Language Learning from Texts: Degrees of Intrinsic Complexity and Their Characterizations*

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Abstract

This paper deals with two problems: 1) what makes languages to be learnable in the limit by natural strategies of varying hardness; 2) what makes classes of languages to be the hardest ones to learn. To quantify hardness of learning, we use *intrinsic complexity* based on reductions between learning problems. Two types of reductions are considered: weak reductions mapping texts (representations of languages) to texts, and strong reductions mapping languages to languages. For both types of reductions, characterizations of complete (hardest) classes in terms of their algorithmic and topological potentials have been obtained. To characterize the strong complete degree, we discovered a new and natural complete class capable of "coding" any learning problem using density of the set of rational numbers. We have also discovered and characterized rich hierarchies of degrees of complexity based on "core" natural learning problems. The classes in these hierarchies contain "multidimensional" languages, where the information learned from one dimension aids to learn other dimensions. In one formalization of this idea, the grammars learned from the dimensions $1, 2, \ldots, k$ specify the "subspace" for the dimension $k + 1$, while the learning strategy for every dimension is predefined. In our other formalization, a "pattern" learned from the dimension k specifies the learning strategy for the dimension $k + 1$. A number of open problems is discussed.

1 Introduction

There are two major objectives our paper attempts to achieve:

a) to discover what makes languages to be learnable in the limit by natural strategies of varying hardness;

b) to discover what makes classes of languages to be the hardest ones to learn.

The theory of learning languages in the limit, which has been quite advanced over the last three decades, suggests several ways to quantify hardness (complexity) of learning. The most popular among them are:

a) counting the number of *mind changes* [BF72, CS83, LZ93] the learner makes before arriving to the final hypothesis;

b) measuring the amount of (so-called *long-term*) memory the learner uses [Kin94, KS95];

c) reductions between different learning problems (classes of languages) and respective degrees of so-called *intrinsic* complexity [FKS95, JS96, JS97].

There have been several other notions of complexity of learning considered in the literature (for example see [Gol67, DS86, Wie86]).

The first two approaches above reveal quite interesting complexity hierarchies among learnable classes of languages ([CS83, LZ93, KS95]). However, a large number of interesting and very different natural classes of learnable classes falls into the category that requires more than uniformly bounded finite number of mind changes, as well as maximum (linear) amount of long-term memory. As it is demonstrated in our paper, intrinsic complexity of language learning, based on the idea of reductions, is perfectly suitable for quantifying hardness of many such natural classes of languages. It can be also successfully utilized to characterize the whole degrees of learnability based on these natural classes.

There are two different approaches to formalizing the concept of intrinsic complexity based on reductions between classes of languages [JS96]. In general terms, a major part of any reduction of one learning problem to another one is a mapping (an operator) that maps a language of the first learning problem to a language of the second one. A language is usually presented to a learner in form of a *text*, an infinite sequence of all elements of the language (possibly, with repetitions). Any non-empty language can be represented by many different texts. If a reduction may translate different texts of the *same* language to texts of *different* languages, we call such a reduction *weak*. If a reduction is required to translate all texts of the input language to texts of the *same* language, we call such a reduction *strong*. Roughly, a weak reduction translates texts to texts, while a strong reduction translates languages to languages. The paper [JS96] reveals significant differences between degrees of intrinsic complexity based on weak and, respectively, strong reductions.

For both types of reductions, we have obtained characterizations of complete degrees in terms of their algorithmic and topological potentials. For the case of strong reductions, we discovered a new natural complete class capable of "coding " (in the limit) any learning problem using density of the set of rational numbers. For weak reducibility, we were able to use the fact that the complete degree contains the class *FINITE* of all finite sets. The characterization for the weak complete degree is very different from any other characterization obtained in the paper - it is based on a requirement of density in terms of Baire topology. Note that a characterization of the complete degree of intrinsic complexity for *function* learning formulated in similar terms was obtained in [KPSW99]. The main difference between our characterization of weak complete degrees and the characterization for function learning in [KPSW99] is the requirement of *standardizability* (see Definition 5) for the hardest classes of languages. This notion, introduced quite long time ago in [Kin75, Fre91, JS94], for different purposes, turned out to be surprisingly useful for the characterization of all degrees in our paper.

For both types of reductions, we have also discovered and characterized rich structures of classes of languages, each of which requires its own specific type of learning strategy. Languages in these classes can be represented in "multidimensional" form, where the information obtained from learning one "dimension" aids in learning other "dimensions". We suggest and discuss several possibilities to formalize such "aid" and the ways it can be used. In the given paper, we concentrate on two following formalizations:

a) the grammars learned from the "dimensions" $L_1, L_2, \ldots, L_k$ specify the "subspace" containing the "sublanguage" L_{k+1};

b) the grammar learned from the "dimension" L_k codes a "pattern" that specifies a learning strategy for the class of languages containing L_{k+1}.

For the first formalization, we have obtained the complete picture of degrees of complexity for the classes of "multidimensional" languages based on combinations of probably the most important known natural classes of learnable languages: *INIT*, *COINIT*, *SINGLE*, *COSINGLE* (see Definition 6). Classes that can be defined under the second formalization turn out to be very complex. Yet we have shown that all of them are incomplete. The general problem whether such classes form a complexity hierarchy remains open.

In short, our major accomplishments are:

1) discovery of the fact that any language learning problem can be algorithmically coded using sets $\{x \mid 0 \leq x \leq r\}$ of *rational* numbers;

2) characterizations of hardest learning problems in terms of their topological and algorithmic potentials;

3) discovery of a complex hierarchy of degrees of "multidimensional" languages; being interesting in its own right, this hierarchy can be used as a *scale* for quantifying hardness of learning complex concepts (for instance, it has been applied to quantify hardness of learning complex geometrical concepts in [JK99]).

2 Notation and Preliminaries

Any unexplained recursion theoretic notation is from [Rog67]. The symbol N denotes the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$. Symbols $\emptyset$, $\subseteq$, $\subset$, $\supseteq$, and $\supset$ denote empty set, subset, proper subset, superset, and proper superset, respectively. $D_0, D_1, \ldots,$ denotes a canonical recursive indexing of all the finite sets [Rog67, Page 70]. We assume that if $D_i \subseteq D_j$ then $i \leq j$ (the canonical indexing defined in [Rog67] satisfies this property). Cardinality of a set S is denoted by $\mathrm{card}(S)$. The maximum and minimum of a set are denoted by $\max(\cdot), \min(\cdot)$, respectively, where $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$. $L_1 \mathbf{\Delta} L_2$ denotes the symmetric difference of L_1 and L_2, that is $L_1 \mathbf{\Delta} L_2 = (L_1 - L_2) \cup (L_2 - L_1)$. For a natural number a, we say that $L_1 =^a L_2$, iff $\mathrm{card}(L_1 \mathbf{\Delta} L_2) \leq a$. We say that $L_1 =^* L_2$, iff $\mathrm{card}(L_1 \mathbf{\Delta} L_2) < \infty$. Thus, we take $n < * < \infty$, for all $n \in N$. If $L_1 =^a L_2$, then we say that L_1 is an a-variant of L_2.

