Learning How to Separate

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Abstract

The main question addressed in the present work is how to find effectively a recursive function separating two sets drawn arbitrarily from a given collection of disjoint sets. In particular, it is investigated when one can find better learners which satisfy additional constraints. Such learners are the following: confident learners which converge on all data-sequences; conservative learners which abandon only definitely wrong hypotheses; set-driven learners whose hypotheses are independent of the order and the number of repetitions of the data-items supplied; learners where either the last or even all hypotheses are programs of total recursive functions.

The present work gives a complete picture of the relations between these notions: the only implications are that whenever one has a learner which only outputs programs of total recursive functions as hypotheses, then one can also find learners which are conservative and set-driven. The following two major results need a nontrivial proof.

(1) There is a class for which one can find, in the limit, recursive functions separating the sets in a confident and conservative way, but one cannot find even partial-recursive functions separating the sets in a set-driven way.

(2) There is a class for which one can find, in the limit, recursive functions separating the sets in a confident and set-driven way, but one cannot find even partial-recursive functions separating the sets in a conservative way.

1 Introduction

Consider the scenario in which a subject is attempting to learn its environment. At any given time, the subject receives a finite piece of data about its

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environment, and based on this finite information, conjectures an explanation about the environment. The subject is said to *learn* its environment just in case the explanations conjectured by the subject become fixed over time, and this fixed explanation is a correct representation of the subject's environment. Inductive Inference, a subfield of computational learning theory, provides a framework for the study of the above scenario when the subject is an algorithmic device. The above model of learning is based on the work initiated by Gold [8] and has been used in inductive inference of both functions and sets. This model is often referred to as *explanatory learning*. We refer the reader to [1,2,5,10,13] for background material in this field.

In recursion theory, recursive separability of disjoint sets has been extensively explored [19]. A prominent fact is that there are disjoint recursively enumerable sets which cannot be separated by a total recursive function which takes 0 on the first and 1 on the second set. Indeed, the following question has been investigated: What are the oracles such that relative to them every two disjoint and recursively enumerable sets are separable? These oracles turned out to be those which allow to compute a complete extension of Peano-Arithmetic [17].

In the present work, we consider a combination of learning and separation. Thus a machine receives, as input, data about two disjoint sets. The machine is then expected to come up, in the limit, with a procedure separating the two input sets. A machine is able to *sep-identify* sets from a class of disjoint sets, if it is able to sep-identify any pair of sets from the class.

Here are some examples. Given a recursively enumerable class $\mathcal{C} = \{L_0, L_1, \ldots\}$ of non-empty disjoint sets, the sep-identifier reads more and more data on the input sets L, L' until it finds a data-item $x \in L'$ and a j such that $x \in L_j$. Then the sep-identifier outputs an index for a partial-recursive function which maps L_j to 1 and all L_i with $i \neq j$ to 0 (and is done).

But it is not required that the class $\mathcal{C}$ is recursively enumerable. One can even sep-identify any given class which consists of one infinite and arbitrarily many finite sets. The sep-identifier, in parallel, reads data and outputs hypotheses. At any intermediate step it does the following. Let H, H' denote the sets of examples seen so far from the sets L, L' to be separated. If $|H| \leq |H'|$, then the sep-identifier outputs the characteristic function of $\mathbb{N} - H$, otherwise it outputs the characteristic function of H'. A hypothesis is revised only if new elements of the currently smaller set show up. As one of the sets L, L' is finite, the sep-identifier therefore converges to one of the following functions: If $|L| \leq |L'|$, then the last hypothesis is the characteristic function of $\mathbb{N} - L$; otherwise the last hypothesis is the characteristic function of L'. Thus, the sep-identifier is successful on $\mathcal{C}$. In contrast to the previous example, this sep-identifier might have to revise the conjectured function finitely often.

One can combine sep-identification with additional constraints which are motivated from corresponding constraints used for notions of learning. The main result of the present work is to give a complete picture of the relations between the following criteria of sep-identification: confident sep-identification, conservative separation, set-driven sep-identification and Popperian sep-identification. Here a confident sep-identifier converges on every input function. A conservative sep-identifier abandons only definitely wrong hypotheses. A set-driven sep-identifier outputs hypotheses depending only on the set of data-items seen so far, but not on their order or quantity. A Popperian sep-identifier only conjectures programs for total functions. In addition, a weak version of Popperian sep-identification is considered, namely where the sep-identifier might preliminarily conjecture some partial-recursive functions but the final hypothesis is then a program for a total function separating the given sets. It is shown that Popperian sep-identification implies the other notions of sep-identification except confident sep-identification; further implications between these five criteria of sep-identification do not exist.

Notation. Any unexplained recursion theoretic notation is from [19]. The symbol $\mathbb{N}$ denotes the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$. Symbols $\emptyset$, $\subseteq$, $\subset$, $\supseteq$, and $\supset$ denote empty set, subset, proper subset, superset, and proper superset, respectively. Cardinality of a set S is denoted by $\mathrm{card}(S)$. Let $\max(A)$ denote the maximum of A and $\min(A)$ the minimum of A; by convention, $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$. The notions $\mathrm{domain}(\eta)$ and $\mathrm{range}(\eta)$ denote the domain and range of partial function η respectively.

The function $\langle \cdot, \cdot \rangle$ is a computable, bijective mapping from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ [19], $\langle x, y \rangle = \frac{1}{2} \cdot (x + y + 1) \cdot (x + y) + y$. Note that $\langle \cdot, \cdot \rangle$ is monotonically increasing in both of its arguments. This notion is extended to triples in a natural way: $\langle x, y, z \rangle = \langle x, \langle y, z \rangle \rangle$.

By φ we denote a fixed *acceptable* programming system for the partial-recursive functions mapping $\mathbb{N}$ to $\mathbb{N}$ [15,19]. An example for an acceptable programming system is any recursive enumeration of all Turing machines. Further examples are standard programming languages such as Basic, C, Fortran, Pascal provided that the data-type of normal variables is $\mathbb{N}$ (without upper bound on the values). By φ_i we denote the partial-recursive function computed by the program with number i in the φ-system. By Φ we denote an arbitrary fixed Blum complexity measure [3,9] for the φ-system. By W_i we denote $\mathrm{domain}(\varphi_i)$. W_i is, then, the recursively enumerable subset of $\mathbb{N}$ accepted (or equivalently, generated) by the φ-program i. Symbols L, L', H, H', with or without subscripts, range over recursively enumerable sets. By $W_{i,s}$ we denote the set $\{x < s : \Phi_i(x) < s\}$.

A non-empty class $\mathcal{C}$ of recursively enumerable sets is said to be recursively enumerable [19] iff there exists a recursive function f such that $\mathcal{C} = \{W_{f(i)} : i \in \mathbb{N}\}$. In this latter case we say that $W_{f(0)}, W_{f(1)}, \ldots$ is a recursive enumeration of $\mathcal{C}$.

K denotes the diagonal halting set, that is $\{x : \varphi_x(x)\downarrow\}$. A pair of disjoint sets, L and L', are said to be recursively separable iff there exists a recursive function f such that for all $x \in L$, $f(x) = 0$ and for all $x \in L'$, $f(x) = 1$. If a pair of disjoint sets is not recursively separable, then the pair is said to be recursively inseparable; for example the sets $\{x \in K : \varphi_x(x) = 0\}$ and

$\{x \in K : \varphi_x(x) = 1\}$ form a recursively inseparable pair of sets [17, Theorem II.2.5].

Following Gold [8], the next definition introduces the concept of a *sequence* of data and of a text for a set.

Definition 1 (a) A *sequence* σ is a mapping from an initial segment of $\mathbb{N}$ into $(\mathbb{N} \cup \{\#\})$. The empty sequence is denoted by λ.

(b) The *content* of a sequence σ, denoted content(σ), is the set of natural numbers in the range of σ. That is, content$(\sigma) = \text{range}(\sigma) - \{\#\}$.

(c) The *length* of σ, denoted by $|\sigma|$, is the number of elements in σ. So, $|\lambda| = 0$ and $|2\,3\,5\,\#\,\#| = 5$.

(d) Let SEQ denote the set of all finite sequences. Let SEQ^2 denote the set of all pairs (σ, σ') such that $\sigma, \sigma' \in \text{SEQ}$, $|\sigma| = |\sigma'|$ and $\text{content}(\sigma) \cap \text{content}(\sigma') = \emptyset$. Furthermore, $\text{SEQ}^2(L, L')$ is the set of all $(\sigma, \sigma') \in \text{SEQ}^2$ such that $\text{content}(\sigma) \subseteq L$ and $\text{content}(\sigma') \subseteq L'$.

(e) An infinite sequence T is called a text for a set L iff content$(T) = L$. A pair (T, T') of infinite sequences is a doubletext (for content(T) and content(T')), if T and T' are texts for disjoint sets.

(f) For $n \leq |\sigma|$, the initial sequence of σ of length n is denoted by $\sigma[n]$. So, $\sigma[0]$ is λ. Similarly $T[n]$ is, for any $n \in \mathbb{N}$, the initial segment of T of length n.

