{(n,i)}$ block or a $\lambda_{(n,i+1)}$ block. ◁ (Subclaim 35)

By Condition (b), f cannot map a $\lambda_{(n,i)}$ block in the scaling segment to any $\lambda_{(n,j)}$ block in the same scaling segment with $j > i+1$. The positions of a $\lambda_{(n,i)}$ block can be mapped in a one-to-one fashion to the positions of another $\lambda_{(n,i)}$ block. The positions of a $\lambda_{(n,i)}$ block can also be mapped one-to-one to the positions of a $\lambda_{(n,i+1)}$ block, as shown below.

$$\begin{array}{cccc} 0^{\lfloor \frac{n+1}{2} \rfloor} & 1^{i} & 0^{\lceil \frac{n+1}{2} \rceil} & 1^{n} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0^{\lfloor \frac{n+1}{2} \rfloor} & 1^{i+1} & 0^{\lceil \frac{n+1}{2} \rceil} & 1^{n} \end{array}$$

◀

The next claim shows that to achieve a doubling of the lead, a single $\lambda_{(n,n-1)}$ block can be mapped to two adjacent $\lambda_{(n,0)}$ blocks.

▷ Claim 36. Let f be any C-quasi-isometric reduction from α to β. Then, for large enough n, f can map the sequence of positions of a $\lambda_{(n,n-1)}$ block into the sequence of positions of exactly k blocks $\lambda_{(n,0)}$ iff $k = 2$.

Proof. If a $\lambda_{(n,n-1)}$ block were mapped to a single $\lambda_{(n,0)}$ block, then the positions of the substrings $0^{\lfloor \frac{n+1}{2} \rfloor}$ and $0^{\lceil \frac{n+1}{2} \rceil}$ must be mapped into the sequence of positions of 0^n. But the positions of the substring 1^{n-1} would have to be mapped into the sequence of positions of 1^n, and this would violate Condition (b) for large enough n. Furthermore, when $n > C+1$, a $\lambda_{(n,n-1)}$ block cannot be mapped across more than two $\lambda_{(n,0)}$ blocks without resulting in an interval of at least $C+1$ positions of ζ_n having no preimage, contradicting Lemma 6. A one-to-one mapping of the positions of a single $\lambda_{(n,n-1)}$ block to the positions of two $\lambda_{(n,0)}$ blocks is depicted in the next figure.

$$\begin{array}{cccc} 0^{\lfloor \frac{n+1}{2} \rfloor} & 1^{n-1} & 0^{\lceil \frac{n+1}{2} \rceil} & 1^n \\ \downarrow & \swarrow & \swarrow & \swarrow \\ 0^n & 1^n & 0^n & 1^n \end{array}$$

Thus, when n is large enough, a single $\lambda_{(n,n-1)}$ block maps to exactly two $\lambda_{(n,0)}$ blocks. ◁

We elaborate on why, for large enough n, the lead of f is quadrupled at the end of the scaling segment. Suppose that f has a nonnegative lead ℓ at the start of the scaling segment. When n is large enough, $\ell \leq nB'_n$. By Claim 34, for $1 \leq i \leq n-2$, the first $nB'_n - \ell$ blocks $\lambda_{(n,i)}$ of the (first part of the) scaling segment of θ_n are mapped to the last $nB'_n - \ell$ blocks $\lambda_{(n,i)}$ of the (first part of the) scaling segment of ζ_n, while the last ℓ blocks $\lambda_{(n,i)}$ are mapped to the first ℓ blocks $\lambda_{(n,i+1)}$. Further, the first $nB'_n - \ell$ blocks $\lambda_{(n,n-1)}$ are mapped to the last $nB'_n - \ell$ blocks $\lambda_{(n,n-1)}$. By Claim 36, each of the last ℓ $\lambda_{(n,n-1)}$ blocks must be mapped to two $\lambda_{(n,0)}$ blocks, while the first $nB'_n - 2\ell$ blocks $\lambda_{(n,0)}$ of the (first part of the) scaling segment of θ_n are mapped in a monotonic and one-block-to-one-block fashion to the remaining $nB'_n - 2\ell$ blocks $\lambda_{(n,0)}$. Thus, if f has a lead of ℓ in the inner join segment preceding the current scaling segment, then it has a lead of 2ℓ at the end of the first part of the scaling segment; after the second part of the scaling segment, a similar argument as before shows that the lead is quadrupled to 4ℓ. We next consider the branching segment.

Branching segment. In this segment, the lead of a C-quasi isometry f is increased by 1 or decreased by up to C when n is large enough. To achieve this, suppose that, after the join segment of θ_n succeeding the scaling segment, there are altogether B blocks in $\zeta_1 \cdots \zeta_{n-1}$ and the current prefix of ζ_n. The branching segment of θ_n is

$$(\lambda_{(n,0)})^{2Bn+1}$$

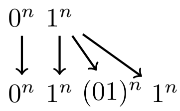
while the branching segment of ζ_n is

$$(\lambda_{(n,0)})^{2Bn}\lambda'_n .$$

▷ Claim 37. Let f be any C-quasi-isometric reduction from α to β. Suppose that n is large enough so that the lead ℓ of f in the sequence of positions of the join segment just before the branching segment is defined and nonnegative. Then, for large enough n, the lead of f in the sequence of positions of the join segment after the branching segment is at least $\ell - C$ and at most $\ell + 1$.

Proof. When $n > C$, the lead of f in the sequence of positions of the join segment just before the branching segment is at most nB, where B is the number of blocks in the prefix of α (or β) built so far. Thus, there are at most nB blocks $\lambda_{(n,0)}$ of the branching segment of ζ_n that have preimages before the start of the branching segment of θ_n. Consequently, the first $\lambda_{(n,0)}$ block of the branching segment of θ_n is mapped into the sequence of positions of the first $nB + 1$ blocks $\lambda_{(n,0)}$ of the branching segment of ζ_n.

By Lemma 5, up to $C + 1$ blocks $\lambda_{(n,0)}$ can be mapped to a single λ'_n block. Hence, the lead of f from the previous inner join segment can be decreased by up to C in the subsequent inner join segment. Further, when n is large enough, a single $\lambda_{(n,0)}$ block can be mapped across $\lambda_{(n,0)}\lambda'_n$. This can be done by mapping the 0^n substring of the $\lambda_{(n,0)}$ block in the branching segment of θ_n to the 0^n substring of the $\lambda_{(n,0)}$ block in the branching segment of ζ_n, and then mapping 1^n to $1^n(01)^n1^n$ by mapping the first third of 1's to the first occurrence of 1^n, the second third of 1's to $(01)^n$ and the last third of 1's to the second occurrence of 1^n. The mapping is illustrated as follows.



The proof of Claim 32 shows that for large enough n, one and only one $\lambda_{(n,0)}$ block can be mapped to a $\lambda_{(n,0)}$ block. Thus, the remaining $\lambda_{(n,0)}$ blocks in the branching segment of θ_n are mapped in a monotonic and one-block-to-one-block fashion to the rest of the $\lambda_{(n,0)}$ blocks in the branching segment of ζ_n. ◁

After appending an n-join segment to the branching segment, the selection segment is the final main segment to be added.