We let $\langle \cdot, \cdot \rangle$ stand for an arbitrary, computable, bijective mapping from $N \times N$ onto N [Rog67]. We assume without loss of generality that $\langle \cdot, \cdot \rangle$ is monotonically increasing in both of its arguments. We define $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$. $\langle \cdot, \cdot \rangle$ can be extended to n-tuples in a natural way (including $n = 1$, where $\langle x \rangle$ may be taken to be x). Projection functions $\pi_1, \ldots, \pi_n$ corresponding to n-tuples can be defined similarly (where the tuple size would be clear from context). Due to the above isomorphism between N^k and N, we often identify the tuple $(x_1, \cdots, x_n)$ with $\langle x_1, \cdots, x_n \rangle$.

By φ we denote a fixed *acceptable* programming system for the partial computable functions mapping N to N [Rog67, MY78]. By φ_i we denote the partial computable function computed by the program with number i in the φ-system. Symbol $\mathcal{R}$ denotes the set of all recursive functions, that is total computable functions. By Φ we denote an arbitrary fixed Blum complexity measure

[Blu67, HU79] for the φ-system. A partial recursive function $\Phi(\cdot,\cdot)$ is said to be a Blum complexity measure for φ, iff the following two conditions are satisfied:

(a) for all i and x, $\Phi(i,x)\downarrow$ iff $\varphi_i(x)\downarrow$.

(b) the predicate: $P(i,x,t) \equiv \Phi(i,x) \leq t$ is decidable.

By convention we use Φ_i to denote the partial recursive function $\lambda x.\Phi(i,x)$. Intuitively, $\Phi_i(x)$ may be thought as the number of steps it takes to compute $\varphi_i(x)$.

By W_i we denote domain(φ_i). W_i is, then, the r.e. set/language ($\subseteq N$) accepted (or equivalently, generated) by the φ-program i. We also say that i is a grammar for W_i. Symbol $\mathcal{E}$ will denote the set of all r.e. languages. Symbol L, with or without decorations, ranges over $\mathcal{E}$. By $\overline{L}$, we denote the complement of L, that is $N - L$. Symbol $\mathcal{L}$, with or without decorations, ranges over subsets of $\mathcal{E}$. By $W_{i,s}$ we denote the set $\{x < s \mid \Phi_i(x) < s\}$.

A class $\mathcal{L} \subseteq \mathcal{E}$ is said to be recursively enumerable (r.e.) [Rog67], iff $\mathcal{L} = \emptyset$ or there exists a recursive function f such that $\mathcal{L} = \{W_{f(i)} \mid i \in N\}$. In this latter case we say that $W_{f(0)}, W_{f(1)}, \ldots$ is a recursive enumeration of $\mathcal{L}$. $\mathcal{L}$ is said to be 1–1 enumerable iff (i) $\mathcal{L}$ is finite or (ii) there exists a recursive function f such that $\mathcal{L} = \{W_{f(i)} \mid i \in N\}$ and $W_{f(i)} \neq W_{f(j)}$, if $i \neq j$. In this latter case we say that $W_{f(0)}, W_{f(1)}, \ldots$ is a 1–1 recursive enumeration of $\mathcal{L}$.

A partial function F from N to N is said to be partial limit recursive, iff there exists a recursive function f from $N \times N$ to N such that for all x, $F(x) = \lim_{y\to\infty} f(x,y)$. Here if $F(x)$ is not defined then $\lim_{y\to\infty} f(x,y)$, must also be undefined. A partial limit recursive function F is called (total) limit recursive function, if F is total. For example, the characteristic function of any recursively enumerable, non-recursive set is, by definition, not a recursive function, but this function is clearly limit recursive. $\downarrow$ denotes defined or converges. $\uparrow$ denotes undefined or diverges.

We now present concepts from language learning theory. The next definition introduces the concept of a *sequence* of data.

Definition 1 (a) A *sequence* σ is a mapping from an initial segment of N into $(N \cup \{\#\})$. The empty sequence is denoted by Λ.

(b) The *content* of a sequence σ, denoted content(σ), is the set of natural numbers in the range of σ.

(c) The *length* of σ, denoted by $|\sigma|$, is the number of elements in σ. So, $|\Lambda| = 0$.

(d) For $n \leq |\sigma|$, the initial sequence of σ of length n is denoted by $\sigma[n]$. So, $\sigma[0]$ is Λ.

Intuitively, #'s represent pauses in the presentation of data. We let σ, τ, and γ, with or without decorations, range over finite sequences. We denote the sequence formed by the concatenation of τ at the end of σ by $\sigma \diamond \tau$. Sometimes we abuse the notation and use $\sigma \diamond x$ to denote the concatenation of sequence σ and the sequence of length 1 which contains the element x. SEQ denotes the set of all finite sequences.

Definition 2 [Gol67] (a) A *text* T for a language L is a mapping from N into $(N \cup \{\#\})$ such that L is the set of natural numbers in the range of T.

(b) The *content* of a text T, denoted by content(T), is the set of natural numbers in the range of T; that is, the language which T is a text for.

(c) $T[n]$ denotes the finite initial sequence of T with length n.

We let T, with or without decorations, range over texts. We let $\mathcal{T}$ range over sets of texts.

A class $\mathcal{T}$ of texts is said to be r.e. iff there exists a recursive function f, and a sequence $T_0, T_1, \ldots$ of texts such that $\mathcal{T} = \{T_i \mid i \in N\}$, and, for all i, x, $T_i(x) = f(i,x)$.

Definition 3 A *language learning machine* [Gol67] is an algorithmic device which computes a mapping from SEQ into N.

We let $\mathbf{M}$, with or without decorations, range over learning machines. $\mathbf{M}(T[n])$ is interpreted as the grammar (index for an accepting program) conjectured by the learning machine $\mathbf{M}$ on the initial sequence $T[n]$. We say that $\mathbf{M}$ converges on T to i, (written $\mathbf{M}(T)\downarrow = i$) iff $(\stackrel{\infty}{\forall} n)[\mathbf{M}(T[n]) = i]$.

There are several criteria for a learning machine to be successful on a language. Below we define identification in the limit introduced by Gold [Gol67].

Definition 4 [Gol67, CS83] Suppose $a \in N \cup \{*\}$.

(a) $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies a text T just in case $(\exists i \mid W_i =^a \text{content}(T))$ $(\stackrel{\infty}{\forall} n)[\mathbf{M}(T[n]) = i]$.

(b) $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies an r.e. language L (written: $L \in \mathbf{TxtEx}^a(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies each text for L.

(c) $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies a class $\mathcal{L}$ of r.e. languages (written: $\mathcal{L} \subseteq \mathbf{TxtEx}^a(\mathbf{M})$) just in case $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies each language from $\mathcal{L}$.

(d) $\mathbf{TxtEx}^a = \{\mathcal{L} \subseteq \mathcal{E} \mid (\exists \mathbf{M})[\mathcal{L} \subseteq \mathbf{TxtEx}^a(\mathbf{M})]\}$.

For $a = 0$, we often write $\mathbf{TxtEx}$ instead of $\mathbf{TxtEx}^0$.

Other criteria of success are finite identification [Gol67], behaviorally correct identification [Fel72, OW82, CL82], and vacillatory identification [OW82, Cas88]. In the present paper, we only discuss results about $\mathbf{TxtEx}^a$-identification.