Intuitively, #'s represent pauses in the presentation of data. We let σ, σ', τ and τ' with or without subscripts, range over finite sequences. We denote the sequence formed by the concatenation of τ at the end of σ by $\sigma\tau$. Furthermore, we use σx to denote the concatenation of sequence σ and the sequence of length 1 which contains the element x.

We now consider the notion of separating sets. Roughly speaking, a sep-identifier for a class of disjoint sets finds for each doubletext for distinct sets in this class a partial-recursive function mapping one set to 0 and the other one to 1. More precisely, this is defined as follows.

Definition 2 (a) A partial-recursive function ψ *separates* sets L and L' if $\psi(x) = 0$ for all $x \in L$ and $\psi(x) = 1$ for all $x \in L'$. If $x \notin L \cup L'$, $\psi(x)$ is either undefined or one of the values 0 and 1.

(b) A *sep-identifier* M is a recursive function from SEQ^2 to $\mathbb{N} \cup \{?\}$. A sep-identifier M converges on a doubletext (T, T') iff there is a length n such that for all $m \geq n$, $M(T[m], T'[m]) = M(T[n], T'[n])$.

(c) A class $\mathcal{C}$ of pairwise disjoint subsets of $\mathbb{N}$ is *sep-identifiable* iff there is a sep-identifier which, for every distinct (and thus disjoint) $L, L' \in \mathcal{C}$, converges on any doubletext for L and L' to an index e of a partial-recursive function which separates L and L'.

(d) Let $M_0, M_1, \ldots$ be a recursive enumeration of all partial-recursive functions from SEQ^2 to $\mathbb{N} \cup \{?\}$ in the sense that $n, \sigma, \sigma' \to M_n(\sigma, \sigma')$ with

$n \in \mathbb{N}$ and $(\sigma, \sigma') \in \mathrm{SEQ}^2$ is a partial-recursive function. The set of the total functions in this list coincides with the set of all sep-identifiers.

Remark 3 It is not more difficult to separate k disjoint sets instead of 2. For example, given 3 sets L, L', L'' by their texts T, T', T'', one can simulate the sep-identifier for each pair of 2 sets coming up with programs e, e', e'' to separate the pairs (L, L'), (L, L'') and (L', L''), respectively. Then one has that the program d separates all three sets where d is given as

$$\varphi_d(x) = \begin{cases} 0, & \text{if } \varphi_e(x) \downarrow = 0 \wedge \varphi_{e'}(x) \downarrow = 0; \\ 1, & \text{if } \varphi_e(x) \downarrow = 1 \wedge \varphi_{e''}(x) \downarrow = 0; \\ 2, & \text{if } \varphi_{e'}(x) \downarrow = 1 \wedge \varphi_{e''}(x) \downarrow = 1; \\ u, & \text{if } \varphi_e(x), \varphi_{e'}(x), \varphi_{e''}(x) \text{ are defined} \\ & \text{and no previous case applies;} \\ \uparrow, & \text{otherwise;} \end{cases}$$

and u is an arbitrary number in $\{0, 1, 2\}$, it does not matter which one. It is easy to verify then that $L \subseteq \varphi_d^{-1}(0)$, $L' \subseteq \varphi_d^{-1}(1)$, $L'' \subseteq \varphi_d^{-1}(2)$ and φ_d is total if the functions $\varphi_e, \varphi_{e'}, \varphi_{e''}$ are total. Similar arguments deal with the case of $4, 5, \ldots$ sets. Thus we deal only with separating pairs of sets.

2 The Criteria of Separation

The following notions restrict the permitted behaviour of the sep-identifier. That is, the sep-identifier M has to satisfy some additional properties. It will be shown that there are classes which are sep-identifiable but where no sep-identifier satisfies any of these additional requirements.

Definition 4 (a) M is *Popperian* iff for all $(\sigma, \sigma') \in \mathrm{SEQ}^2$, either $M(\sigma, \sigma') =$? or $M(\sigma, \sigma')$ is an index of a total function.

(b) M is *conservative on* $(\sigma, \sigma') \in \mathrm{SEQ}^2$ iff the following holds for all $m \leq |\sigma|$, $n < m$ and $e \in \mathbb{N}$: whenever $e = M(\sigma[n], \sigma'[n])$, $\mathrm{content}(\sigma[m]) \subseteq \varphi_e^{-1}(0)$ and $\mathrm{content}(\sigma'[m]) \subseteq \varphi_e^{-1}(1)$ then $M(\sigma[m], \sigma'[m]) \in \{?, e\}$. M is *conservative* iff M is conservative on all $(\sigma, \sigma') \in \mathrm{SEQ}^2$.

(c) M is *set-driven* iff it holds for all $(\sigma, \sigma'), (\tau, \tau') \in \mathrm{SEQ}^2$ with $\mathrm{content}(\sigma) = \mathrm{content}(\tau)$ and $\mathrm{content}(\sigma') = \mathrm{content}(\tau')$ that $M(\sigma, \sigma') = M(\tau, \tau')$.

(d) M is *confident* iff M converges on all doubletexts, even on doubletexts not for sets in $\mathcal{C}$.

(e) M is a *recsep-identifier* for a class $\mathcal{C}$ of disjoint sets iff M is a sep-identifier for $\mathcal{C}$ which converges on every doubletext for distinct sets in $\mathcal{C}$ to an index of a total function.

Remark 5 These notions for behaviours of sep-identifiers are parallel to the corresponding notions of traditional learners of the same name as introduced in [2,4,10,16,18,21].

If a class $\mathcal{C}$ of disjoint sets is learnable under such a criterion, then it is also sep-identifiable under the same criterion. For example, consider Popperian learning where a class $\mathcal{C}$ is Popperian learnable in the limit iff there is a recursive function M such that

- M maps every string in SEQ either to the symbol ? or to an index of a total recursive $\{0, 1\}$-valued function;
- If $L \in \mathcal{C}$, then for every text T for L, there is an index e computing the characteristic function of L such that $M(T[n]) = e$, for almost all n.

Then one can transform this M into a sep-identifier N by defining that, for all $(\sigma, \sigma') \in \text{SEQ}^2$, $N(\sigma, \sigma') = M(\sigma')$. The sep-identifier N converges on every doubletext (T, T') for distinct sets L and $L' \in \mathcal{C}$ to the characteristic function of L' which separates L and L'. Furthermore, whenever N outputs a hypothesis $e \in \mathbb{N}$, φ_e is a total function. That is, N inherits the property of being Popperian from M. Similarly one can show that conservatively, set-driven and confidently learnable classes $\mathcal{C}$ of disjoint sets are also conservatively, set-driven and confidently sep-identifiable, respectively.

The converse direction does not hold. For a given enumeration of machines containing all learners, one can choose for the eth learner a non-empty recursive set $L_e \subseteq \{\langle e, x\rangle : x \in \mathbb{N}\}$ not learned by it; the choice is arbitrary if the eth machine is not a learner because of being partial. The sets $L_0, L_1, \ldots$ exist because no learner learns all recursive sets, even not all recursive subsets of $\{\langle e, x\rangle : x \in \mathbb{N}\}$. The class $\{L_0, L_1, L_2, \ldots\}$ has the following sep-identifier M: $M(\sigma, \sigma')$ outputs the characteristic function of the set $\{\langle e, x\rangle : x \in \mathbb{N}\}$ if e is the unique number such that there is a pair of the form $\langle e, x\rangle \in \text{content}(\sigma')$. If there is no such e or if there are several, then $M(\sigma, \sigma')$ outputs ?. The sep-identifier M satisfies all the restrictions postulated in Definition 4. For more connections between learning sets and learning how to separate, the reader should consult the technical report [11].

Remark 6 Blum and Blum [2] considered the model of learning extensions of partial-recursive functions. The separations considered in the present work can be viewed as a special case of this type of learning, since one could map the class $\mathcal{C}$ to the class $\mathcal{F}$ of all functions $\Psi_{L,L'}$ (for distinct $L, L' \in \mathcal{C}$) with $\Psi_{L,L'}$ being 0 on L and being 1 on L' and being undefined everywhere else. Now $\mathcal{C}$ is (conservatively) sep-identifiable iff $\mathcal{F}$ is (conservatively) learnable in the model of Blum and Blum [2]. Let $\mathcal{C}$ be a class which is sep-identifiable but not conservatively sep-identifiable. Then $\mathcal{F}$ corresponding to this class $\mathcal{C}$ witnesses that, in the model of Blum and Blum, some class of partial-recursive functions is learnable in the limit but is not conservatively learnable. This gives a contrast to the case of learning total recursive functions where Stephan and Zeugmann [20] showed that conservativeness is not restrictive.

Although every separation problem is the special case of a learning problem in the model of Blum and Blum [2], there is no general correspondence between these worlds. For example, there are reliably but not consistently learnable classes of functions while Theorem 8 below shows that these notions coincide in the case of separating sets.