Selection segment. This segment filters out quasi-isometries whose lead in the previous inner join segment is not an element of S_n. The selection segment for θ_n is

$$\lambda_{(n,1)}(\lambda_{(n,0)})^{\max(S_n)}$$

while the selection segment for ζ_n is a concatenation

$$\lambda_{(n,0)} \cdots \lambda_{(n,1)} \cdots \lambda_{(n,0)} \cdots \lambda_{(n,0)}\lambda_{(n,1)}$$

of $\max(S_n) + 1$ n-blocks such that the i-th block is $\lambda_{(n,0)}$ if $i - 1 \notin S_n$ and is $\lambda_{(n,1)}$ if $i - 1 \in S_n$.

▷ Claim 38. Let f be any C-quasi-isometric reduction from α to β. Suppose that n is large enough so that the lead ℓ of f in the sequence of positions of the join segment just before the selection segment is defined and nonnegative. Then, for large enough n, f maps the sequence of positions of the $\lambda_{(n,1)}$ block in the selection segment of θ_n into the sequence of positions of exactly one of the $\lambda_{(n,1)}$ blocks in the selection segment of ζ_n. In particular, $\ell \in S_n$.

Proof. By Claim 31, for large enough n, f cannot map the $\lambda_{(n,1)}$ block of the selection segment of θ_n to any position in the first join segment of ζ_{n+1}. Thus, $\ell \leq \max(S_n)$. By Claim 31 again, f must map the $\lambda_{(n,1)}$ block of the selection segment of θ_n to some $\lambda_{(n,1)}$ block in the selection segment of ζ_n, say the c-th block of the selection segment of ζ_n, for large enough n. By Condition (b), if $n > C$, then f does not map any $\lambda_{(n,0)}$ block of the selection segment of θ_n to a position before the image of the $\lambda_{(n,1)}$ block under f. This implies that $\ell = c - 1 \in S_n$. ◁

Putting everything together, the structures of θ_n and ζ_n for $n \geq 2$ look as follows:

$$\begin{aligned}
\theta_n &= v_{n,1}\ s_{n,1}\ v_{n,2}\ s_{n,2}\ v_{n,3}\ t_n\ v_{n,4}\ u_n \\
&= (\lambda_{(n,0)})^{3nB_1^n} && \leftarrow v_{n,1} \quad \text{(Join)} \\
&\quad (\lambda_{(n,1)})^{nB_2^n}(\lambda_{(n,2)})^{nB_2^n}\cdots(\lambda_{(n,n-1)})^{nB_2^n}(\lambda_{(n,0)})^{2nB_2^n} && \leftarrow s_{n,1} \quad \text{(Scaling)} \\
&\quad (\lambda_{(n,0)})^{3nB_3^n} && \leftarrow v_{n,2} \quad \text{(Join)} \\
&\quad (\lambda_{(n,1)})^{nB_4^n}(\lambda_{(n,2)})^{nB_4^n}\cdots(\lambda_{(n,n-1)})^{nB_4^n}(\lambda_{(n,0)})^{2nB_4^n} && \leftarrow s_{n,2} \quad \text{(Scaling)} \\
&\quad (\lambda_{(n,0)})^{3nB_5^n} && \leftarrow v_{n,3} \quad \text{(Join)} \\
&\quad (\lambda_{(n,0)})^{2nB_6^n+1} && \leftarrow t_n \quad \text{(Branching)} \\
&\quad (\lambda_{(n,0)})^{3nB_7^n} && \leftarrow v_{n,4} \quad \text{(Join)} \\
&\quad \lambda_{(n,1)}(\lambda_{(n,0)})^{\max(S_n)} && \leftarrow u_n \quad \text{(Selection)}
\end{aligned} \tag{3}$$

$$\begin{aligned}
\zeta_n &= v'_{n,1}\ s'_{n,1}\ v'_{n,2}\ s'_{n,2}\ v'_{n,3}\ t'_n\ v'_{n,4}\ u'_n \\
&= (\lambda_{(n,0)})^{3nB_1^n} && \leftarrow v'_{n,1} \quad \text{(Join)} \\
&\quad (\lambda_{(n,1)})^{nB_2^n}(\lambda_{(n,2)})^{nB_2^n}\cdots(\lambda_{(n,n-1)})^{nB_2^n}(\lambda_{(n,0)})^{2nB_2^n} && \leftarrow s'_{n,1} \quad \text{(Scaling)} \\
&\quad (\lambda_{(n,0)})^{3nB_3^n} && \leftarrow v'_{n,2} \quad \text{(Join)} \\
&\quad (\lambda_{(n,1)})^{nB_4^n}(\lambda_{(n,2)})^{nB_4^n}\cdots(\lambda_{(n,n-1)})^{nB_4^n}(\lambda_{(n,0)})^{2nB_4^n} && \leftarrow s'_{n,2} \quad \text{(Scaling)} \\
&\quad (\lambda_{(n,0)})^{3nB_5^n} && \leftarrow v'_{n,3} \quad \text{(Join)} \\
&\quad (\lambda_{(n,0)})^{2nB_6^n}\lambda'_n && \leftarrow t'_n \quad \text{(Branching)} \\
&\quad (\lambda_{(n,0)})^{3nB_7^n} && \leftarrow v'_{n,4} \quad \text{(Join)} \\
&\quad \lambda_{(n,0)}\cdots\lambda_{(n,1)}\cdots\lambda_{(n,0)}\cdots\lambda_{(n,0)}\lambda_{(n,1)} && \leftarrow u'_n \quad \text{(Selection)}
\end{aligned}$$

where B_i^n is the number of blocks in α (or β) preceding the segment where the parameter B_i^n is first used.

Suppose that f is a C-quasi-isometry from α to β. As explained earlier, if n is sufficiently large, then the lead ℓ of f in the first join segment of θ_{n+1} is contained in S_n. The lead is then quadrupled after the scaling segment. Further, by Claim 37, the lead ℓ' of f in the final join segment of θ_{n+1} is at least $4\ell - C$, at most $4\ell + 1$ and is contained in S_{n+1}. To establish that $\alpha \not\leq_{mqi} \beta$, we prove that there is a constant c such that the coefficients of the expressions for ℓ and ℓ' as linear combinations of powers of 4 agree except on the smallest c powers of 4. Thus, the branch on the Kleene tree represented by ℓ can be properly extended by a branch represented by a prefix of ℓ', so by repeatedly calculating the leads of f for increasing values of n, one may construct an infinite branch of the tree.

▷ Claim 39. There exists a nonnegative constant c such that if $n > c$, $\ell = \sum_{m=1}^{n} b_m \cdot 4^{n-m} \in S_{n+1}$ and $\ell' = \sum_{m=1}^{n+1} b'_m \cdot 4^{n+1-m} \in S_{n+2}$, where $b_m \in \{0,1\}$ and $b'_m \in \{0,1\}$, then for all $m \in \{1,\ldots,n-c\}$, $b_m = b'_m$.