The following definition is a generalization of the definition of limiting standardizability considered in [Kin75, Fre91, JS94].

Definition 5 Let $a \in N \cup \{*\}$. A class $\mathcal{L}$ of recursively enumerable sets is called *a-limiting standardizable* iff there exists a partial limiting recursive function F such that

(a) For all i such that $W_i =^a L$ for some $L \in \mathcal{L}$, $F(i)$ is defined.

(b) For all $L, L' \in \mathcal{L}$, for all i, j such that $W_i =^a L$ and $W_j =^a L'$,

$$F(i) = F(j) \Leftrightarrow L = L'.$$

[Kin75, Fre91, JS94] $\mathcal{L}$ is called *limiting standardizable* iff $\mathcal{L}$ is 0-limiting standardizable.

Thus, informally, a class $\mathcal{L}$ of r.e. languages is limiting standardizable if all the infinitely many grammers $i \in N$ of each language $L \in \mathcal{L}$ can be mapped ("standardized") in the limit to some unique grammar (natural number). Notice that it is not required that this "standard grammar" must be a grammar of L again. However, standard grammars for *different* languages from $\mathcal{L}$ have to be pairwise different.

The following basic classes of languages will be used frequently in the following.

Definition 6 $\mathit{SINGLE} = \{L \mid (\exists i)[L = \{i\}]\}$.

$\mathit{COSINGLE} = \{L \mid (\exists i)[L = N - \{i\}]\}$.

$\mathit{INIT} = \{L \mid (\exists i)[L = \{x \mid x \leq i\}]\}$.

$\mathit{COINIT} = \{L \mid (\exists i)[L = \{x \mid x \geq i\}]\}$.

$\mathit{FINITE} = \{L \mid L \textit{ is a finite subset of } N\}$.

3 Weak and Strong Reductions

We first present some technical machinery.

We write $\sigma \subseteq \tau$ if σ is an initial segment of τ, and $\sigma \subset \tau$ if σ is a proper initial segment of τ. Likewise, we write $\sigma \subset T$ if σ is an initial finite sequence of text T. Let finite sequences σ^0, σ^1, $\sigma^2, \ldots$ be given such that $\sigma^0 \subseteq \sigma^1 \subseteq \sigma^2 \subseteq \cdots$ and $\lim_{i\to\infty} |\sigma^i| = \infty$. Then there is a unique text T such that for all $n \in N$, $\sigma^n = T[|\sigma^n|]$. This text is denoted by $\bigcup_n \sigma^n$. Let $\mathbf{T}$ denote the set of all texts, that is, the set of all infinite sequences over $N \cup \{\#\}$.

We define an *enumeration operator* (or just operator), Θ, to be an algorithmic mapping from SEQ into SEQ such that for all σ, $\tau \in$ SEQ, if $\sigma \subseteq \tau$, then $\Theta(\sigma) \subseteq \Theta(\tau)$. We further assume that for all texts T, $\lim_{n\to\infty} |\Theta(T[n])| = \infty$. By extension, we think of Θ as also defining a mapping from $\mathbf{T}$ into $\mathbf{T}$ such that $\Theta(T) = \bigcup_n \Theta(T[n])$.

A final notation about the operator Θ. If for a language L, there exists an L' such that for each text T for L, $\Theta(T)$ is a text for L', then we write $\Theta(L) = L'$, else we say that $\Theta(L)$ is undefined. The reader should note the overloading of this notation because the type of the argument to Θ could be a sequence, a text, or a language; it will be clear from the context which usage is intended.

We let $\Theta(\mathcal{T}) = \{\Theta(T) \mid T \in \mathcal{T}\}$, and $\Theta(\mathcal{L}) = \{\Theta(L) \mid L \in \mathcal{L}\}$.

We also need the notion of an infinite sequence of grammars. We let α, with or without decorations, range over infinite sequences of grammars. From the discussion in the previous section it is clear that infinite sequences of grammars are essentially infinite sequences over N. Hence, we adopt the machinery defined for sequences and texts over to finite sequences of grammars and infinite sequences of grammars. So, if $\alpha = i_0, i_1, i_2, i_3, \ldots$, then $\alpha[3]$ denotes the sequence i_0, i_1, i_2, and $\alpha(3)$ is i_3. Furthermore, we say that α converges to i if there exists an n such that, for all $n' \geq n$, $i_{n'} = i$.

We say that an infinite sequence α of grammars is $\mathbf{TxtEx}^a$-*admissible* for text T just in case α witnesses $\mathbf{TxtEx}^a$-identification of text T. So, if $\alpha = i_0,\ i_1,\ i_2, \ldots$ is a $\mathbf{TxtEx}^a$-admissible sequence for T, then α converges to some i such that $W_i =^a \text{content}(T)$; that is, the limit i of the sequence α is a grammar for an a-variant of the language content(T).

We now formally introduce our reductions.

Definition 7 [JS96] Let $a \in N \cup \{*\}$. Let $\mathcal{L}_1 \subseteq \mathcal{E}$ and $\mathcal{L}_2 \subseteq \mathcal{E}$ be given. Let $\mathcal{T}_1 = \{T \mid T \text{ is a text for } L \in \mathcal{L}_1\}$. Let $\mathcal{T}_2 = \{T \mid T \text{ is a text for } L \in \mathcal{L}_2\}$. We say that $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$ just in case there exist operators Θ and Ψ such that for all $T \in \mathcal{T}_1$ and for all infinite sequences α of grammars the following hold:

(a) $\Theta(T) \in \mathcal{T}_2$ and

(b) if α is a $\mathbf{TxtEx}^a$-admissible sequence for $\Theta(T)$, then $\Psi(\alpha)$ is a $\mathbf{TxtEx}^a$-admissible sequence for T.

We say that $\mathcal{L}_1 \equiv^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$ iff $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$ and $\mathcal{L}_2 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_1$.

Intuitively, $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$ just in case there exists an operator Θ that transforms texts for languages in $\mathcal{L}_1$ into texts for languages in $\mathcal{L}_2$ and there exists another operator Ψ that behaves as follows: if Θ transforms text T (for a language in $\mathcal{L}_1$) to text T' (for a language in $\mathcal{L}_2$), then Ψ transforms $\mathbf{TxtEx}^a$-admissible sequences for T' into $\mathbf{TxtEx}^a$-admissible sequences for T. Thus, informally, the operator Ψ has "to work" only on $\mathbf{TxtEx}^a$-admissible sequences for such texts T'. In other words, if α is a sequence of grammars which is *not* $\mathbf{TxtEx}^a$-admissible for any text T' in $\{\Theta(T) \mid \text{content}(T) \in \mathcal{L}_1\}$, then $\Psi(\alpha)$ can be defined *arbitrarily.* This property will be used implicitly at all places below where we have to define operators Ψ witnessing (together with operators Θ) some reducibility. Note that this approach both simplifies the corresponding definitions and preserves the computability of the so defined operators.