Definition 7 [2,6,7] Let L, L' be disjoint sets and $(\sigma, \sigma') \in \mathrm{SEQ}^2(L, L')$.

(a) (σ, σ') is a *stabilizing sequence* for M on (L, L') iff for all $(\tau, \tau') \in \mathrm{SEQ}^2(L, L')$ such that $\sigma \subseteq \tau$ and $\sigma' \subseteq \tau'$, $M(\sigma, \sigma') = M(\tau, \tau')$.
(b) (σ, σ') is a *locking sequence* for M on (L, L') iff (σ, σ') is a stabilizing sequence for M on (L, L') and $\varphi_{M(\sigma,\sigma')}$ separates L and L'.

Using standard arguments, as for example in [2], one can show that if M sep-identifies $\{L, L'\}$, then there is a locking sequence for M on (L, L').

A sep-identifier M is *consistent on* (σ, σ') iff either $M(\sigma, \sigma') = ?$ or all $x \in \mathrm{content}(\sigma)$ satisfy $\varphi_{M(\sigma,\sigma')}(x) = 0$ and all $x \in \mathrm{content}(\sigma')$ satisfy $\varphi_{M(\sigma,\sigma')}(x) = 1$. A sep-identifier M is *consistent* iff it is consistent on all $(\sigma, \sigma') \in \mathrm{SEQ}^2$. A sep-identifier M is *reliable* iff for all doubletexts (T, T') where M converges to a natural number e, the partial-recursive function φ_e separates $\mathrm{content}(T)$ and $\mathrm{content}(T')$. Given a Popperian sep-identifier M for a class $\mathcal{C}$, one can build a new sep-identifier N which is consistent on every input $(\sigma, \sigma') \in \mathrm{SEQ}^2$. This can be done as follows. Note that the programs output by M form a recursively enumerable set of programs, $\{p_0, p_1, \ldots\}$, for a class of recursive functions (as M is Popperian). Without loss of generality, one may assume that this class contains a program for the characteristic function of every finite set. Now, N on any input $(\sigma, \sigma') \in \mathrm{SEQ}^2$, can output the first program p_i in the list which is consistent with the input (that is, $\mathrm{content}(\sigma) \subseteq \varphi_{p_i}^{-1}(0)$, and $\mathrm{content}(\sigma') \subseteq \varphi_{p_i}^{-1}(1)$). Clearly, N is consistent. Furthermore, N is a sep-identifier for any pair of languages, for which M is a sep-identifier (as N sep-identifies any pair of languages which can be separated by some program in the list).

One can easily see that every consistent N is also reliable since whenever N converges on a doubletext (T, T') to an index for ψ, then ψ maps $\mathrm{content}(T)$ to 0 and $\mathrm{content}(T')$ to 1.

The next result shows that the converse of above two results also holds, and thus it is not necessary to consider consistent and reliable learners beyond this result.

Theorem 8 *A class $\mathcal{C}$ is Popperian sep-identifiable iff it is consistently sep-identifiable iff it is reliably sep-identifiable.*

Proof. By the comments preceding Theorem 8, it is sufficient to show that $\mathcal{C}$ is Popperian sep-identifiable whenever $\mathcal{C}$ is reliably sep-identifiable.

Let M be a reliable sep-identifier for $\mathcal{C}$. Consider the following class

$$\mathcal{F} = \{F_{(\sigma,\sigma')} : (\sigma, \sigma') \in \mathrm{SEQ}^2\}$$

where the membership of x in a set $F_{(\sigma,\sigma')}$ is defined according to the first case below which applies:

- If $x \in \text{content}(\sigma')$ then $x \in F_{(\sigma,\sigma')}$;
- If $x \in \text{content}(\sigma)$ or $M(\sigma,\sigma') = ?$ then $x \notin F_{(\sigma,\sigma')}$;
- If $x \notin \text{content}(\sigma) \cup \text{content}(\sigma')$ and there is an s such that for all $t < s$, $M(\sigma\#^t, \sigma' x^t) = M(\sigma,\sigma')$ and $M(\sigma x^s, \sigma'\#^s) \neq M(\sigma,\sigma')$, then $x \in F_{(\sigma,\sigma')}$;
- If $x \notin \text{content}(\sigma) \cup \text{content}(\sigma')$ and there is a t such that for all $s \leq t$, $M(\sigma x^s, \sigma'\#^s) = M(\sigma,\sigma')$ and $M(\sigma\#^t, \sigma' x^t) \neq M(\sigma,\sigma')$, then $x \notin F_{(\sigma,\sigma')}$.

Let $(\sigma,\sigma') \in \text{SEQ}^2$. If $M(\sigma,\sigma') = ?$ then only the first two cases are relevant and $F_{(\sigma,\sigma')} = \text{content}(\sigma')$. Otherwise $M(\sigma,\sigma')$ outputs a number $e \in \mathbb{N}$. Since M is reliable, there is no $x \notin \text{content}(\sigma) \cup \text{content}(\sigma')$ for which M converges on both doubletexts $(\sigma\#^\infty, \sigma' x^\infty)$ and $(\sigma x^\infty, \sigma'\#^\infty)$ to e. Thus the above case-distinction defines for every x, whether x belongs to $F_{(\sigma,\sigma')}$ or not. It is easy to see that these computations are uniform and that $\mathcal{F}$ is contained in an indexed family $\{L_0, L_1, \ldots\}$. Recall that $\{L_0, L_1, \ldots\}$ is an indexed family iff the function $i, x \to L_i(x)$ is total recursive in both inputs [10, Exercise 4–7 on page 85]. Let $ind(e)$ be a program for the characteristic function of L_e.

Now a Popperian sep-identifier N for $\mathcal{F}$ outputs on input $(\sigma,\sigma') \in \text{SEQ}^2$ an index $ind(e)$, where e is the least number such that L_e is consistent with $(\sigma,\sigma') \in \text{SEQ}^2$ (that is, $\text{content}(\sigma) \subseteq \mathbb{N} - L_e$, and $\text{content}(\sigma') \subseteq L_e$). The sep-identifier N is total because the set $F_{(\sigma,\sigma')}$ is consistent with the input (σ,σ') and $F_{\sigma,\sigma'} \in \{L_0, L_1, \ldots\}$.

It remains to prove that N actually works for $\mathcal{C}$ and not only for $\mathcal{F}$. So let L, L' be disjoint sets in $\mathcal{C}$. By assumption M converges on every doubletext for L and L'. Thus there exists a locking-sequence $(\sigma,\sigma') \in \text{SEQ}^2(L, L')$ such that $M(\sigma\tau, \sigma'\tau') = M(\sigma,\tau)$ for all $(\tau,\tau') \in \text{SEQ}^2(L, L')$. It holds that $M(\sigma x^s, \sigma'\#^s) = M(\sigma,\sigma')$ for all $x \in L$ and all s. Similarly $M(\sigma\#^s, \sigma' x^s) = M(\sigma,\sigma')$ for all $x \in L'$ and all s. It follows that $L \subseteq \mathbb{N} - F_{(\sigma,\sigma')}$ and $L' \subseteq F_{(\sigma,\sigma')}$. Since $F_{(\sigma,\sigma')}$ belongs to the indexed family, there exists an e such that $L_e = F_{(\sigma,\sigma')}$. Then N converges to a canonical index of the characteristic function of a set $L_{e'}$ with $e' \leq e$. Since M is consistent, this function separates L and L'. This completes the proof. ∎

The main result of the paper is that there are only three implications within the set of properties defined in Definition 4. These implications are the ones caused by the fact that the Popperian sep-identifier N defined in Theorem 8 is a conservative and set-driven recsep-identifier.

Theorem 9 *Every Popperian sep-identifiable class is also conservatively sep-identifiable, set-driven sep-identifiable and recsep-identifiable; further implications do not hold.*

$$
\begin{tikzcd}
\text{Recsep} & \text{Set-Driven Sep} & \text{Conservative Sep} & \\
 & & & \text{Confident Sep} \\
 & \text{Popperian Sep} \arrow[uul] \arrow[uu] \arrow[uur] & &
\end{tikzcd}
$$

Furthermore, for every criterion I mentioned in Definition 4 there is class $\mathcal{C}$ which is not I-sep-identifiable but J-sep-identifiable for all criteria J mentioned in Definition 4 which does not imply the criterion I. The class $\mathcal{C}$ can be chosen such that every set in $\mathcal{C}$ is recursive.

The remaining part of the paper is used to prove Theorem 9. In three cases, the classes to witness the result are easy to construct.