Proof. Let c be a positive constant such that $C < 4^c$. We show that for all $m \in \{1,\ldots,n-c\}$, $b_m = b'_m$. As shown earlier, $4\ell - C \leq \ell' \leq 4\ell + 1$. Substituting the expressions for ℓ and ℓ' as linear combinations of powers of 4 into the inequality $4\ell - C \leq \ell'$,

$$\sum_{m=1}^{n} b_m 4^{n-m+1} - 4^c < \sum_{m=1}^{n} b_m 4^{n-m+1} - C \leq \sum_{m=1}^{n+1} b'_m 4^{n-m+1}.$$

Dividing both sides of the inequality by 4^{c+1},

$$\sum_{m=1}^{n} b_m 4^{n-m-c} - \frac{1}{4} < \sum_{m=1}^{n+1} b'_m 4^{n-m-c}\,.$$

Splitting the sums on both sides according to whether the powers of 4 are nonnegative or negative,

$$\sum_{m=1}^{n-c} b_m 4^{n-m-c} + \sum_{m=n-c+1}^{n} b_m 4^{n-m-c} - \frac{1}{4} < \sum_{m=1}^{n-c} b'_m 4^{n-m-c} + \sum_{m=n-c+1}^{n+1} b'_m 4^{n-m-c}\,.$$

Rearranging,

$$\sum_{m=1}^{n-c} b_m 4^{n-m-c} + \sum_{m=n-c+1}^{n} b_m 4^{n-m-c} - \frac{1}{4} - \sum_{m=n-c+1}^{n+1} b'_m 4^{n-m-c} < \sum_{m=1}^{n-c} b'_m 4^{n-m-c}\,.$$

Since

$$\sum_{m=n-c+1}^{n} b_m 4^{n-m-c} - \frac{1}{4} - \sum_{m=n-c+1}^{n+1} b'_m 4^{n-m-c} \geq -\frac{1}{4} - \sum_{m=1}^{\infty} 4^{-m} = \frac{7}{12}\,,$$

it follows that

$$\sum_{m=1}^{n-c} b'_m 4^{n-m-c} > \sum_{m=1}^{n-c} b_m 4^{n-m-c} - \frac{7}{12}\,.$$

Taking the ceiling on both sides,

$$\begin{aligned}
\sum_{m=1}^{n-c} b'_m 4^{n-m-c} &= \left\lceil \sum_{m=1}^{n-c} b'_m 4^{n-m-c} \right\rceil \\
&\geq \left\lceil \sum_{m=1}^{n-c} b_m 4^{n-m-c} - \frac{7}{12} \right\rceil \\
&= \sum_{m=1}^{n-c} b_m 4^{n-m-c}\,. \qquad (4)
\end{aligned}$$

From the inequality $\ell' \leq 4\ell + 1$,

$$\sum_{m=1}^{n-c} b'_m 4^{n-m+1} + \sum_{m=n-c+1}^{n+1} b'_m 4^{n-m+1} \leq \sum_{m=1}^{n} b_m 4^{n-m+1} + 1\,.$$

Dividing both sides by 4^{c+1} and applying the floor function,

$$\begin{aligned}
\sum_{m=1}^{n-c} b'_m 4^{n-m-c} &= \left\lfloor \sum_{m=1}^{n-c} b'_m 4^{n-m-c} + \sum_{m=n-c+1}^{n+1} b'_m 4^{n-m-c} \right\rfloor \\
&\leq \left\lfloor \sum_{m=1}^{n} b_m 4^{n-m-c} + \frac{1}{4^{c+1}} \right\rfloor \\
&= \sum_{m=1}^{n} b_m 4^{n-m-c}\,. \qquad (5)
\end{aligned}$$

Inequalities (4) and (5) together give that for each $m \in \{1, \ldots, n-c\}$, $b'_m = b_m$. $\triangleleft$

▷ Claim 40. $\alpha \not\leq_{mqi} \beta$.

Proof. Suppose that f is a C-quasi-isometry from α to β. We show that an infinite branch of the Kleene tree can be determined recursively in f. Let c be the constant in the statement of Claim 39. Fix $n_0 > c$ large enough so that whenever $n \geq n_0$, f has nonnegative lead in the last join segment of θ_n. Then, calculate the sequence $(\ell_n)_{n=n_0}^{\infty}$ of leads of f in the last join segment of θ_n for $n \geq n_0$, expressing each ℓ_n in the shape $\sum_{m=1}^{n} b_m^n 4^{n-m}$, where $b_m^n \in \{0,1\}$. By Claim 39, $b_m^n = b_m^{n+1}$ for all $m \in \{1,\ldots,n-c\}$, and so by Claim 38,

$$b_1^{n_0} \cdots b_{n_0-c}^{n_0} b_{n_0-c+1}^{n_0+1} b_{n_0-c+2}^{n_0+2} \cdots b_{n_0-c+k}^{n_0+k} \cdots$$

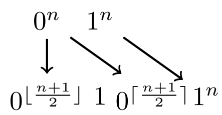
is an infinite branch of the Kleene tree. But since the tree has no infinite recursive branches, f must be nonrecursive. ◁

To finish the proof, we observe that a quasi-isometric reduction from α to β can be obtained from an infinite branch of the Kleene tree.

▷ Claim 41. $\alpha \leq_{qi} \beta$.

Proof. Fix an infinite branch $\mathcal{B}(1)\mathcal{B}(2)\cdots$ of the Kleene tree. For each $n \in \mathbb{N}$, we describe a mapping from θ_n to ζ_n, using the structures depicted in Equation (3) as a reference. This will give a quasi-isometric reduction from α to β. The mappings to be defined are strictly increasing.

First, since $\theta_1 = \zeta_1 = \lambda_{(n,0)}$, we can map θ_1 to ζ_1 in a strictly increasing fashion. The lead in the next segment is 0. For $n \geq 2$, map each join segment $v_{n,i}$ of θ_n to the corresponding join segment $v'_{n,i}$ of ζ_n, shifted by the lead at the current step. That is, suppose the lead is ℓ_1 and the number of blocks in α before this segment is B_{2i-1}^n. Then, for $1 \leq i \leq 3nB_{2i-1}^n - \ell_1$, map the i-th block of the join segment of θ_n into the $(i+\ell_1)$-th block of the corresponding join segment of ζ_n. Map the last ℓ_1 blocks $\lambda_{(n,0)}$ of the join segment into the first ℓ_1 blocks of the following segment in ζ_n—which must be $\lambda_{(n,0)}$ or $\lambda_{(n,1)}$ blocks. Note that a $\lambda_{(n,0)}$ block can be mapped into a $\lambda_{(n,1)}$ block.



The lead in the next segment is ℓ_1.