If $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$ then, intuitively, the problem of $\mathbf{TxtEx}^a$-identifying $\mathcal{L}_2$ is at least as hard as the problem of $\mathbf{TxtEx}^a$-identifying $\mathcal{L}_1$, since the solvability of the former problem implies the solvability of the latter one. That is, given any machine $\mathbf{M}_2$ which $\mathbf{TxtEx}^a$-identifies $\mathcal{L}_2$, it is easy to construct a machine $\mathbf{M}_1$ which $\mathbf{TxtEx}^a$-identifies $\mathcal{L}_1$. To see this suppose Θ and Ψ witness $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$. $\mathbf{M}_1(T)$, for a text T is defined as follows. Let $p_n = \mathbf{M}_2(\Theta(T)[n])$, and $\alpha = p_0, p_1, \ldots$. Let $\alpha' = \Psi(\alpha) = p'_0, p'_1, \ldots$. Then let $\mathbf{M}_1(T) = \lim_{n\to\infty} p'_n$. Consequently, $\mathcal{L}_2$ may be considered as a "hardest" problem for $\mathbf{TxtEx}^a$-identification if for *all* classes $\mathcal{L}_1 \in \mathbf{TxtEx}^a$, $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$ holds. If $\mathcal{L}_2$ itself belongs to $\mathbf{TxtEx}^a$, then $\mathcal{L}_2$ is said to be *complete*. We now formally define these notions of hardness and completeness for the above reduction.

Definition 8 [JS96] Let $a \in N \cup \{*\}$. Let $\mathcal{L} \subseteq \mathcal{E}$ be given.
(a) If for all $\mathcal{L}' \in \mathbf{TxtEx}^a$, $\mathcal{L}' \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}$, then $\mathcal{L}$ is $\leq^{\mathbf{TxtEx}^a}_{\text{weak}}$-*hard*.
(b) If $\mathcal{L}$ is $\leq^{\mathbf{TxtEx}^a}_{\text{weak}}$-hard and $\mathcal{L} \in \mathbf{TxtEx}^a$, then $\mathcal{L}$ is $\leq^{\mathbf{TxtEx}^a}_{\text{weak}}$-*complete*.

It should be noted that if $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$ by operators Θ and Ψ, then there is no requirement that Θ maps all texts for each language in $\mathcal{L}_1$ into texts for a *unique* language in $\mathcal{L}_2$. If we further place such a constraint on Θ, we get the following stronger notion.

Definition 9 [JS96] Suppose $a \in N \cup \{*\}$. Let $\mathcal{L}_1 \subseteq \mathcal{E}$ and $\mathcal{L}_2 \subseteq \mathcal{E}$ be given. We say that $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}_2$ just in case there exist operators Θ, Ψ witnessing that $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}_2$, and for all $L_1 \in \mathcal{L}_1$, there exists an $L_2 \in \mathcal{L}_2$, such that ($\forall$ texts T for L_1)[$\Theta(T)$ is a text for L_2].

We say that $\mathcal{L}_1 \equiv^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}_2$ iff $\mathcal{L}_1 \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}_2$ and $\mathcal{L}_2 \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}_1$.

We can similarly define $\leq^{\mathbf{TxtEx}^a}_{\text{strong}}$-hardness and $\leq^{\mathbf{TxtEx}^a}_{\text{strong}}$-completeness.

Proposition 1 *([JS96])* $\leq^{\mathbf{TxtEx}^a}_{\text{weak}}$, $\leq^{\mathbf{TxtEx}^a}_{\text{strong}}$ *are reflexive and transitive.*

The above proposition holds for most natural learning criteria. It is also easy to verify the next proposition stating that strong reducibility implies weak reducibility.

Proposition 2 *[JS96] Suppose $a \in N \cup \{*\}$. Let $\mathcal{L} \subseteq \mathcal{E}$ and $\mathcal{L}' \subseteq \mathcal{E}$ be given. Then $\mathcal{L} \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}' \Rightarrow \mathcal{L} \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}'$.*

Proposition 3 *(based on [JS97]) Suppose $a \in N \cup \{*\}$. Suppose $\mathcal{L} \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}'$, via Θ and Ψ. Then, for all $L, L' \in \mathcal{L}$, $L \subseteq L' \Rightarrow \Theta(L) \subseteq \Theta(L')$.*

We will be using Proposition 3 implicitly when we are dealing with strong reductions. Since, for $\mathcal{L} \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}'$ via Θ and Ψ, for all $L \in \mathcal{L}$, $\Theta(L)$ is defined (= some $L' \in \mathcal{L}'$), when considering strong reductions, we often consider Θ as mapping sets to sets instead of mapping sequences to sequences. This is clearly without loss of generality, as one can easily convert such Θ to Θ as in Definition 9 of strong-reduction.

4 A Natural Strongly Complete Class and a Characterization of Strongly Complete Classes

In this section we exhibit a natural class which is $\leq^{\mathbf{TxtEx}^a}_{\text{strong}}$-complete for all $a \in N$ (see Theorem 2). Corollary 1 to Theorem 2 then shows an even simpler class, $\mathit{RINIT}_{0,1}$ defined below, as $\leq^{\mathbf{TxtEx}}_{\text{strong}}$-complete. We also characterize the $\leq^{\mathbf{TxtEx}^a}_{\text{strong}}$-complete degree, for all $a \in N$, in Theorem 3.

Let $\mathbf{rat}$ denote the set of all non-negative rational numbers. For $s, r \in \mathbf{rat}$, let $\mathbf{rat}_{s,r} = \{x \in \mathbf{rat} \mid s \leq x \leq r\}$. For allowing us to consider r.e. sets of rational numbers, let coderat$(\cdot)$ denote an effective bijective mapping from $\mathbf{rat}$ to N.

Definition 10 Suppose $r \in \mathbf{rat}_{0,1}$.
Let $X_r = \{\text{coderat}(x) \mid x \in \mathbf{rat} \text{ and } 0 \leq x \leq r\}$.
Let $X^{cyl}_r = \{\text{coderat}(2w + x) \mid x \in \mathbf{rat},\ w \in N \text{ and } 0 \leq x \leq r\}$.

Notice that the factor 2 in the definition of X_r^{cyl} is used for technical reasons only – since 1 belongs to both $\mathbf{rat}_{0,1}$ and $\{1 + x \mid x \in \mathbf{rat}_{0,1}\}$.

Definition 11 Suppose $s, r \in \mathbf{rat}_{0,1}$ and $s < r$.
Let $RINIT_{s,r} = \{X_w \mid w \in \mathbf{rat}_{s,r}\}$.
Let $RINIT^{cyl}_{s,r} = \{X^{cyl}_w \mid w \in \mathbf{rat}_{s,r}\}$.

Our main goal in this section is to show that the class $RINIT_{0,1}$ is complete. Informally, we have to demonstrate that every language learning problem can be effectively coded as a sequence of increasing rationals that stabilizes to one rational in the interval $[0, 1]$. More specifically, we code by rationals the sequence of hypotheses outputted by a (modified) learning device being fed an arbitrary text of a learnable language. First, we prove a simple technical Proposition 4 that gives us opportunity to algorithmically generate sequences of rationals that tend to get closer to each other still keeping previously chosen distances between them; these sequences are necessary for coding. Using Theorem 1 gives us opportunity to use learning machines $\mathbf{M}$ that have special properties: their outputs do not depend on arrangement and order of language elements in the input. Using such a machine Proposition 7 allows us to construct a "learning device" H that stabilizes its conjectures on certain "full locking sequences" for the underlying languages. Using the functions provided by Proposition 4, one can map sequences of conjectures produced by H on inputs stabilizing to "full locking sequences" to sequences of rationals stabilizing to a rational representing a language in $RINIT_{0,1}$.