Proposition 10 *Let $\mathcal{C}$ contain all sets $L_{e,y} = \{\langle e, y\rangle\} \cup \{\langle e, x+2\rangle : \varphi_e(x) = y\}$ where $e \in \mathbb{N}$, $y \in \{0, 1\}$ and φ_e is a $\{0, 1\}$-valued total recursive function. Then $\mathcal{C}$ has a conservative, confident and set-driven recsep-identifier but is not Popperian sep-identifiable.*

Proof. The sep-identifier M outputs on input $(\sigma, \sigma') \in \mathrm{SEQ}^2$ the index $h(e, y)$ for the function

$$
\varphi_{h(e,y)}(\langle e', x\rangle) = \begin{cases} 0 & \text{if } e' \neq e \text{ or } x = 1 - y; \\ 1 & \text{if } e' = e \text{ and } x = y; \\ \varphi_e(x-2) & \text{if } e' = e,\ y = 1 \text{ and } x \geq 2; \\ 1 - \varphi_e(x-2) & \text{if } e' = e,\ y = 0 \text{ and } x \geq 2; \end{cases}
$$

if there are a unique $e \in \mathbb{N}$ and $y \in \{0, 1\}$ such that $\langle e, y\rangle \in \mathrm{content}(\sigma')$. Otherwise $M(\sigma, \sigma')$ is ?. Whenever $L' \in \mathcal{C}$ and (T, T') is a doubletext with $\mathrm{content}(T') = L'$, M converges to $h(e, y)$ which is an index of the characteristic function of L'. Thus M witnesses that $\mathcal{C}$ is sep-identifiable. Since M outputs on every doubletext at most one index different from ?, M is conservative. It is easy to see that M is set-driven and confident.

However, there is no Popperian sep-identifier for $\mathcal{C}$. This holds because otherwise one could obtain from such a sep-identifier a recursive enumeration of all $\{0, 1\}$-valued recursive functions as follows. Suppose $\{p_0, p_1, \ldots\}$ is an enumeration of all the programs in the range of M. Without loss of generality assume that all the programs in the above list are $\{0, 1\}$-valued. Define $\varphi_{q_i}(x) = \varphi_{p_i}(x+2)$. Then, $\{q_0, q_1, \ldots\}$ gives a recursive enumeration of programs for the class of all $\{0, 1\}$-valued recursive functions. However, such an enumeration not exist [17, Proposition II.2.1]. Thus there is no Popperian sep-identifier for $\mathcal{C}$. ∎

Proposition 11 *Let $\mathcal{C}$ contain all sets $L_{e,y} = \{\langle e, y\rangle\} \cup \{\langle e, x+2\rangle : \varphi_e(x) = y\}$ where $e \in \mathbb{N}$, $y \in \{0, 1\}$ and φ_e is a $\{0, 1\}$-valued function which is undefined* *on at most one input. Then $\mathcal{C}$ has a conservative, confident and set-driven sep-identifier but neither a recsep-identifier nor a Popperian sep-identifier.*

Proof. The conservative, confident and set-driven sep-identifier is the same one as in Proposition 11. However, due to enriching the class, the property of being a recsep-identifier is lost.

Now, assume by way of contradiction that there is a recsep-identifier N for $\mathcal{C}$. Let e be an index of a $\{0,1\}$-valued function φ_e which is defined at all but at most one input. Now define a doubletext (T, T') for $L = \{\langle e, 0\rangle\} \cup \{\langle e, x+2\rangle : \varphi_e(x) = 0\}$ and $L' = \{\langle e, 1\rangle\} \cup \{\langle e, x+2\rangle : \varphi_e(x) = 1\}$. Feeding this doubletext (T, T') into N, one finds, in the limit, a program e' such that $\varphi_{e'}$ is a total function separating L and L'. Then one can compute from e' a further program e'' such that $\varphi_{e''}(x) = \varphi_{e'}(\langle e, x+2\rangle)$. The function $\varphi_{e''}$ is a total extension of φ_e. But it is well-known that there is no procedure to obtain such an e'' from e, even in the limit. This follows, for example, from a result of Kummer and Stephan [14, Proof of Theorem 8.1]. They constructed a family of partial-recursive functions $\varphi_{g(0)}, \varphi_{g(1)}, \ldots$, each of which is defined on all but at most one input, such that every learner finding in the limit, from e and a graph of total extension of $\varphi_{g(e)}$, an index for this total extension, has high Turing degree [14, Proof of Theorem 8.1]. ∎

The following auxiliary result is used to prove Proposition 13 below.

Proposition 12 *If A is infinite and M a confident sep-identifier, then M fails to sep-identify a class of two disjoint finite subsets of A.*

Proof. Let $A = \{a_0, a_1, \ldots\}$. Now one tries to construct inductively over n a doubletext (T, T') on which M diverges if the construction goes through for all n successfully. It is intended that $T = \lim_n \sigma_n$ and $T' = \lim_n \sigma'_n$. At the beginning, $\sigma_0 = \lambda$ and $\sigma'_0 = \lambda$. For $n = 0, 1, \ldots$ one does the following: If $M(\sigma_n a_n^m, \sigma'_n \#^m) \neq M(\sigma_n, \sigma'_n)$ for some m, then one takes $\sigma_{n+1} = \sigma a_n^m$ and $\sigma'_{n+1} = \sigma'_n \#^m$. Otherwise, if there is an m such that $M(\sigma_n \#^m, \sigma'_n a_n^m) \neq M(\sigma_n, \sigma'_n)$ then one takes $\sigma_{n+1} = \sigma \#^m$ and $\sigma'_{n+1} = \sigma'_n a_n^m$. If there is in both cases no such m, then the construction terminates without giving the desired doubletext. If the construction runs for all n, then M changes its hypothesis infinitely often in contradiction to M being confident. Thus there is an n where the construction terminates. It follows that M converges to the same index on the doubletexts $(\sigma_n a_n^\infty, \sigma'_n \#^\infty)$ and $(\sigma_n \#^\infty, \sigma'_n a_n^\infty)$. Thus M fails to sep-identify at least one of the classes $\{\text{content}(\sigma_n a_n), \text{content}(\sigma'_n)\}$ and $\{\text{content}(\sigma_n), \text{content}(\sigma'_n a_n)\}$. ∎

Proposition 13 *There is a class of finite sets which is Popperian sep-identifiable but not confidently sep-identifiable.*

Proof. The proof is a variant of the proof that the class of finite sets have a Popperian learner but not a confident learner [18, Proposition 4.6.2A]. The class will be a subclass of the class of finite sets and is Popperian sep-identifiable by Remark 5. $\mathcal{C}$ is constructed by induction over x starting with the empty class before stage 0.

In stage x, it is tested whether there are two finite sets L and L' disjoint from each other and the finitely many finite sets already in $\mathcal{C}$ such that M_x does not sep-identify $\{L, L'\}$. If such L and L' exist, then they are put into $\mathcal{C}$. Otherwise, $\mathcal{C}$ remains unchanged. But in that case, it follows from Proposition 12 that M_x is not a confident sep-identifier. ∎

3 Diagonalizing Against Set-Driven Separation

The following technical result is based on a method of Jockusch [12]. Its main objective is to build a partial-recursive function ψ which is total on $U_{x,y} = \{\langle x, y, z\rangle : z \in \mathbb{N}\}$, if W_y is infinite, and which does not have a total recursive extension on $U_{x,y}$, if W_y is finite. In Theorem 15 below, the auxiliary partial-recursive function ψ will be used to define a class which is not sep-identifiable according to a certain criterion.

Proposition 14 *Let $U_{x,y} = \{\langle x, y, z\rangle : z \in \mathbb{N}\}$. There exists a partial-recursive $\{0,1\}$-valued function ψ such that, for all x, y,*

(A) *If W_y is infinite then ψ is total on $U_{x,y}$, that is, $U_{x,y} \subseteq domain(\psi)$;*
(B) *If W_y is finite, then there is no recursive function Ψ which coincides with ψ on $domain(\psi) \cap U_{x,y}$.*

Furthermore, ψ takes on each set $U_{x,y}$ both values 0 and 1.

Proof. The function ψ is defined as

$$\psi(\langle x, y, z\rangle) = \begin{cases} z & \text{if } z \leq 1; \\ 1 - b & \text{if } z > 1 \text{ and the computation} \\ & \text{of } \varphi_z(\langle x, y, z\rangle) \text{ terminates} \\ & \text{with output } b \text{ before } z \text{ elements are} \\ & \text{enumerated into } W_y, \text{ where } b \in \{0,1\}; \\ 1 & \text{if } z > 1 \text{ and the previous case} \\ & \text{does not hold and } W_y \text{ contains} \\ & \text{at least } z \text{ elements;} \\ \uparrow & \text{otherwise;} \end{cases}$$

where the first case is just inserted in order to get that ψ takes both values, 0 and 1.

Verification of (A). If W_y is infinite, then ψ is defined on all $\langle x, y, z\rangle$ since in the case that $\psi(\langle x, y, z\rangle)$ is not defined according the second case, then it is defined according to the third case eventually.