Next we describe the mapping for scaling part $s_{n,i}$ with lead ℓ_2. For each $k \leq n-1$, map the first $nB_{2i}^n - \ell_2$ blocks $\lambda_{(n,k)}$ of $s_{n,i}$ to the corresponding last $nB_{2i}^n - \ell_2$ blocks $\lambda_{(n,k)}$ of $s'_{n,i}$. Then, for each $k \leq n-2$, map the last ℓ_2 blocks $\lambda_{(n,k)}$ of $s_{n,i}$ to the first ℓ_2 blocks $\lambda_{(n,k+1)}$ of $s'_{n,i}$. Observe that each block $\lambda_{(n,k)}$ can be mapped to a block $\lambda_{(n,k+1)}$ in a strictly increasing manner. Further, map each of the last ℓ_2 blocks $\lambda_{(n,n-1)}$ to exactly two blocks $\lambda_{(n,0)}$ of $s'_{n,i}$. Map the first $2nB_{2i}^n - 2\ell_2$ blocks $\lambda_{(n,0)}$ to the remaining $2nB_{2i}^n - 2\ell_2$ blocks $\lambda_{(n,0)}$ of $s'_{n,i}$. Map the last $2\ell_2$ blocks $\lambda_{(n,0)}$ to the first $2\ell_2$ blocks of the following join segment in ζ_n. The lead of the next join segment is $2\ell_2$.

For the branching segment, suppose that the current lead is ℓ_3. If $\mathcal{B}(n-1) = 1$, map the $(2nB_6^n - \ell_3)$-th $\lambda_{(n,0)}$ block to the concatenation $\lambda_{(n,0)}\lambda'_n$ of two blocks in $t'_n v_{n,4}$. Otherwise, map the $(2nB_6^n - \ell_3 + 1)$-st $\lambda_{(n,0)}$ block to the λ'_n block in t'_n. Map the rest of the $\lambda_{(n,0)}$ blocks such that f is strictly increasing. Then, the lead of the next join segment is $\ell_3 + \mathcal{B}(n-1)$.

For the selection segment, suppose that the current lead is ℓ_4. Map the $\lambda_{(n,1)}$ block to the (ℓ_4+1)-st block. By induction, $\ell_4 \in S_n$ and so the (ℓ_4+1)-st block in the selection segment of ζ_n is $\lambda_{(n,1)}$. Recall that a $\lambda_{(n,0)}$ block can be mapped in a strictly increasing fashion to a $\lambda_{(n,1)}$ block. Thus, the remaining $\lambda_{(n,0)}$ blocks can be mapped to the subsequent $\lambda_{(n,0)}$ or $\lambda_{(n,1)}$ blocks in a strictly increasing manner. ◁

From Claims 40 and 41, we conclude that $\alpha \leq_{qi} \beta$ but $\alpha \not\leq_{mqi} \beta$. Hence, mqi-reducibility is strictly a stronger notion than general quasi-isometric reducibility. ◀

5 Automatic Quasi-Isometric Reductions

Automatic structures were introduced independently by Hodgson in 1983 [5] and by Khoussainov and Nerode in 1995 [9]. Since then, automatic structures have been well studied, with many surveys written on this topic [3, 8, 22, 24].

The results in the previous section focus mainly on quasi-isometric reductions which are recursive. In this section, we extend the previous results to the area of automata theory by studying quasi-isometric reductions which are automatic. We first define formally the notion of automatic quasi-isometric reductions, by replacing the recursive infinite strings in Definition 4 with an isomorphic automatic structure.

We let the domain be some regular set $D \subseteq \Sigma^*$ for some finite alphabet Σ and let $<_{llex}$ be the length lexicographic ordering. We define the successor function $succ : D \to D$ such that $succ(x) = \min_{llex}\{y \in D : x <_{llex} y\}$. Addition by a constant is then defined using the successor function such that for any $x \in D$ and $k \in \mathbb{N}_0$, $x \oplus k := succ^k(x)$. Similarly, $x \ominus k := succ^{-k}(x)$ is defined whenever there are at least k elements of D that are length-lexicographically less than x. Then, given a constant $k \in \mathbb{N}_0$, the successor function and addition by constant k can be performed automatically.

From the previous paragraph, $(D, <_{llex}, succ)$ is an automatic structure that is isomorphic to $(\mathbb{N}, <, succ)$, where $succ$ denotes the successor function in the respective domain. Then, quasi-isometry results proven earlier for infinite strings also apply to the corresponding automatic structures by replacing $(\mathbb{N}, <, succ)$ accordingly with $(D, <_{llex}, succ)$. In particular, Proposition 3 holds when we replace $\mathbb{N}, <$ and $succ$ with $D, <_{llex}$ and $succ$ respectively. Hence, we can redefine the notion of quasi-isometric reducibility between two automatic colourings of an automatic domain as follows.

▶ **Definition 42.** *Let $C \in \mathbb{N}$, $D \subseteq \Sigma^*$ be regular and α, β be automatic colourings of D, that is, automatic functions from D to some finite sets. A C-*quasi-isometric reduction *from α to β is a colour-preserving function $f : D \to D$ such that for all $x, y \in D$,*

(a) *$f(\min_{llex} D) \leq_{llex} \min_{llex} D \oplus C$ and $f(x) \ominus C \leq_{llex} f(x \oplus 1) \leq_{llex} f(x) \oplus C$; and*

(b) *$x \oplus C <_{llex} y \Rightarrow f(x) <_{llex} f(y)$.*

Where appropriate, we may drop the constant C and simply call f a quasi-isometric reduction, or a quasi-isometry, from α to β. We are particularly interested in the case where f is automatic.

▶ **Definition 43** (Automatic Quasi-Isometric Reducibility)**.** *Let $D \subseteq \Sigma^*$ be a regular set. An automatic colouring α of D is* automatically quasi-isometrically reducible*, or* aqi-reducible*, to another automatic colouring β of D iff there exists a quasi-isometric reduction f from α to β such that f is automatic.*

Note that Definition 42 uses the alternate definition of quasi-isometry given in Definition 4, which is a simpler but equivalent version of Definition 2. The advantage of using this simpler definition is that it allows automatic quasi-isometry to be defined for a broader range of domains. For example, the underlying metric space (D, d_{llex}) of the automatic structure $(D, <_{llex}, succ)$ is not always automatic as subtraction is not automatic for many regular languages.

For our results, we define the following property about domain D.

▶ **Definition 44** ([7]). *Given an infinite regular set $D \subseteq \Sigma^*$, we define* the growth of D *as the function $growth_D : \mathbb{N}_0 \to \mathbb{N}_0$ such that $growth_D(n) = |\{\sigma \in D : |\sigma| \leq n\}|$. We say that D has:*

- linear growth *if $growth_D(n) = \Theta(n)$;*
- superlinear growth *if $growth_D(n) = \omega(n)$;*
- polynomial growth *if $growth_D(n) = \Theta(n^c)$ for some $c \in \mathbb{R}$ such that $c \geq 0$;*
- exponential growth *if $growth_D(n) = \Omega(c^n)$ for some $c \in \mathbb{R}$ such that $c > 1$.*

Results in automata theory show that if D is regular, then the growth is either polynomial or exponential, with nothing in between [1, 6, 7, 14, 20, 25, 26]. Furthermore, if the growth is polynomial, then it must be $\Theta(n^c)$ for some $c \in \mathbb{N}$ [25].