In some cases below, in the pairing function we will be using finite sets as arguments (for example $\langle S, l\rangle$). This is for ease of notation: $\langle S, l\rangle$ should be understood as $\langle x, l\rangle$, where x is a canonical code [Rog67] for the finite set S (i.e. $D_x = S$).

Proposition 4 *There exist recursive functions F and ϵ from $\mathbf{rat}_{0,1}$ to $\mathbf{rat}_{0,1}$ such that,*
(i) for all $x \in \mathbf{rat}_{0,1}$, $\epsilon(x) > 0$, and
(ii) for all rationals, x, y, where $0 \le x < y \le 1$,

$$F(x) + \epsilon(x) < F(y).$$

Moreover, $F(1) + \epsilon(1) \le 1$.

PROOF. Let $q_0, q_1, \ldots$, be some 1–1 recursive enumeration of all the rational numbers between 0 and 1 (both inclusive), such that $q_0 = 0$ and $q_1 = 1$.

We define, inductively on i, $F(q_i)$ and $\epsilon(q_i)$.

Let $F(0) = 1/8$ and $\epsilon(0) = 1/8$. Let $F(1) = 7/8$, $\epsilon(1) = 1/8$.

Induction Hypothesis: Suppose we have defined $F(q_i)$ and $\epsilon(q_i)$, for $i \le k$. Then for all $j, j' \le k$, $[q_j < q_{j'} \Rightarrow F(q_j) + \epsilon(q_j) < F(q_{j'})]$. Note that the induction hypothesis is clearly true for $k = 1$.

Now suppose that $F(q_i)$ and $\epsilon(i)$ have been defined for $i \le k$.

We now define $F(q_{k+1})$ and $\epsilon(q_{k+1})$ as follows.

Let $p_1 = \max(\{q_i \mid i \le k \ \wedge \ q_i < q_{k+1}\})$. Let $p_2 = \min(\{q_i \mid i \le k \ \wedge \ q_i > q_{k+1}\})$.

By induction hypothesis, $F(p_1) + \epsilon(p_1) < F(p_2)$.

Let $F(q_{k+1}) = F(p_1) + \epsilon(p_1) + [F(p_2) - (F(p_1) + \epsilon(p_1))]/3$, and $\epsilon(q_{k+1}) = [F(p_2) - (F(p_1) + \epsilon(p_1))]/3$.

It is easy to verify that the induction hypothesis is satisfied. The proposition follows. ∎

Intuitively, one may consider $x \to [F(x), F(x) + \epsilon(x)]$ as a mapping from $\mathbf{rat}_{0,1}$ to nontrivial closed intervals of rationals (within $[0, 1]$) such that intervals do not overlap, and the interval for smaller rational is below the interval for larger rational.

Fix F, ϵ as in the above proposition.

For $S \in \mathit{FINITE}$, let $\mathit{code}(S) = \sum_{x \in S} 2^{-x-1}$. Note that $0 \leq \mathit{code}(S) < 1$.

Note that, if $\min(S - S') < \min(S' - S)$, then $\mathit{code}(S) > \mathit{code}(S')$ (here $\min(\emptyset) = \infty$).

For $S \in \mathit{FINITE}$ and $l \in N$, let $G(\langle S, l \rangle) = F(\mathit{code}(S)) + \epsilon(\mathit{code}(S)) - \frac{\epsilon(\mathit{code}(S))}{l+2}$.

Proposition 5 *G is a recursive mapping from N to $\mathbf{rat}_{0,1}$. Moreover, if $\min(S - S') < \min(S' - S)$ or $S = S'$ and $l > l'$, then $G(\langle S, l \rangle) > G(\langle S', l' \rangle)$.*

PROOF. Follows from definition of G. ∎

Definition 12 [Ful90, BB75] A machine $\mathbf{M}$ is said to be *rearrangement independent* iff for all $\sigma, \tau \in \mathrm{SEQ}$, if $\mathrm{content}(\sigma) = \mathrm{content}(\tau)$, and $|\sigma| = |\tau|$, then $\mathbf{M}(\sigma) = \mathbf{M}(\tau)$.

A machine $\mathbf{M}$ is said to be *order independent* iff for all texts T and T', if $\mathrm{content}(T) = \mathrm{content}(T')$, then either both $\mathbf{M}(T)$ and $\mathbf{M}(T')$ are undefined, or both are defined and $\mathbf{M}(T) = \mathbf{M}(T')$.

Note that rearrangement independent machines base their output only on the content and length of the input. Thus for $l \geq \mathrm{card}(S)$, we define $\beta^{S,l}$ as the lexicographically least σ of length l such that $\mathrm{content}(\sigma) = S$.

Theorem 1 *(based on [Ful90]) Suppose $a \in N \cup \{*\}$ and $\mathcal{L} \in \mathbf{TxtEx}^a$. Then there exists a rearrangement independent and order independent machine $\mathbf{M}$ such that $\mathcal{L} \subseteq \mathbf{TxtEx}^a(\mathbf{M})$.*

Definition 13 [Ful90, BB75] $\sigma \in \mathrm{SEQ}$ is said to be a *stabilizing sequence* for $\mathbf{M}$ on L, iff $\mathrm{content}(\sigma) \subseteq L$, and for all τ such that $\sigma \subseteq \tau$ and $\mathrm{content}(\tau) \subseteq L$, $\mathbf{M}(\sigma) = \mathbf{M}(\tau)$.

$\sigma \in \mathrm{SEQ}$ is said to be a $\mathbf{TxtEx}^a$*-locking sequence* for $\mathbf{M}$ on L, iff σ is a stabilizing sequence for $\mathbf{M}$ on L, and $W_{\mathbf{M}(\sigma)} =^a L$.

Lemma 1 *(based on [BB75, JORS99]) Suppose $a \in N \cup \{*\}$. If $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies L, then there exists a stabilizing sequence for $\mathbf{M}$ on L, and every stabilizing sequence for M on L is a $\mathbf{TxtEx}^a$-locking sequence for $\mathbf{M}$ on L.*

Definition 14 Suppose $\mathbf{M}$ is a rearrangement independent and order independent learning machine. Let $S \in \mathit{FINITE}$ and $l \in N$.

(a) $\langle S, l \rangle$ is said to be a *full-stabilizing-sequence* for $\mathbf{M}$ on L iff:

(i) $l > \max(S)$,

(ii) $(\forall x < l)[x \in L \Leftrightarrow x \in S]$,

(iii) $\beta^{S,2l}$ is a stabilizing sequence for $\mathbf{M}$ on L.

(b) Suppose $a \in N \cup \{*\}$. $\langle S, l \rangle$ is said to be a $\mathbf{TxtEx}^a$*-full-locking-sequence* for $\mathbf{M}$ on L, iff $\langle S, l \rangle$ is a full-stabilizing-sequence for $\mathbf{M}$ on L, and $W_{\mathbf{M}(\beta^{S,2l})} =^a L$.