Verification of (B). If W_y is finite and Ψ is a $\{0, 1\}$-valued recursive function, then Ψ has a program z with $z > \text{card}(W_y)$. In particular, $\psi(\langle x, y, z\rangle)$ is defined according to the second case and different from Ψ: $\psi(\langle x, y, z\rangle) = 1 - \varphi_z(\langle x, y, z\rangle) = 1 - \Psi(\langle x, y, z\rangle)$. ∎

Theorem 15 *There is a class $\mathcal{C}$ which is not set-driven sep-identifiable although $\mathcal{C}$ has a confident and conservative recsep-identifier.*

Proof. Let the set $U_{x,y}$, function ψ, and the conditions (A) and (B) satisfied by ψ be as in Proposition 14. Let $M_0, M_1, \ldots$ be the enumeration of partial-recursive functions from Definition 2. Furthermore, let

$$f(u) = \max(\{\varphi_v(w) : v, w \leq u \wedge \varphi_v(w) \text{ is defined}\}).$$

The function f is total and approximable from below by the total recursive sequence f_s with

$$f_s(u) = \max(\{\varphi_v(w) : v, w \leq u \wedge \varphi_v(w) \text{ terminates in up to } s \text{ steps}\}).$$

The class $\mathcal{C}$ is now intended to be defined such that it contains for all total and set-driven M_x disjoint counterexample-sets L_x and L'_x such that M_x fails to sep-identify them. So, whenever M_x is total and set-driven, one searches for y and L_x, L'_x such that (C) holds and, in the case that (C) cannot be satisfied, (D) holds. It will be shown below that it is always possible to satisfy either (C) or (D); from this it follows that $\mathcal{C}$ is not set-driven sep-identifiable. The conditions for y, L_x and L'_x are the following.

(C) W_y is infinite, $L_x = \{u \in U_{x,y} : \psi(u) \downarrow= 0\}$, $L'_x = \{u \in U_{x,y} : \psi(u) \downarrow= 1\}$ and M_x does not sep-identify $\{L_x, L'_x\}$;

(D) W_y is finite, L_x and L'_x are disjoint subsets of $U_{x,y}$, $\langle x, y, 0\rangle \in L_x$, $\langle x, y, 1\rangle \in L'_x$, $\text{card}(L_x) \leq f(y)$, $\text{card}(L'_x) \leq f(y)$ and M_x does not sep-identify $\{L_x, L'_x\}$.

The further parts of the proof do the following.

- A conservative and confident sep-identifier M is constructed. The construction is based on the property that L_x, L'_x either are subsets of $U_{x,y}$ of cardinality below $f(y)$ or are of the form $\{u \in U_{x,y} : \psi(u) \downarrow= b\}$ for $b = 0, 1$ with $U_{x,y} \subseteq \text{domain}(\psi)$.
- It is shown that there is no set-driven sep-identifier for $\mathcal{C}$. This is done by showing that whenever M_x is set-driven and total, then either (C) or (D) applies so that M_x is diagonalized against explicitly.

Construction of M. $M(\sigma, \sigma')$ checks whether there are unique parameters x, y, b such that $x, y \in \mathbb{N}$, $b \in \{0, 1\}$ and $\langle x, y, b\rangle \in \mathrm{content}(\sigma')$. If not, $M(\sigma, \sigma') = ?$. If so, M outputs the hypothesis $\mathrm{e}(\sigma'[n])$ (defined below), for the least n such that (I) $n \leq |\sigma'|$, (II) $\langle x, y, b\rangle \in \mathrm{content}(\sigma'[n])$ and (III) either $\mathrm{card}(\mathrm{content}(\sigma'[n])) > f_{|\sigma'|}(y)$ or no inconsistency between the data (σ, σ') and $\varphi_{e(\sigma'[n])}$ can be found by simulating $\varphi_{e(\sigma'[n])}$ for $|\sigma'|$ many steps.

The program $e(\tau')$ on input u does the following.

1. Search for $\langle x, y, b\rangle$ with $b \in \{0, 1\}$ such that $\langle x, y, b\rangle \in \mathrm{content}(\tau')$. If $\langle x, y, b\rangle$ does not exist or is not unique, then $\varphi_{e(\tau')}(u)$ is undefined.
2. If $u \in \mathrm{content}(\tau')$ then $\varphi_{e(\tau')}(u) = 1$.
3. If $u \notin U_{x,y}$ then $\varphi_{e(\tau')}(u) = 0$.
4. Search for the first $s \geq |\tau'|$ such that either $\mathrm{card}(\mathrm{content}(\tau')) \leq f_s(y)$ or $\psi(u)$ has been computed in up to s computation steps.
5. If s is found in step 4 and $\mathrm{card}(\mathrm{content}(\tau')) \leq f_s(y)$ then $\varphi_{e(\tau')}(u) = 0$.
6. If s is found in step 4, $\mathrm{card}(\mathrm{content}(\tau')) > f_s(y)$ and $b = 0$, then $\varphi_{e(\tau')}(u) = 1 - \psi(u)$.
7. If s is found in step 4, $\mathrm{card}(\mathrm{content}(\tau')) > f_s(y)$ and $b = 1$, then $\varphi_{e(\tau')}(u) = \psi(u)$.
8. Otherwise $\varphi_{e(\tau')}(u)$ is undefined.

M is conservative. The algorithm abandons a hypothesis $e(\tau')$ only if either $\varphi_{e(\tau')}$ is explicitly inconsistent with the data seen so far or it turns out that the data for the second set does not have a unique $\langle x, y, b\rangle$, with $b \in \{0, 1\}$ – but then $\varphi_{e(\tau')}$ is also inconsistent with the data seen so far. So M is conservative.

M is confident. Let (T, T') be any doubletext and let $L' = \mathrm{content}(T')$. Assume that M does not converge to ?. Then there is a unique $\langle x, y, b\rangle \in \mathrm{content}(T')$ with $x, y \in \mathbb{N}$ and $b \in \{0, 1\}$.

If L' has $f(y) + 1$ or more elements, then there is a least n such that $\langle x, y, b\rangle \in \mathrm{content}(T'[n])$ and $\mathrm{card}(\mathrm{content}(T'[n])) > f(y)$. The algorithm of M will never select any $e(T'[m])$ with $m > n$. Furthermore, whenever it abandons an $e(T'[m])$ with $m < n$, it never takes this hypothesis again. So the algorithm converges to an index $e(T'[m])$ with $m \leq n$.

Otherwise L' has at most $f(y)$ many elements. Let n be the first number such that $f_n(y) = f(y)$ and $\mathrm{content}(T'[n]) = L'$. It follows that $\varphi_{e(T'[n])}$ is the characteristic function of L' which is consistent with $(T[m], T'[m])$ for all m. Therefore, $e(T'[n])$ is never abandoned whenever it is taken and M converges to $e(T'[m])$ for an $m \leq n$. It follows from the case distinction that M always converges.

M is a recsep-identifier for $\mathcal{C}$. Let L' be in $\mathcal{C}$, take x, y such that $L' \subseteq U_{x,y}$ and consider any doubletext (T, T') with $L' = \mathrm{content}(T')$. Let n be the number such that $\varphi_{e(T'[n])}$ is the final hypothesis of M on (T, T'). Now it is shown that the algorithm to compute $\varphi_{e(T'[n])}(u)$ is defined for every u and that it is correct.

If $u \notin U_{x,y}$ or $u \in \text{content}(T'[n])$ then the algorithm terminates already in line 2 or 3 and is correct for u. Otherwise it finds an s in line 4 according to one of the following two cases: In the case that $\text{card}(\text{content}(T'[n])) \leq f(y)$, then $\text{card}(\text{content}(T'[n])) \leq f_s(y)$ for an s. Furthermore, if the output for the final hypothesis on input u is wrong, the hypothesis would be revised and not be the last one. Hence the last hypothesis of M is correct at u.

Otherwise $\text{card}(\text{content}(T'[n])) > f(y)$ and $L' = \{u \in U_{x,y} : \psi(u) \downarrow= b\}$ since L' has come into $\mathcal{C}$ by condition (C). So W_y is infinite and $U_{x,y} \subseteq \text{domain}(\psi)$. In particular the computation $\psi(u)$ terminates after some time s. So s is found and $\varphi_{e(T'[n])}(u)$ defined according to one of the lines 6 and 7 and is correct. In particular, $\varphi_{e(T'[n])}$ is the characteristic function of L' and therefore sep-identifies $\{L, L'\}$.