Besides the growth, which counts the number of strings *up to* length n, another useful property of a regular language is the number of strings at *exactly* length n.

▶ **Lemma 45** ([25, Lemma 1]). *Suppose that D contains $uv_1^*w_1 \dots v_c^*w_c$ as a subset, where for each $1 \leq i \leq c$, v_i is a non-empty string and the first character after v_i^*, if it exists, is not the same as the first character of v_i. If $n = |uw_1 \dots w_c| + k|v_1| \dots |v_c|$ for some $k \in \mathbb{N}$, then the number of strings in D of length n is $\Omega(n^{c-1})$.*

▶ **Lemma 46** ([25, Lemma 4]). *Suppose that a language D is a finite union of the regular expressions of the form $uv_1^*w_1 \dots v_j^*w_j$ where $j \leq c$. Then, the number of strings in D of length n is $O(n^{c-1})$.*

We will now present our main results for this section, which shows that whether automatic quasi-isometric reducibility exists between two automatic colourings depends on the growth of the domain (among other factors).

For the next theorem, we define $\beta : D \to \Gamma$ to be *eventually periodic* if and only if there are $k \in \mathbb{N}$ and $h \in D$ such that for all $x \geq_{llex} h$, $\beta(x \oplus k) = \beta(x)$.

▶ **Theorem 47.** *Let $D \subseteq \Sigma^*$ be an infinite regular set, and let α, β be automatic colourings of D such that β is eventually periodic. The following are equivalent:*

(a) *Every colour in α also occurs in β and every colour which occurs infinitely often in α also occurs infinitely often in β;*
(b) *α is aqi-reducible to β;*
(c) *α is quasi-isometrically reducible to β.*

Proof. $(a) \Rightarrow (b)$: We define the quasi-isometric reduction f from α to β such that:

$$f(x) = \begin{cases} \min_{llex}\{y \in D : \alpha(x) = \beta(y)\} & \text{if } \alpha(x) \text{ occurs only finitely often in } \alpha; \\ \min_{llex}\{y \in D : \alpha(x) = \beta(y) \text{ and } x <_{llex} y\} & \text{if } \alpha(x) \text{ occurs infinitely often in } \alpha. \end{cases}$$

Note that f is automatic since it is first-order defined from automatic parameters D, α, β and $<_{llex}$. Furthermore, f is clearly colour-preserving. It remains to show that f satisfies Conditions (a) and (b) of Definition 42.

Let $x_0 = \min_{llex} D$. Since β is eventually periodic, there are $k, h \in \mathbb{N}$ such that for all $y \geq_{llex} x_0 \oplus h$, $\beta(y \oplus k) = \beta(y)$. Furthermore, there exists $\ell \geq h$ such that for all $x \geq x_0 \oplus \ell$,

$\alpha(x)$ occurs infinitely often in α. Choose $C' = \max(\{t \in \mathbb{N}_0 : x_0 \oplus t = f(x)$ for some $x <_{llex} x_0 \oplus \ell\} \cup \{k, \ell\})$.

We first show that for any $x \in D$, $x \leq_{llex} f(x) \oplus C'$ and $f(x) \leq_{lleq} x \oplus C'$. If $x <_{llex} x_0 \oplus \ell$, then $x \leq_{llex} f(x) \oplus C'$ and $f(x) \leq_{lleq} x \oplus C'$ by definition of C'. Otherwise, then $\alpha(x)$ occurs infinitely often and for all $x' \geq x$, $\beta(x' \oplus k) = \beta(x')$. Hence, $x <_{llex} f(x) \leq_{llex} x \oplus k$. Then, $x \leq_{llex} f(x) \oplus C'$ and $f(x) \leq_{llex} x \oplus C'$.

We now show that f is a C-quasi-isometric reduction with $C = 2C' + 1$. The first half of Condition (a) of Definition 42 follows directly from our claim that $f(x) \leq_{llex} x \oplus C'$ for all $x \in D$. To prove the second half, we have

$$f(x) \leq_{llex} x \oplus C' <_{llex} x \oplus 1 \oplus C' \leq_{llex} f(x \oplus 1) \oplus 2C',$$

and so $f(x \oplus 1) \geq_{llex} f(x) \ominus C$. And similarly,

$$f(x \oplus 1) \leq_{llex} x \oplus 1 \oplus C' <_{llex} f(x) \oplus 2C' \oplus 1 = f(x) \oplus C.$$

To prove Condition (b) of Definition 42, suppose that $x \oplus C <_{llex} y$. Note that

$$f(x) \oplus C' \leq_{llex} x \oplus 2C' <_{llex} y \leq_{llex} f(y) \oplus C'.$$

Then, $f(x) <_{llex} f(y)$.

$(b) \Rightarrow (c)$: This follows from the definition of automatic quasi-isometric reducibility.

$(c) \Rightarrow (a)$: Let f be a C-quasi-isometric reduction from α to β. Since f is colour-preserving, then clearly, every colour in α also occurs in β. Furthermore, by Lemma 5, f is finite-to-one. Hence, every colour which occurs infinitely often in α also occurs infinitely often in β. ◀

We next show that eventual periodicity is related to linear growth by the following proposition.

▶ **Proposition 48.** *Suppose an infinite regular set $D \subseteq \Sigma^*$ has linear growth. Then, any automatic colouring $\beta : D \to \Gamma$ is eventually periodic.*

Proof. Since D has linear growth, then by Lemmas 45 and 46, it must be of the form $D = \bigcup_{i=1}^{t} u_i v_i^* w_i \cup D_{fin}$ where D_{fin} is a finite set, $u_i, w_i \in \Sigma^*$ and $v_i \in \Sigma^+$. Without loss of generality, we can assume that $|u_1| = \ldots = |u_t|$ and $|v_1| = \ldots = |v_t|$, since v_i can be replaced by $v_i^{|v_1|\ldots|v_{i-1}||v_{i+1}|\ldots|v_t|}$ and rotated as needed so that $|u_1| = \ldots = |u_t|$.

Now consider the set D_c of all the strings $x \in D$ with $\beta(x) = c$. Since β is regular, then so is D_c. Thus, D_c satisfies the pumping lemma for any large enough pumping constant k. We consider the following version of the pumping lemma: for any string $x \in D_c$ of length at least $3k$, there is a representation $x = uvw$ such that $|u|, |w| \geq k$, $0 < |v| \leq k$ and $uv^*w \subseteq D_c$. We choose a pumping constant k such that $k \geq |u_i v_i|, |v_i w_i|$ for any $1 \leq i \leq t$. Then, any x of length at least $3k$ must be equal to $u_i v_i^j w_i$ for some $1 \leq i \leq t$ and $j \in \mathbb{N}$. Furthermore, as $D_c \subseteq D$, the representation $x = uvw$ must satisfy that u_i is a prefix of u, w_i is a suffix of w and $v = v_i^{k'}$ for some $k' \leq k$. Hence, the string $u_i v_i^{j+k!} w_i$ must be a pumped word of $u_i v_i^j w_i$, and so $u_i v_i^{j+k!} w_i \in D_c$ as well.