Intuitively, $\langle S, l \rangle$ is a full-stabilizing-sequence ($\mathbf{TxtEx}^a$-full-locking-sequence) for $\mathbf{M}$ on L, if $\beta^{S,2l}$ is a stabilizing sequence ($\mathbf{TxtEx}^a$-locking sequence) for $\mathbf{M}$ on L, and $\beta^{S,2l}$ contains exactly the elements in L which are less than l.

Proposition 6 *Suppose $a \in N \cup \{*\}$ and $\mathbf{M}$ is a rearrangement independent and order independent machine, which $\mathbf{TxtEx}^a$-identifies L. Then there exists a full-stabilizing-sequence for $\mathbf{M}$ on L. Moreover, every full-stabilizing-sequence for $\mathbf{M}$ on L is a $\mathbf{TxtEx}^a$-full-locking-sequence for $\mathbf{M}$ on L.*

PROOF. Suppose $\mathbf{M}$ $\mathbf{TxtEx}^a$-identifies L. Suppose σ is a stabilizing-sequence for $\mathbf{M}$ on L. Let $l = 1 + \max(\{|\sigma|\} \cup \mathrm{content}(\sigma))$, and $S = \{x \mid x < l \ \wedge \ x \in L\}$. It follows that $\beta^{S,2l}$ is also a stabilizing-sequence for $\mathbf{M}$ on L. Thus, $\langle S, l\rangle$ is a full-stabilizing-sequence for $\mathbf{M}$ on L. The second part of the proposition follows from Lemma 1. ∎

Definition 15 We say that $\langle S, l\rangle$ is the *least full-stabilizing-sequence* for $\mathbf{M}$ on L, iff $\langle S, l\rangle$ is a full-stabilizing-sequence for $\mathbf{M}$ on L which minimizes l.

Proposition 7 *Suppose* $\mathbf{M}$ *is a rearrangement independent and order independent machine. Then, there exists a recursive function H mapping SEQ to N, such that*

(i) For all $\sigma \in SEQ$, if $H(\sigma) = \langle S, l\rangle$, then $\max(S) < l$.

(ii) For all $\sigma \subseteq \tau$, $G(H(\tau)) \geq G(H(\sigma))$.

(iii) For all texts T, $H(T) = \lim_{n\to\infty} H(T[n])$ converges to the least full-stabilizing-sequence for $\mathbf{M}$ *on* $\mathrm{content}(T)$, *if any.*

PROOF. Define $H(\sigma)$ as follows:

For $l \leq 1 + \max(\mathrm{content}(\sigma) \cup \{|\sigma|\})$, let $S_l^\sigma = \mathrm{content}(\sigma) \cap \{x \mid x < l\}$.

Let $H(\sigma) = \langle S_l^\sigma, l\rangle$, for the least $l \leq 1 + \max(\mathrm{content}(\sigma) \cup \{|\sigma|\})$, such that

$$(\forall \tau \mid \beta^{S_l^\sigma,2l} \subseteq \tau \ \wedge \ \mathrm{content}(\tau) \subseteq \mathrm{content}(\sigma) \ \wedge \ |\tau| \leq |\sigma|)[\mathbf{M}(\beta^{S_l^\sigma,2l}) = \mathbf{M}(\tau)]$$

Note that there exists an l as above, since $l = 1 + \max(\mathrm{content}(\sigma) \cup \{|\sigma|\})$, satisfies the requirements.

Using Proposition 5, we claim that H satisfies the properties above. (i) is trivially true. Clearly, $H(T)$ converges to the least full-stabilizing-sequence for $\mathbf{M}$ on $\mathrm{content}(T)$, if any. Thus, (iii) is satisfied. Now we consider the monotonicity requirement (ii). Suppose $\sigma \subseteq \tau$. Suppose $H(\sigma) = \langle S_l^\sigma, l\rangle$ and $H(\tau) = \langle S_{l'}^\tau, l'\rangle$.

(1) Clearly, $S_w^\sigma \subseteq S_w^\tau$, for all w.

(2) If $l' < l$, then $S_{l'}^\tau$ must be a proper superset of $S_{l'}^\sigma$ (otherwise $\langle S_{l'}^\tau, l'\rangle$ would have been a candidate for consideration as full-stabilizing-sequence even for input σ). Thus, $G(\langle S_{l'}^\tau, l'\rangle) > G(\langle S_l^\sigma, l\rangle)$, by Proposition 5.

(3) If $l' \geq l$, then $S_{l'}^\tau \supseteq S_l^\sigma$. Thus, $G(\langle S_{l'}^\tau, l'\rangle) \geq G(\langle S_l^\sigma, l\rangle)$, by Proposition 5. ∎

Theorem 2 *For any $a \in N$, $RINIT_{0,1}^{cyl}$ is $\leq_{\mathrm{strong}}^{\mathbf{TxtEx}^a}$-complete.*

PROOF. Clearly $RINIT_{0,1}^{cyl} \in \mathbf{TxtEx} \subseteq \mathbf{TxtEx}^a$.

Suppose $\mathcal{L} \in \mathbf{TxtEx}^a$. Let $\mathbf{M}$ be a rearrangement independent and order independent machine which $\mathbf{TxtEx}^a$-identifies $\mathcal{L}$.

Let H be as in Proposition 7.

Let Θ be defined as follows.

Let $\Theta(\sigma) = X_{G(H(\sigma))}^{cyl}$. Note that for $L \in \mathbf{TxtEx}^a(\mathbf{M})$, $\Theta(L) = X_{G(\langle S,l\rangle)}^{cyl}$, where $\langle S, l\rangle$, is the least full-stabilizing-sequence for $\mathbf{M}$ on L (by Proposition 7).

Ψ is defined as follows. Suppose a sequence α of grammars converges to a grammar p. (If there is no such p, then it does not matter what Ψ outputs on sequence α). Suppose $x \in \mathbf{rat}_{0,1}$ is the maximum rational number (if any) such that $\mathrm{coderat}(2w + x) \in W_p$, for at least $2a + 1$ different $w \in N$. (If there is no such x, then it does not matter what Ψ outputs on sequence α). Suppose $S \in FINITE, l \in N$ (if any) are such that $x = G(\langle S, l\rangle)$. (If there are no such S, l, then it does not matter what Ψ outputs on sequence α). Then, $\Psi(\alpha)$ converges to $\mathbf{M}(\beta^{S,2l})$. It is easy to verify that Θ and Ψ witness that $\mathbf{TxtEx}^a(\mathbf{M}) \leq_{\mathrm{strong}}^{\mathbf{TxtEx}^a} RINIT_{0,1}^{cyl}$.

This completes the proof of Theorem 2. ∎

Corollary 1 *$RINIT_{0,1}$ is $\leq^{\mathbf{TxtEx}}_{\text{strong}}$-complete.*

Proof. By Theorem 2, $RINIT^{cyl}_{0,1}$ is $\leq^{\mathbf{TxtEx}}_{\text{strong}}$-complete. Now we show that $RINIT^{cyl}_{0,1} \leq^{\mathbf{TxtEx}}_{\text{strong}} RINIT_{0,1}$ as follows.

Define Θ as follows. $\Theta(X) = X \cap \mathbf{rat}_{0,1}$.