$\mathcal{C}$ is not set-driven sep-identifiable. Consider any total and set-driven M_x and assume that L_x, L'_x cannot be taken according to (C). Now it is shown that they can then be found according to (D). Consider the sets

$$\begin{aligned} U_{x,y,b} &= \{u \in U_{x,y} : \psi(u) \downarrow= b\}, \\ V_x &= \{y : (\exists(\sigma,\sigma') \in \text{SEQ}^2(U_{x,y,0}, U_{x,y,1}))\,(\forall u \in U_{x,y}) \\ &\qquad [\langle x,y,0\rangle \in \text{content}(\sigma) \wedge \langle x,y,1\rangle \in \text{content}(\sigma') \;\wedge \\ &\qquad (\psi(u) \downarrow= 0 \Rightarrow M_x(\sigma u, \sigma'\#) = M_x(\sigma,\sigma'))\;\wedge \\ &\qquad (\psi(u) \downarrow= 1 \Rightarrow M_x(\sigma\#, \sigma' u) = M_x(\sigma,\sigma'))\;\wedge \\ &\qquad (M_x(\sigma u, \sigma'\#) = M_x(\sigma,\sigma') \vee M_x(\sigma\#, \sigma' u) = M_x(\sigma,\sigma'))\,]\,\}. \end{aligned}$$

The set V_x is a Σ_2^0-set as it is defined with an existential quantifier followed by a universal one and the conditions inside are Π_1. Moreover, whenever W_y is infinite, then M_x sep-identifies the class $\{U_{x,y,0}, U_{x,y,1}\}$ and there is a locking-sequence $(\sigma,\sigma') \in \text{SEQ}^2(U_{x,y,0}, U_{x,y,1})$ witnessing this fact. Without loss of generality, $\langle x,y,0\rangle \in \text{content}(\sigma)$ and $\langle x,y,1\rangle \in \text{content}(\sigma')$. Then (σ,σ') also witnesses that $y \in V_x$. Since $\{y : W_y \text{ is infinite}\}$ is not Σ_2^0 and the Σ_2^0-sets are closed under finite variants, there are infinitely many $y \in V_x$ such that W_y is finite.

For every $y \in V_x$ such that W_y is finite, the sets $U_{x,y,0}$ and $U_{x,y,1}$ form a recursively inseparable pair by condition (B) in Proposition 14. In particular the sets $\{u \in U_{x,y} : M_x(\sigma u, \sigma'\#) = M_x(\sigma,\sigma')\}$ and $\{u \in U_{x,y} : M_x(\sigma\#, \sigma' u) = M_x(\sigma,\sigma')\}$ cannot partition $U_{x,y}$ and must have an infinite intersection.

Therefore, the following function is partial-recursive and defined for all $y \in V_x$, where W_y is finite: $\varphi_e(y) = \text{card}(\sigma\sigma' u)$ for the first $\langle \sigma,\sigma',u\rangle$ found such that $(\sigma,\sigma') \in \text{SEQ}^2(U_{x,y,0}, U_{x,y,1})$, $u \in U_{x,y}$, $\langle x,y,0\rangle \in \text{content}(\sigma)$, $\langle x,y,1\rangle \in \text{content}(\sigma')$, $u \notin \text{content}(\sigma\sigma')$ and $M_x(\sigma u, \sigma'\#) = M_x(\sigma\#, \sigma' u)$.

In particular, there is an $y > e$ such that W_y is finite and $\varphi_e(y)$ is defined. It holds that $f(y) \geq \varphi_e(y)$. Since M_x is set-driven, M_x converges on double-texts (T, T') for $\text{content}(\sigma u)$ and $\text{content}(\sigma')$ and (T'', T''') for $\text{content}(\sigma)$ and content($\sigma' u$) to the same index of a partial-recursive function θ. Since u occurs in T and T''', θ has to map u to 0 and 1, respectively. So, M_x fails to sep-identify one of the classes $\{\text{content}(\sigma u), \text{content}(\sigma')\}$ and $\{\text{content}(\sigma), \text{content}(\sigma' u)\}$. This class then satisfies condition (D) and $\mathcal{C}$ contains sets L_x, L'_x witnessing that M_x is not a sep-identifier for $\mathcal{C}$. ∎

4 Diagonalizing Against Conservative Separation

Theorem 16 *There is a class which has a confident and set-driven recsep-identifier but is not conservatively sep-identifiable.*

Proof. Let $O_{x,y} = \{\langle x, y, 2z+1\rangle : z \in \mathbb{N}\}$, $E_{x,y} = \{\langle x, y, 2z\rangle : z \in \mathbb{N}\}$ and $U_{x,y} = O_{x,y} \cup E_{x,y}$. Let $M_0, M_1, \dots$ be an enumeration of total machines never outputting ? such that for every $\mathcal{C}$ which is conservatively sep-identifiable, there exists an M_x which conservatively sep-identifies $\mathcal{C}$. Note that such an enumeration can be easily obtained from the enumeration in Definition 2, using the technique in [10, Proposition 4.15] and the fact that this construction is compatible with conservativeness.

Furthermore, it is easy to adapt Proposition 14 such that it holds with $O_{x,y}$ in place of $U_{x,y}$. Namely, there is a partial-recursive $\{0,1\}$-valued function ψ such that, for all x, y,

(A) if W_y is infinite then ψ is total on $O_{x,y}$, that is, $O_{x,y} \subseteq \text{domain}(\psi)$;
(B) If W_y is finite, then there is no recursive function Ψ which coincides with ψ on $\text{domain}(\psi) \cap O_{x,y}$.

The function ψ takes on each set $O_{x,y}$ both values 0 and 1. Here we assume $\psi(\langle x, y, 2*0+1\rangle) = 0$ and $\psi(\langle x, y, 2*1+1\rangle) = 1$, based on construction given for the proof of Proposition 14.

Construction of $\mathcal{C}$. Let $\text{ConsM} = \{x : (\forall y)\,(\forall \text{ finite and disjoint } L_x, L'_x \subseteq U_{x,y})\,[M_x \text{ is conservative on all } (\sigma, \sigma') \in \text{SEQ}^2(L_x, L'_x)]\}$.

Note that the complement of ConsM is recursively enumerable. We will later construct a recursive f such that for all x and y, $W_{f(x,y)}$ is a recursive subset of $E_{x,y}$. In addition, for all x, we will define L_x and L'_x. We will ensure that, for all x, there exists a y such that following properties are satisfied:

(C) $L_x, L'_x \subseteq U_{x,y}$ and L_x, L'_x are not empty.
(D) M_x is not a conservative sep-identifier for $\{L_x, L'_x\}$.
(E) If $x \in \text{ConsM}$ and $(L_x \cup L'_x) \cap W_{f(x,y)} \neq \emptyset$, then $\text{card}(L_x \cup L'_x) \leq 2 + \min((L_x \cup L'_x) \cap W_{f(x,y)})$ and $(L_x \cup L'_x) \cap W_{f(x,y), \max(L_x \cup L'_x)} \neq \emptyset$.
(F) If $x \in \text{ConsM}$ and $(L_x \cup L'_x) \cap W_{f(x,y)} = \emptyset$, then W_y is infinite, $L_x = (\psi^{-1}(0) \cap O_{x,y}) \cup (E_{x,y} - W_{f(x,y)})$ and $L'_x = \psi^{-1}(1) \cap O_{x,y}$.

(G) If $x \notin \text{ConsM}$, then $L_x = \text{content}(\sigma) \cup \{d\}$ and $L'_x = \text{content}(\sigma')$, where $(\sigma, \sigma') \in \text{SEQ}^2$ is the least pair such that $\text{content}(\sigma)$ and $\text{content}(\sigma')$ are disjoint non-empty subsets of $U_{x,y}$, M_x is not conservative on (σ, σ'), and $d \in O_{x,y}$ is the least number such that x is enumerated into the complement of ConsM within d steps and $d > \max(\text{content}(\sigma) \cup \text{content}(\sigma'))$.

Now let $\mathcal{C} = \{L_x : x \in \mathbb{N}\} \cup \{L'_x : x \in \mathbb{N}\}$.

Intuitively, if M_x is not conservative (on $U_{x,y}$), then one can detect it, and use appropriate diagonalizing L_x, L'_x (see property (G) above). On the other hand, if M_x is conservative, then for an appropriate y, we place elements of $W_{f(x,y)}$ in $L_x \cup L'_x$ to denote whether W_y is finite or infinite (see properties (E), (F) above). These properties, then allow us to construct a confident and set-driven recsep-identifier for $\mathcal{C}$. Moreover, we ensure that M_x is not a conservative sep-identifier for $\{L_x, L'_x\}$, using an appropriate construction.

By (D), $\mathcal{C}$ is not conservatively sep-identifiable. Using (C), (E), (F) and (G) above, we construct the following machine which is a confident and set-driven recsep-identifier for $\mathcal{C}$.

Construction of M(σ, σ').