Next, we claim that there is a constant m depending only on k such that $u_i v_i^{j+k!} w_i = u_i v_i^j w_i \oplus m$ for all i and large enough j. First, observe that if $u_i v_i^j w_i$ is the h-th string in D of its length, then $u_i v_i^{j+k!} w_i$ is also the h-th string in D of its length. Furthermore, the number of strings in D of length $|u_i v_i^j w_i|$ to $|u_i v_i^{j+k!} w_i| - 1$ is a constant m depending only on k. Hence, it follows that $u_i v_i^{j+k!} w_i = u_i v_i^j w_i \oplus m$ for all i and large enough j.

Therefore, we can conclude that for large enough x, if $x \in D_c$, then $x \oplus m$ is also in D_c. In other words, if $\beta(x) = c$, then $\beta(x \oplus m) = \beta(x) = c$. Note that this is true for all colours $c \in \Gamma$. Hence, for each colour $c \in \Gamma$, there is a string $x_c \in D$ and a constant $m_c \in \mathbb{N}$ such that for all $x \geq_{llex} x_c$ with $\beta(x) = c$, we have that $\beta(x) = \beta(x \oplus m_c) = c$. Then, by letting $M = \prod_{c \in \Gamma} m_c$ and $x_0 = \max_{llex}\{x_c : c \in \Gamma\}$, we have that for all $x \geq_{llex} x_0$, $\beta(x) = \beta(x \oplus M)$. So, β is eventually periodic. ◀

Hence, we have the following results.

▶ **Corollary 49.** *Suppose an infinite regular set $D \subseteq \Sigma^*$ has linear growth, and let α, β be automatic colourings of D. The following are equivalent:*

(a) *Every colour in α also occurs in β and every colour which occurs infinitely often in α also occurs infinitely often in β;*

(b) *α is aqi-reducible to β;*

(c) *α is quasi-isometrically reducible to β.*

The above result does not hold if D has superlinear growth. We first show a counterexample where Condition (a) holds, but Condition (c) does not hold, and so Condition (b) does not hold.

▶ **Theorem 50.** *Suppose an infinite regular set $D \subseteq \Sigma^*$ has superlinear growth. There exist automatic colourings α, β of D such that the following statements are true:*

(a) *Every colour in α also occurs in β and every colour which occurs infinitely often in α also occurs infinitely often in β;*

(b) *α is* not *quasi-isometrically reducible to β.*

Proof. We define α and β as follows. Let $\alpha(x) = 1$ if x is the length-lexicographically minimum string of its length and $\alpha(x) = 0$ otherwise. Let $\beta(x) = 1 - \alpha(x)$. The range of α and β is the same, which is $\{0, 1\}$, and both 0 and 1 occur infinitely often in both α and β. Hence, α and β satisfy statement (a).

We now show that α is not quasi-isometrically reducible to β using Corollary 7. We first claim that there exists $K \in \mathbb{N}$ such that for any $x \in D$, there is some $x' \in D$ such that $x \leq_{llex} x' <_{llex} x \oplus K$ and $\alpha(x') = 0$. Let $S = \alpha^{-1}(0)$. Since α is automatic, then S must be regular. Furthermore, since D has superlinear growth, then by definition of α, S must be infinite. So, by the pumping lemma, S must have a subset of the form uv^*w for some strings $u, w \in \Sigma^*$ and $v \in \Sigma^+$. Let $K = |uvw| + 2$. For each $x \in D$, there is an element $x'' \in S$ such that $|x| < |x''| \leq |x| + |uvw|$. Then, $x'' \geq_{llex} x$. If we also have that $x'' <_{llex} x \oplus K$, then we are done. Otherwise, $x'' \geq_{llex} x \oplus K$ and so $|x| \leq |x \oplus K \ominus 1| \leq |x''| \leq |x| + K - 2$. Note that for any length n, there is at most one string $y \in D$ of length n such that $\alpha(y) \neq 0$. Since $S = \alpha^{-1}(0)$, there are at most $K - 1$ elements of $D \setminus S$ of length $|x|$ to $|x| + K - 2$. Hence, there are at most $K - 1$ elements of $D \setminus S$ which are length-lexicographically between x and $x \oplus K \ominus 1$ inclusive. Since there are K elements in D which are length-lexicographically between x and $x \oplus K \ominus 1$ inclusive, one of them must be in S. Therefore, for any $x \in D$, there is some $x' \in S$ such that $x \leq_{llex} x' <_{llex} x \oplus K$. Moreover, by definition of S, $\alpha(x') = 0$.

Now suppose that there exists a C-quasi-isometry f from α to β for some C. Since the quasi-isometry defined by the automatic structure $(D, <_{llex}, succ)$ is isomorphic to the quasi-isometry between strings defined by $(\mathbb{N}, <, succ)$, Corollary 7 implies that for any $x \in D$, there is some $y \leq_{llex} y' < y \oplus KC$ such that $\beta(y') = 0$. On the other hand, this implies that there are at most KC strings of each length, which implies that there are at most KCn strings of length up to $n - 1$. This contradicts the assumption that D has superlinear growth. Hence, α is not quasi-isometrically reducible to β. ◀

Furthermore, if the growth of D is also polynomial (on top of being superlinear), we can show that there exist automatic colourings α, β such that α is quasi-isometrically reducible to β but not automatically. That is, we can separate the notion of quasi-isometric reducibility from its automatic counterpart.

▶ **Theorem 51.** *Suppose an infinite regular set $D \subseteq \Sigma^*$ has superlinear but polynomial growth. There exist automatic colourings α, β of D such that the following statements are true:*
(a) *α is* not *aqi-reducible to β;*
(b) *α is quasi-isometrically reducible to β.*

Proof. Suppose that D has polynomial growth with degree $c \geq 2$. Then, by Lemmas 45 and 46, D must have a subset of the form $uv_1^* w_1 v_2^* w_2 \ldots v_c^* w_c$ where for each $1 \leq i \leq c$, v_i is a non-empty string and the first character after v_i^*, if it exists, is not the same as the first character of v_i.

Define $t = v_1^{|v_2| \cdot \ldots \cdot |v_c|}$. Then, $ut^* w_1 \ldots w_c$ is a subset of D and $|t| = |v_1| \cdot \ldots \cdot |v_c| > 0$. Let $w = w_1 \ldots w_c$ and $\alpha, \beta : D \to \{0, 1\}$ be defined such that:

- $\alpha(x) = 1$ if and only if $x \in u(t^2)^* w$;
- $\beta(x) = 1$ if and only if $x \in u(t^4)^* w$.