Let Ψ be defined as follows. Suppose a sequence α of grammars converges to a grammar p. Then, $\Psi(\alpha)$ converges to a grammar for $\{\text{coderat}(2w+x) \mid w \in N \ \wedge\ x \in \mathbf{rat}_{0,1} \ \wedge\ \text{coderat}(x) \in W_p\}$. It is easy to verify that Θ and Ψ witness that $RINIT^{cyl}_{0,1} \leq^{\mathbf{TxtEx}}_{\text{strong}} RINIT_{0,1}$. ∎

Why $RINIT_{0,1}$ is complete and, say, $INIT$ is not? From the first glance, strategies learning both classes seem to be identical: being fed the input text, pick the largest number in it to represent the language to be learned. However, there is a subtle difference. Numbers in any language in $INIT$ can be listed in the ascending order, while for the rationals in languages from $RINIT_{0,1}$ this is not possible. Learning, say, the language $\{0, 1, 2, 3, 4, 5, 6\}$, being fed the number 3, we need at most three "mind changes" to arrive at the correct hypothesis. On the other hand, learning the language $X_{2/3}$, we always choose the largest number in the input as our conjecture, however, 1/2 being such a number in the initial fragment of the input does not impact in any way the number of mind changes that will yet occur before we arrive at the final conjecture 2/3 – it depends entirely on the input. This lack of any conceivable bound on the number of remaining mind changes differentiates $RINIT_{0,1}$ from all other, non-complete, classes observed in our paper.

Theorem 3 *For any $a \in N$ and any $\mathcal{L} \in \mathbf{TxtEx}^a$, $\mathcal{L}$ is $\leq^{\mathbf{TxtEx}^a}_{\text{strong}}$-complete iff there exists a recursive function H from $\mathbf{rat}_{0,1}$ to N such that:*

(a) $\{W_{H(r)} \mid r \in \mathbf{rat}_{0,1}\} \subseteq \mathcal{L}$.
(b) If $0 \leq r < r' \leq 1$, then $W_{H(r)} \subset W_{H(r')}$.
(c) $\{W_{H(r)} \mid r \in \mathbf{rat}_{0,1}\}$ is a-limiting standardizable.

Proof. For the whole proof, for $q \in \mathbf{rat}_{0,1}$, let T_q denote a text, obtained effectively from q, for X^{cyl}_q.

$\Rightarrow$: Suppose $\mathcal{L}$ is $\leq^{\mathbf{TxtEx}^a}_{\text{strong}}$-complete. Then, $RINIT^{cyl}_{0,1} \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}$, say via Θ, Ψ.

Define H and E as follows.

$W_{H(q)} = \text{content}(\Theta(T_q))$, for $q \in \mathbf{rat}_{0,1}$. Clearly, $\{W_{H(r)} \mid r \in \mathbf{rat}_{0,1}\} \subseteq \mathcal{L}$.

E defined below will witness the a-limiting standardizability of $\{W_{H(r)} \mid r \in \mathbf{rat}_{0,1}\}$. $E(p)$ is defined as follows. Suppose $\alpha_p = p, p, p, \ldots$. Suppose $\Psi(\alpha_p)$ converges to w. Then $E(p) =$ maximum rational number $r \in \mathbf{rat}_{0,1}$ (if any) such that, for at least $2a+1$ different natural numbers m, $\text{coderat}(2m+r) \in W_w$.

It is easy to verify that H satisfies parts (a) and (b) of the theorem and E witnesses the a-limiting standardizability as required in part (c).

$\Leftarrow$: Suppose that H is as given in the theorem, and E witnesses the a-limiting standardizability as given in condition (c) of the theorem.

Then, define Θ and Ψ witnessing $RINIT^{cyl}_{0,1} \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{L}$ as follows.

$\Theta(L) = \bigcup\{W_{H(q)} \mid \text{coderat}(q) \in L \ \wedge\ q \in \mathbf{rat}_{0,1}\}$.

Let p_q denote a grammar (obtained effectively from q), for $\text{content}(\Theta(T_q))$.

Define Ψ as follows. Suppose a sequence α of grammars converges to a grammar i. Then, $\Psi(\alpha)$ converges to a grammar for X_q^{cyl}, such that $E(i) = E(p_q)$ (if there is any such $q \in \mathbf{rat}_{0,1}$).

It is easy to verify that Θ and Ψ witness that $RINIT_{0,1}^{cyl} \leq_{\text{strong}}^{\mathbf{TxtEx}^a} \mathcal{L}$. Since $RINIT_{0,1}^{cyl}$ is $\leq_{\text{strong}}^{\mathbf{TxtEx}^a}$-complete by Theorem 2, we have that $\mathcal{L}$ is also $\leq_{\text{strong}}^{\mathbf{TxtEx}^a}$-complete. ∎

5 Strong Degrees and Their Characterizations

In this section we establish and characterize a rich structure of degrees of strong reducibility (or, simply, strong degrees), where every degree represents some natural type of learning strategies and reflects topological and algorithmic structures of the languages within it.

Our characterizations of degrees are of two types. Characterizations of the first type, (see Theorems 4, 6, 8, 10, 12, 16, 21) specify language classes in and below a given degree. Every such characterization specifies a class of natural strategies learning all languages in the given degree and failing to learn (at least some) languages in the degrees above or incomparable with the given degree. In certain sense, such a characterization establishes the scope of learnability defined by the degree.

Characterizations of the second type (see Theorems 5, 7, 9, 11, 13, 17, 22) specify algorithmic and set-theoretical restrictions on all classes of languages in a given degree and all degrees above imposed by learnability of hardest classes in the given degree.

Every class $\mathcal{L}$ of languages observed in this paper naturally specifies all classes in the strong degree of this class (that is, all classes that are strongly reducible to the given class, and to which the given class is strongly reducible). We will denote the strong degree of a class $\mathcal{L}$ of languages using the same name as for the class $\mathcal{L}$ itself (for example, *INIT* will stand both for the class $\mathcal{L} = \mathit{INIT}$, as well as for the whole degree of all classes of languages which are $\equiv_{\text{strong}}^{\mathbf{TxtEx}}$ to *INIT*). Which connotation is being used will be always clear from the context.

The structure of degrees developed in this section can be represented in the form of a complex directed graph. The lowest, or, rather, starting points of our hierarchies, are the degrees *SINGLE*, *COSINGLE*, *INIT*, and *COINIT*, that contain well-known classes of languages learnable by some "simplest" strategies. All of these degrees are proven in [JS96] to be pairwise different.

A natural class of languages to consider is also *FINITE*. However, this class was shown in [JS96] to be in the same strong degree as *INIT*. The paper [JS96] contains a number of other natural classes of languages, all of which belong to the degrees *SINGLE*, *COSINGLE*, *INIT*, or *COINIT*. This enables us to concentrate on classes *SINGLE*, *COSINGLE*, *INIT*, and *COINIT* as the "backbone" of our hierarchy.

There certainly exist some other classes and degrees that can be deemed "natural". Two examples of such classes are

$$\mathit{FINITE}_n = \{L \mid L \subseteq N, card(L) \leq n\}$$

and

$$\mathit{COCFINUP}_n = \{L \mid N - L = \{i, i+1, \ldots, i+j\} \text{ for some } j < n\}.$$

($\mathit{COCFINUP}_n$ is the short-cut for "co-(contiguous *FINITE* up to n)"). The corresponding strong degrees form natural growing hierarchies relative to n. However, these classes just "stretch" *SINGLE* and, respectively, *COSINGLE* to "up to n" elements. These hierarchies may deserve separate exploration, but we leave them beyond the scope of this paper, concentrating on classes not impacted by uniform bounds on mind changes.