1. Let $A = \text{content}(\sigma)$ and let $B = \text{content}(\sigma')$.
2. Determine x, y, x', y' such that A and B are non-empty subsets of $U_{x,y}$ and $U_{x',y'}$, respectively.
3. If A or B are empty or x, y, x', y' do not exist then output ?.
4. Else If $(x, y) \neq (x', y')$, then output a program for characteristic function of $U_{x',y'}$.
5. Else If $x \notin \text{ConsM}$ as witnessed within $\max(A \cup B)$ steps, then let
   $(\tau, \tau') \in \text{SEQ}^2$ be the least pair such that $\text{content}(\tau)$ and $\text{content}(\tau')$ are non-empty disjoint subsets of $U_{x,y}$ and M_x is not conservative on (τ, τ') and
   $d \in O_{x,y}$ be the least number such that x is enumerated into the complement of ConsM within d steps and $d > \max(\text{content}(\tau) \cup \text{content}(\tau'))$.
   (* Such τ, τ', d can be effectively found from x, using the fact that $x \notin \text{ConsM}$. *)
   If $A \subseteq \text{content}(\tau)$, then output characteristic function of $\text{content}(\tau')$. Else output characteristic function of $\text{content}(\tau) \cup \{d\}$.
   (* This step was designed to satisfy property (G). *)
6. Else If $(A \cup B) \cap W_{f(x,y),\max(A \cup B)} \neq \emptyset$, then
   If $\text{card}(A \cup B) \leq 2 + \min((A \cup B) \cap W_{f(x,y)})$, output a program for the characteristic function of B.
   Else output ?.
   (* This step was designed to satisfy property (E). *)
7. Else

Let $b = 0$ if $\langle x, y, 1\rangle \in A$ and $b = 1$ otherwise.
Output a program for the (possibly partial) function η defined as:

$$\eta(u) = \begin{cases} 0 & \text{if } u \notin U_{x,y}; \\ b & \text{if } u \in E_{x,y}; \\ \psi(u) & \text{if } x \in O_{x,y} \text{ and } b = 0; \\ 1 - \psi(u) & \text{if } x \in O_{x,y} \text{ and } b = 1. \end{cases}$$

(* This step was designed to satisfy property (F). *)
End.

M is a set-driven and confident recsep-identifierfor $\mathcal{C}$. It follows from the definition that M is set-driven. Suppose that a doubletext (T, T') for L and L' is given to M. We now show that M will converge on (T, T') (and thus M is confident). Furthermore, if L, L' are members of $\mathcal{C}$, then M on (T, T') will converge to a program for a recursive function separating (L, L'). Now consider the first case which applies. So x, y, x', y' exist implicitly in Cases 2, 3, 4 and 5; $(x, y) = (x', y')$ in Cases 3, 4 and 5; $x \in$ ConsM in Cases 4 and 5.

Case 1: There are no unique x, y, x', y' such that $L \subseteq U_{x,y}$ and $L' \subseteq U_{x',y'}$.

In this case, M converges on (T, T') to ? according to step 3. Note that Case 1 also covers the case where L or L' are empty.

Case 2: $(x, y) \neq (x', y')$,

In this case, by step 4, M on (T, T') converges to a program for a recursive function separating L and L'.

Case 3: $x \notin$ ConsM.

In this case, by step 5, clearly M converges on (T, T'). Furthermore, if both L and L' are members of $\mathcal{C}$, then using property (G), M converges to a program for a recursive function separating L and L'.

Case 4: $(L \cup L') \cap W_{f(x,y),\max(L\cup L')} \neq \emptyset$.

In this case, clearly for large enough n, $M(T[n], T'[n])$ will output programs based on step 6. Thus M clearly converges on (T, T'). Furthermore, if L, L' are members of $\mathcal{C}$, then using property (E), we have $\mathrm{card}(L \cup L') \leq 2 + \min((L \cup L') \cap W_{f(x,y)})$, and thus M will converge to a program for recursive function separating L, L'.

Case 5: $(L \cup L') \cap W_{f(x,y),\max(L\cup L')} = \emptyset$.

In this case, for large enough n, $M(T[n], T'[n])$, will output based on step 7. Thus M converges on (T, T'). Furthermore, if L, L' are members of $\mathcal{C}$, then we must have $(L \cup L') \cap W_{f(x,y)} = \emptyset$ by property (E). Now using property (F), we have that W_y is infinite. Thus M converges to a program for η which is total because ψ is total on $O_{x,y}$. The definition of ψ says that $\psi(\langle x, y, 1\rangle) = 0$. So the parameter b is chosen appropriately whenever sufficiently many data-items have been seen and η separates L, L'.

The function f. We now continue with the definition of function f. Intuitively, for each $x \in$ ConsM, we try to fool M_x into making an error (by trying to force infinitely many mind changes) while separating L_x, L'_x, where $L_x = (\psi^{-1}(0) \cap O_{x,y}) \cup (E_{x,y} - W_{f(x,y)})$, and $L'_x = \psi^{-1}(1) \cap O_{x,y}$. We will argue that either we succeed in doing so, for some y with W_y being infinite (and thus we have a diagonalization using property (F)), or we can use property (E) for diagonalization (for this, we will use the fact that $\{y : W_y \text{ is infinite}\}$ is Π_2 complete).

Construction of f. We now define $W_{f(x,y)}$. Note that $W_{f(x,y)}$ will be a subset of $E_{x,y}$ (we will also argue below that $W_{f(x,y)}$ is recursive). Later, we will also define suitable L_x and L'_x, and show that (C) to (G) are satisfied.

Initially let $\sigma_0 = \sigma'_0 = \lambda$. Let $W^s_{f(x,y)}$ denote the set of those elements which are enumerated into $W_{f(x,y)}$ before stage s. Go to stage 0.

Stage s

1. Dovetail steps 2 and 3, until search in one of them succeeds. If search in step 2 succeeds (before the search in step 3), then go to step 4. If search in step 3 succeeds (before the search in step 2), then go to step 5.
2. Search for $z \in E_{x,y}$ such that $z > \max(\text{content}(\sigma_s) \cup \text{content}(\sigma'_s) \cup \{s\})$ and $\varphi_{M_x(\sigma_s,\sigma'_s)}(z)\downarrow = 0$.
3. Search for $(\tau_s, \tau'_s) \in \text{SEQ}^2$ such that the following conditions are satisfied.
   $\sigma_s \subseteq \tau_s$ and $\text{content}(\tau_s) \subseteq (\psi^{-1}(0) \cap O_{x,y}) \cup (E_{x,y} - W^s_{f(x,y)})$.
   $\sigma'_s \subseteq \tau'_s$ and $\text{content}(\tau'_s) \subseteq \psi^{-1}(1) \cap O_{x,y}$.
   $M_x(\sigma_s, \sigma'_s) \neq M_x(\tau_s, \tau'_s)$.
4. Enumerate z into $W_{f(x,y)}$.
   Search for $(\tau_s, \tau'_s) \in \text{SEQ}^2$ such that the following conditions are satisfied.
   $\sigma_s \subseteq \tau_s$ and $\text{content}(\tau_s) \subseteq (\psi^{-1}(0) \cap O_{x,y}) \cup E_{x,y} - (W^s_{f(x,y)} \cup \{z\})$.
   $\sigma'_s \subseteq \tau'_s$ and $\text{content}(\tau'_s) \subseteq \psi^{-1}(1) \cap O_{x,y}$.
   $M_x(\sigma_s, \sigma'_s) \neq M_x(\tau_s, \tau'_s)$.
   If and when such τ_s and τ'_s are found, go to step 5.
5. Let σ_{s+1} be an extension of τ_s and σ'_{s+1} be an extension of τ'_s, such that
   $|\sigma_{s+1}| = |\sigma'_{s+1}|$ and
   $\text{content}(\sigma_{s+1}) = \text{content}(\tau_s) \cup (\psi^{-1}(0) \cap O_{x,y} \cap \{r : r \leq s\}) \cup ([E_{x,y} - (W^s_{f(x,y)} \cup \{z\})] \cap \{r : r < s\})$, and
   $\text{content}(\sigma'_{s+1}) = \text{content}(\tau'_s) \cup (\psi^{-1}(1) \cap O_{x,y} \cap \{r : r \leq s\})$.
   Go to stage $s+1$.

End stage s

Definition of L_x, L'_x and Verification of the properties (C) through (G). Note that either $W_{f(x,y)}$ is finite, or there exist infinitely many stages, and $s \in W_{f(x,y)}$ iff $s \in W^s_{f(x,y)}$ (note that by step 2, we choose z to be larger than s; some of these z may be placed into $W_{f(x,y)}$). Thus $W_{f(x,y)}$ is recursive.

For each $x \in \mathbb{N}$, we now consider the following cases.

Case 1: $x \notin \mathrm{ConsM}$.

In this case, let $(\sigma, \sigma') \in \mathrm{SEQ}^2$ be the least pair such that, for some y, content(σ) and content(σ') are non-empty, disjoint subsets of $U_{x,y}$ and M_x is not conservative on (σ, σ'). Let $L_x = \mathrm{content}(\sigma) \cup \{d\}$ and $L'_x = \mathrm{content}(\sigma')$, where $d \in O_{x,y}$ is the least number such that x is enumerated into the complement of ConsM in less than d steps and d is larger than any element of $\mathrm{content}(\sigma) \cup \mathrm{content}(\sigma')$.

Thus, properties (C), (D) and (G) are satisfied, and (E) and (F) do not apply.

Case 2: $x \in \mathrm{ConsM}$ and there exists a y such that W_y is infinite and M_x is not a sep-identifier for $\{(\psi^{-1}(0) \cap O_{x,y}) \cup E_{x,y} - W_{f(x,y)}, \psi^{-1}(1) \cap O_{x,y}\}$.