We first prove statement (a) and show that α is not aqi-reducible to β. Suppose that f is an automatic C-quasi-isometry from α to β. By Lemma 45, there are $\Omega(n^{c-1})$ strings in D of length $n = |uw| + (2i+1)|t|$ for any $i \in \mathbb{N}_0$. Since $|ut^{2i}w| < n < |ut^{2(i+1)}w|$, there are $\Omega(n^{c-1})$ strings in D which are length-lexicographically between $ut^{2i}w$ and $ut^{2(i+1)}w$. Note that $c \geq 2$, and so there exists some i_0 such that for all $i \geq i_0$, $ut^{2i}w \oplus C <_{llex} ut^{2(i+1)}w$. Then, $f(ut^{2i}w) <_{llex} f(ut^{2(i+1)}w)$.

Define $g : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(ut^{2i}w) = ut^{4g(i)}w$ for any $i \in \mathbb{N}_0$. Such g is well-defined by the definitions of α and β, as well as the colour-preserving property of f. Furthermore, since f is automatic, then so is g. From the previous paragraph, we know that for all $i \geq i_0$, $g(i+1) \geq g(i) + 1$. Then, for any $i \in \mathbb{N}_0$, $g(i_0 + i) \geq g(i_0) + i$. Therefore, for any $x \in \mathbb{N}_0$,

$$f(ut^{2(i_0+i)}w) = ut^{4(g(i_0+i))}w \geq_{llex} ut^{4(g(i_0)+i)}w.$$

This contradicts the assumption that f is automatic. Hence, an automatic quasi-isometry from α to β cannot exist.

We now prove statement (b) and show that α is quasi-isometrically reducible to β.

As shown earlier, Lemma 45 implies that there are $\Omega(i^{c-1})$ strings in D which are length-lexicographically between $ut^{2i}w$ and $ut^{2(i+1)}w$ for any $i \in \mathbb{N}_0$. Furthermore, by Lemma 46, there are at most $(2|t|+1)O(i^{c-1}) = O(i^{c-1})$ strings in D which are length-lexicographically between $ut^{2i}w$ and $ut^{2(i+1)}w$. Hence, there exist some $d_1, d_2 \in \mathbb{R}^+$ such that for all large enough i, we have

$$d_1 i^{c-1} \leq |\{x \in D : ut^{2i}w <_{llex} x <_{llex} ut^{2(i+1)}w\}| \leq d_2 i^{c-1}. \tag{6}$$

By a similar argument, there exist some $d_3, d_4 \in \mathbb{R}$ such that for all large enough i, we have

$$d_3 i^{c-1} \leq |\{y \in D : ut^{4i}w <_{llex} y <_{llex} ut^{4(i+1)}w\}| \leq d_4 i^{c-1}. \tag{7}$$

Then, there is some $i_0 \in \mathbb{N}$ such that (6) and (7) are both true for all $i \geq i_0$.

We define $f : D \to D$ as follows. For any $i \geq i_0$:

- $f(ut^{2i}w) = ut^{4i}w$;
- $f(ut^{2i}w \oplus 1), f(ut^{2i}w \oplus 2), \ldots, f(ut^{2(i+1)}w \ominus 1)$ are distributed evenly between $ut^{4i}w$ and $ut^{4(i+1)}w$ in a non-decreasing manner.

Furthermore, for any $x <_{llex} ut^{2i_0}w$, $f(x)$ is the length-lexicographically minimum string in D such that $\beta(x) = \alpha(x)$.

Let $C = \max\{\lceil d_4/d_1 \rceil, \lceil d_2/d_3 \rceil, j_0\}$ where $ut^{4i_0}w = \min_{llex} D \oplus j_0$. We show that f is a C-quasi-isometry. First, we show that for any $x <_{llex} ut^{2i_0}w$, $f(x) <_{llex} ut^{4i_0}w$. Note that $i_0 \geq 1$. Then, $uw <_{llex} ut^{3i_0}w <_{llex} ut^{4i_0}w$. Moreover, $\beta(uw) = 1$ and $\beta(ut^{3i_0}w) = 0$. So, for any $x <_{llex} ut^{2i_0}w$, the length-lexicographically minimum $z \in D$ such that $\beta(z) = \alpha(x)$ satisfies $z <_{llex} ut^{4i_0}w$. Hence, $f(x) = z <_{llex} ut^{4i_0}w$.

Then, by definition of C, $f(\min_{llex} D) <_{llex} ut^{4i_0}w \leq_{llex} \min_{llex} D \oplus C$ since $\min_{llex} D <_{llex} ut^{2i_0}w$. Similarly, if $x <_{llex} ut^{2i_0}w$, then by definition of C, $f(x) \ominus C \leq_{llex} f(x \oplus 1) \leq_{llex} f(x) \oplus C$. Now suppose that $x \geq_{llex} ut^{2i_0}w$. Clearly, $f(x \oplus 1) \geq_{llex} f(x) \geq_{llex} f(x) \ominus C$. Let i be the largest i' such that $ut^{2i'}w \leq_{llex} x$. Note that $i \geq i_0$. So, from (6) and (7), the ratio

$$\frac{|\{y \in D : ut^{4i}w <_{llex} y <_{llex} ut^{4(i+1)}w\}|}{|\{x \in D : ut^{2i}w <_{llex} x <_{llex} ut^{2(i+1)}w\}|} \leq \frac{d_4}{d_1}.$$

Hence, by definition of f, $f(x \oplus 1) \leq_{llex} f(x) \oplus \lceil d_4/d_1 \rceil \leq_{llex} f(x) \oplus C$. Therefore, f satisfies Condition (a) of Definition 42.

We now show that f satisfies Condition (b) of Definition 42. Suppose that $x \oplus C <_{llex} y$. By definition of C, $y \geq_{llex} ut^{2i_0}w$. First, consider the case where $x <_{llex} ut^{2i_0}w$, so that $f(x) <_{llex} ut^{4i_0}w$. Observe that f is non-decreasing from $ut^{2i_0}w$ onwards, and so $f(y) \geq_{llex} f(ut^{2i_0}w) = ut^{4i_0}w$. Then, we have $f(x) <_{llex} ut^{4i_0}w \leq_{llex} f(y)$.

Now we consider the case that $x \geq_{llex} ut^{2i_0}w$. Let i be the largest i' such that $ut^{2i'}w \leq_{llex} x$. Clearly, $i \geq i_0$. If $x = ut^{2i}w$ or $ut^{2(i+1)}w \leq_{llex} y$, then clearly, $f(x) <_{llex} y$. So, we can assume that $ut^{2i}w <_{llex} x <_{llex} y <_{llex} ut^{2(i+1)}w$. By (6) and (7), the ratio

$$\frac{|\{x \in D : ut^{2i}w <_{llex} x <_{llex} ut^{2(i+1)}w\}|}{|\{y \in D : ut^{4i}w <_{llex} y <_{llex} ut^{4(i+1)}w\}|} \leq \frac{d_2}{d_3}.$$

Hence, there are at most $\lceil d_2/d_3 \rceil \leq C$ consecutive strings $x', x' \oplus 1, \ldots, x' \oplus j$ between $ut^{2i}w$ and $ut^{2(i+1)}w$ which are mapped to the same y' between $ut^{4i}w$ and $ut^{4(i+1)}w$. So, since $y >_{llex} x \oplus C$, then $f(y) >_{llex} f(x)$. ◀

On the other hand, if the growth of D is not polynomial, then it must be exponential. We can separate the notions of quasi-isometric reducibility from its automatic counterparts for some domains D with exponential growth.

▶ **Example 52.** Let $D \subseteq \{0,1\}^*$ be defined as follows: D contains all $\sigma \in \{0,1\}^*$ of even length, and all $\tau \in 0^*1^*$ of odd length. There exist automatic colourings α, β of D such that the following statements are true:

(a) α is *not* aqi-reducible to β;
(b) α is quasi-isometrically reducible to β.

We define $\alpha, \beta : D \to \{0,1\}$ as follows:

- $\alpha(\sigma) = 1$ if and only if $\sigma \in 1^*$ and $|\sigma|$ is odd.
- $\beta(\sigma) = 1$ if and only if $\sigma \in 0^*$ and $|\sigma|$ is odd.

We first prove that α is not aqi-reducible to β. Suppose, for the sake of contradiction, that f is a C-quasi-isometric reduction from α to β which is automatic. Since f is colour-preserving, then for any $n \in \mathbb{N}_0$, $f(1^{2n+1}) = 0^{2m+1}$ for some $m \in \mathbb{N}_0$. In particular, observe that $f(1^{2n+1}) = 0^{\Theta(n)}$. Now consider $f(0^{2n+1})$ for large enough $n \in \mathbb{N}_0$. Note that $0^{2n+1} \oplus (2n+1) = 1^{2n+1}$. By Condition (b) of Definition 42, $f(0^{2n+1}) \oplus \lfloor (2n+1)/(C+1) \rfloor \leq f(1^{2n+1})$. On the other hand, by Condition (a) of Definition 42, $f(1^{2n+1}) \leq_{llex} f(0^{2n+1}) \oplus (2n+1)C$.

Hence, $f(0^{2n+1}) = f(1^{2n+1}) \ominus \Theta(n) = 0^{\Theta(n)} \ominus \Theta(n)$. In other words, $f(0^{2n+1}) = 1^m 0\sigma$ for some $\sigma \in \{0,1\}^*$ such that $|\sigma| = \Theta(\log n)$ and $m + |\sigma| = \Theta(n)$.

Now consider the set $S = f(\{0^{2n+1} : n \in \mathbb{N}_0\})$. Since f is automatic, then the set S is regular, and thus satisfies the pumping lemma. Take some $f(0^{2n'+1})$ for some large enough n'. As shown earlier, $f(0^{2n'+1}) = 1^{m'}0\sigma'$ for some $m' \in \mathbb{N}_0$ and $\sigma' \in \{0,1\}^*$ such that $|\sigma'| = \Theta(\log n')$. So, by the pumping lemma, there are some $u, w \in \{0,1\}^*$ and $v \in \{0,1\}^+$ such that $\sigma' = uvw$ and $1^{m'}0uv^*w \subseteq S$. On the other hand, for all large enough n, $f(0^{2n+1})$ has the form $1^m 0\sigma$ where $|\sigma| = \Theta(\log n)$ and $m + |\sigma| = \Theta(n)$. This contradicts that S contains all words $1^{m'}0uv^i w$ where $i \in \mathbb{N}_0$. Hence, α is not aqi-reducible to β.

It remains to show that α is quasi-isometrically reducible to β. Let $f : D \to D$ be defined as follows. Firstly, $f(\epsilon) = f(0) = \epsilon$. For any $n \in \mathbb{N}_0$:

- $f(1^{2n+1}) = 0^{2n+1}$;
- $f(1^{2n+1} \oplus 1), f(1^{2n+1} \oplus 2), \ldots, f(1^{2(n+1)+1} \ominus 1)$ are distributed evenly between 0^{2n+1} and $0^{2(n+1)+1}$ in a non-decreasing manner.

We shall show that f is a 1-quasi-isometric reduction from α to β.

Note that $\min_{llex} D = \epsilon$. Clearly, $f(\epsilon) = \epsilon \leq_{llex} \epsilon \oplus 1$. Furthermore, f is non-decreasing and so for any $x \in D$, $f(x) \ominus 1 \leq_{llex} f(x) \leq_{llex} f(x \oplus 1)$. To prove the remaining inequality in Condition (a) of Definition 42, note that for any $n \in \mathbb{N}_0$, $0^{2(n+1)+1} = 0^{2n+1} \oplus (2n+2+2^{2n+2})$ and $1^{2(n+1)+1} = 1^{2n+1} \oplus (2n+4+2^{2n+2})$. Hence, there are $2n+1+2^{2n+2}$ strings length-lexicographically between 0^{2n+1} and $0^{2(n+1)+1}$, and $2n+3+2^{2n+2}$ strings between 1^{2n+1} and $1^{2(n+1)+1}$. Since there are fewer strings between 0^{2n+1} and $0^{2(n+1)+1}$ than between 1^{2n+1} and $1^{2(n+1)+1}$, by definition of f, for any $x \geq_{llex} 1$, $f(x \oplus 1) = f(x)$ or $f(x) \oplus 1$. So, $f(x \oplus 1) \leq f(x) \oplus 1$ for all $x \in D$. Therefore, f satisfies Condition (a) of Definition 42 with $C = 1$.

To prove that f satisfies Condition (b) of Definition 42, note that for any $n \in \mathbb{N}_0$, $2n+1+2^{2n+2} \geq 5$ and so the number of strings between 1^{2n+1} and $1^{2(n+1)+1}$ is less than two times the number of strings between 0^{2n+1} and $0^{2(n+1)+1}$. Hence, each $y \in D$ is the image of at most 2 strings $x \in D$. Since f is non-decreasing, then $f(x) <_{llex} f(y)$ whenever $x \oplus 1 <_{llex} y$.

Hence, f satisfies Conditions (a) and (b) of Definition 42, and is a 1-quasi-isometric reduction from α to β.

6 Conclusions and Future Investigations

The present paper introduced finer-grained notions of quasi-isometries between infinite strings, in particular requiring the reductions to be recursive. We showed that permutation quasi-isometric reductions are provably more restrictive than one-one quasi-isometric reductions, which are in turn provably more restrictive than many-one quasi-isometric reductions. One result was that general many-one quasi-isometries are strictly more powerful than recursive many-one quasi-isometries, which answers Khoussainov and Takisaka's open problem.

This work also presented some results on the structures of the permutation, one-one and many-one quasi-isometric degrees. It was shown, for example, that there are two infinite strings whose many-one quasi-isometric degrees have a unique common upper bound. It was also proven that the partial order Σ^ω_{mqi} is non-dense with respect to pairs of mqi-degrees. We conclude with the simple observation that the class of mqi-degrees does not form a lattice; in particular, the mqi-degrees $[0^\omega]_{mqi}$ and $[1^\omega]_{mqi}$ do not have a common lower bound.

Furthermore, this paper also studied quasi-isometries which are automatic. The main results show that automatic quasi-isometry can be separated from general quasi-isometry depending on the growth of the domain.

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