In this case, let $L_x = (\psi^{-1}(0) \cap O_{x,y}) \cup E_{x,y} - W_{f(x,y)}$, and $L'_x = \psi^{-1}(1) \cap O_{x,y}$. Now, M_x is not a sep-identifier for (L_x, L'_x).

Thus, (C), (D) and (F) are satisfied, and (E) and (G) do not apply.

Case 3: $x \in \mathrm{ConsM}$ and for all y such that W_y is infinite, M_x is a sep-identifier for $\{(\psi^{-1}(0) \cap O_{x,y}) \cup E_{x,y} - W_{f(x,y)}, \psi^{-1}(1) \cap O_{x,y}\}$.

In the following we will select finite L_x, L'_x with $L_x \cap W_{f(x,y)} \neq \emptyset$, for some y, satisfying conditions (C), (D) and (E).

Now we deal with Case 3 in detail: Let $I_1 = \{y : (\exists s)\,[$ in the construction of $W_{f(x,y)}$, step 4 of stage s is started but does not end $]\}$. Note that, $\{y : W_y$ is infinite$\} \subseteq I_1$. (Reason: For W_y being infinite, M_x is a sep-identifier for $\{(\psi^{-1}(0) \cap O_{x,y}) \cup E_{x,y} - W_{f(x,y)}, \psi^{-1}(1) \cap O_{x,y}\}$, by hypothesis of the case. Thus there are only finitely many stages in the construction, and for every stage entered, step 2 or step 3 must succeed).

Furthermore, I_1 is recursively enumerable relative to the oracle K. Thus, for every $y \in I_1$ one can find s, z and σ_s, σ'_s (depending on y) using the oracle K, where in the definition of $W_{f(x,y)}$, s is the stage in which step 4 is started but does not end, and z is as defined in step 4 of stage s.

Using the oracle K, one can also test whether the following two conditions hold:

(P1) $M_x(\sigma_s z d^n, \sigma'_s \#^{n+1}) = M_x(\sigma_s, \sigma'_s)$, for all $d \in \psi^{-1}(0) \cap O_{x,y}$ and all n;
(P2) $M_x(\sigma_s z \#^n, \sigma'_s d^{n+1}) = M_x(\sigma_s, \sigma'_s)$, for all $d \in \psi^{-1}(1) \cap O_{x,y}$ and all n.

Let $I_2 = \{y \in I_1 : $ (P1) and (P2) are satisfied$\}$. Note that I_2 is recursively enumerable relative to the oracle K. Note that, if W_y is infinite and M_x is a conservative sep-identifier for $\{(\psi^{-1}(0)\cap O_{x,y})\cup E_{x,y}-W_{f(x,y)}, \psi^{-1}(1)\cap O_{x,y}\}$, then $\varphi^{-1}_{M_x(\sigma_s,\sigma'_s)}(0) \supseteq (\psi^{-1}(0)\cap O_{x,y})\cup\{z\}$ and $\varphi^{-1}_{M_x(\sigma_s,\sigma'_s)}(1) \supseteq \psi^{-1}(1)\cap O_{x,y}$ (here $\varphi_{M_x(\sigma_s,\sigma'_s)}(z)=0$, since $y\in I_1$, and thus step 2 had succeeded in stage s). Thus, y must satisfy (P1) and (P2). Thus, $I_2 \supseteq \{y : W_y \text{ is infinite}\}$. Since I_2 is recursively enumerable relative to oracle K, and $\{y : W_y \text{ is infinite}\}$ is Π_2-complete, there must exist a y such that W_y is finite, and $y\in I_2$. For the following, fix such a y, and corresponding s, z, σ_s and σ'_s, where s is the stage in which step 4 of $W_{f(x,y)}$ starts but does not finish, and z is as defined in step 4 of stage s. Let $A=\varphi^{-1}_{M_x(\sigma_s,\sigma'_s)}(0)$, and $B=\varphi^{-1}_{M_x(\sigma_s,\sigma'_s)}(1)$.

Case 3.1: At least one of the sets $(\psi^{-1}(0)\cap O_{x,y})-A$ and $(\psi^{-1}(1)\cap O_{x,y})-B$ is infinite.

If $\text{card}((\psi^{-1}(0)\cap O_{x,y})-A)=\infty$, then let $d\in(\psi^{-1}(0)\cap O_{x,y})-A$ be such that $z\in W_{f(x,y),d}$. Now, M_x is not a sep-identifier for $\{\text{content}(\sigma_s)\cup\{z,d\}, \text{content}(\sigma'_s)\}$, since by property (P1), $M_x(\sigma_s z d^n, \sigma'_s\#^{n+1}) = M_x(\sigma_s,\sigma'_s)$, for all n, and $\varphi_{M_x(\sigma_s,\sigma'_s)}$ does not separate $(\text{content}(\sigma_s)\cup\{z,d\})$ and $\text{content}(\sigma'_s)$. Thus, we define $L_x=\text{content}(\sigma_s)\cup\{z,d\}$ and $L'_x=\text{content}(\sigma'_s)$. Note that $\text{card}(L_x\cup L'_x)\le 2+\text{card}(\text{content}(\sigma_s)\cup\text{content}(\sigma'_s))\le 2+z$, z is the only element of $(L_x\cup L'_x)\cap W_{f(x,y)}$, and $z\in L_x\cap W_{f(x,y),\max(L_x\cup L'_x)}$.

Similarly, if $\text{card}((\psi^{-1}(1)\cap O_{x,y})-B)=\infty$, then we can reason as above by taking $d\in(\psi^{-1}(1)\cap O_{x,y})-B$, $L_x=\text{content}(\sigma_s)\cup\{z\}$ and $L'_x=\text{content}(\sigma'_s)\cup\{d\}$, and using (P2) instead of (P1).

Thus, properties (C), (D) and (E) are satisfied, and (F) and (G) do not apply.

Case 3.2: $\text{content}(\sigma_s)\not\subseteq A$ or $\text{content}(\sigma'_s)\not\subseteq B$.

Let $d\in(\psi^{-1}(0)\cap O_{x,y})-\text{content}(\sigma_s)$ be such that $z\in W_{f(x,y),d}$. Let $L_x=\text{content}(\sigma_s)\cup\{z,d\}$, $L'_x=\text{content}(\sigma'_s)$. Now, M_x is not a sep-identifier for (L_x,L'_x), since by property (P1), $M_x(\sigma_s z d^n,\sigma'_s\#^{n+1})=M_x(\sigma_s,\sigma'_s)$, for all n, and $\varphi_{M_x(\sigma_s,\sigma'_s)}$ does not separate L_x, L'_x.

Thus, properties (C), (D) and (E) are satisfied and (F) and (G) do not apply.

Case 3.3: $\text{content}(\sigma_s)\subseteq A$, $\text{content}(\sigma'_s)\subseteq B$ and the two sets $(\psi^{-1}(0)\cap O_{x,y})-A$ and $(\psi^{-1}(1)\cap O_{x,y})-B$ are both finite.

Since by condition (B) no total function coincides with ψ on $O_{x,y}$, we must have that $A\cap O_{x,y}$ and $B\cap O_{x,y}$ are not recursive. Since $x\in\text{ConsM}$, the set

$$
\begin{aligned}
C \;=\{d\in O_{x,y} \;:\; (\exists n,m)\,&[M_x(\sigma_s z d^n,\sigma'_s\#^{n+1})\neq M_x(\sigma_s,\sigma'_s)]\\
&\wedge\;[M_x(\sigma_s z\#^m,\sigma'_s d^{m+1})\neq M_x(\sigma_s,\sigma'_s)]\}
\end{aligned}
$$

is disjoint from A and B. However, $\text{card}(O_{x,y}-(A\cup B\cup C))=\infty$, due to non-recursiveness of $A\cap O_{x,y}$ and $B\cap O_{x,y}$. Thus, there exists a $d\in O_{x,y}-(A\cup B\cup C)$, such that $z\in W_{f(x,y),d}$. If for all n, $M_x(\sigma_s z d^n,\sigma'_s\#^{n+1})=M_x(\sigma_s,\sigma'_s)$, then let $L_x = \mathrm{content}(\sigma_s) \cup \{z, d\}$, $L'_x = \mathrm{content}(\sigma'_s)$. Otherwise, for all n, $M_x(\sigma_s z \#^n, \sigma'_s d^{n+1}) = M_x(\sigma_s, \sigma'_s)$ — in this case let $L_x = \mathrm{content}(\sigma_s) \cup \{z\}$, $L'_x = \mathrm{content}(\sigma'_s) \cup \{d\}$.

Now, M_x is not a sep-identifier for (L_x, L'_x), since $\varphi_{M_x(\sigma_s, \sigma'_s)}$ does not separate L_x and L'_x.

Thus, properties (C), (D) and (E) are satisfied, and (F) and (G) do not apply.

From the above cases 1, 2, 3.1, 3.2 and 3.3, we have that (C) to (G) are satisfied. This completes the proof of the theorem. ∎

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