We let $\mathit{BASIC} = \{\mathit{INIT}, \mathit{COINIT}, \mathit{SINGLE}, \mathit{COSINGLE}\}$.

We begin with characterizations of degrees $\mathit{SINGLE}, \mathit{INIT}, \mathit{COINIT}, \mathit{COSINGLE}$.

Theorem 4 *$\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{SINGLE}$ iff there exist F, a partial recursive function from $\mathrm{FINITE} \times N$ to N, and G, a partial limit recursive mapping from N to N, such that*

(a) For any language $L \in \mathcal{L}$,

(i) there exists a finite $S \subseteq L$, and $j \in N$ such that $F(S, j)\downarrow$; and

(ii) for all $S, S' \subseteq L$, for all $j, j' \in N$, $[[F(S, j)\downarrow$ and $F(S', j')\downarrow] \Rightarrow F(S, j) = F(S', j')]$.

For $L \in \mathcal{L}$, we abuse notation slightly and let $F(L) = F(S, j)$, such that $S \subseteq L$, $j \in N$ and $F(S, j)\downarrow$.

(b) For all $L \in \mathcal{L}$, $G(F(L))$ converges to a grammar for L.

Proof. (Only if direction) Suppose $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{SINGLE}$ via Θ and Ψ. Define F and G as follows.

$F(S, j) = \min(\bigcup\{\mathrm{content}(\Theta(\tau)) \mid |\tau| \leq j \text{ and } \mathrm{content}(\tau) \subseteq S\})$ (where $\min(\emptyset)$ is undefined).

Define $G(w)$ as follows: Let p be a grammar for $\{w\}$. Let $G(w) =$ limit (if any) of $\Psi(\alpha_p)$, where $\alpha_p = p, p, p, \ldots$.

It is easy to verify that F, G satisfy requirements (a) and (b) of the theorem.

(If direction) Suppose F, G are given as in the theorem. Define Θ as follows.

$\Theta(L) = \{F(S, j) \mid S \text{ is finite, } S \subseteq L, j \in N, \text{ and } F(S, j)\downarrow\}$.

Define Ψ as follows. Suppose a sequence α of grammars converges to a grammar p. Then, $\Psi(\alpha)$ converges to $G(\min(W_p))$ (if defined).

It is easy to verify that Θ, Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{SINGLE}$. ∎

Theorem 5 *$\mathit{SINGLE} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ iff there exists a recursive function H such that*

(a) $\{W_{H(i)} \mid i \in N\} \subseteq \mathcal{L}$,

(b) $W_{H(i)} \neq W_{H(j)}$, for $i \neq j$, and

(c) $\{W_{H(i)} \mid i \in N\}$ is limiting standardizable.

Proof. For the whole proof, let T_i denote a text, obtained effectively from i, for $\{i\}$.

(Only if direction) Suppose $\mathit{SINGLE} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ via Θ, Ψ.

Define H and E as follows.

$W_{H(i)} = \mathrm{content}(\Theta(T_i))$.

$E(p)$ is defined as follows. Suppose $\alpha_p = p, p, p, \ldots$. Suppose $\Psi(\alpha_p)$ converges to w. Then $E(p) = \min(W_w)$ (if any).

It is easy to verify that H satisfies requirements (a) and (b) of the theorem, and E witnesses that requirement (c) is satisfied.

(If direction) Suppose that H, E are given such that H satisfies requirements (a) and (b) in the theorem and E witnesses the satisfaction of requirement (c).

Define Θ as follows.

$\Theta(L) = \bigcup_{i \in L} W_{H(i)}$.

Let p_i denote a grammar (obtained effectively from i), for $\mathrm{content}(\Theta(T_i))$.

Define Ψ as follows. Suppose a sequence α of grammars converges to a grammar q. Then, $\Psi(\alpha)$ converges to a grammar for $\{i\}$, such that $E(q) = E(p_i)$ (if there is any such i).

It is easy to verify that Θ and Ψ witness that *SINGLE* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$. ∎

Notation and definitions below provide us with terminology and apparatus for characterizations of *INIT* and *COINIT*.

Definition 16 F, a partial recursive mapping from $\mathit{FINITE} \times N$ to N, is called an *up-mapping* iff for all finite sets S, S', for all $j, j' \in N$:

$$\text{If } S \subseteq S' \text{ and } j \leq j', \text{ then } F(S,j)\downarrow \Rightarrow [F(S',j')\downarrow \geq F(S,j)].$$

For an up-mapping F and $L \subseteq N$, we abuse notation slightly and let $F(L)$ denote $\lim_{S\to L, j\to\infty} F(S,j)$ (where by $S \to L$ we mean: take any sequence of finite sets $S_1, S_2, \ldots$, such that $S_i \subseteq S_{i+1}$ and $\bigcup S_i = L$, and then take the limit over these S_i's).

Note that $F(L)$ may be undefined in two ways:
(1) $F(S,j)$ may take arbitrary large values for $S \subseteq L$, and $j \in N$, or
(2) $F(S,j)$ may be undefined for all $S \subseteq L$, $j \in N$.

Definition 17 F, a partial recursive mapping from $\mathit{FINITE} \times N$ to N, is called a *down-mapping* iff for all finite sets S, S' and $j, j' \in N$,

$$\text{If } S \subseteq S' \text{ and } j \leq j', \text{ then } F(S,j)\downarrow \Rightarrow [F(S',j')\downarrow \leq F(S,j)].$$

For a down-mapping F and $L \subseteq N$, we abuse notation slightly and let $F(L) = \lim_{S\to L, j\to\infty} F(S,j)$.

The following results characterize strong degrees below and above *INIT* and *COINIT*.

Theorem 6 *$\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ INIT iff there exist F, a partial recursive up-mapping, and G, a partial limit recursive mapping from N to N, such that for all $L \in \mathcal{L}$,*
(a) $F(L)\downarrow < \infty$.
(b) $G(F(L))$ converges to a grammar for L.

Proof. (Only if direction) Suppose $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ *INIT* via Θ and Ψ. Define F and G as follows.

$F(S,j) = \max(\bigcup\{\text{content}(\Theta(\tau)) \mid |\tau| \leq j \text{ and content}(\tau) \subseteq S\})$. Clearly, F is a partial recursive up-mapping.

Define $G(w)$ as follows. Let p be a grammar for $\{x \mid x \leq w\}$. Let $G(w) =$ limit (if any) of $\Psi(\alpha_p)$, where $\alpha_p = p, p, p, \ldots$.

It is easy to verify that F, G satisfy requirements (a) and (b) of the theorem.

(If direction) Suppose F (a partial recursive up-mapping) and G (a partial limit recursive mapping) satisfying requirements (a) and (b) in the theorem are given.

Define Θ as follows.

$\Theta(L) = \{x \mid (\exists \text{ finite } S \subseteq L)(\exists j)[x \leq F(S,j)\downarrow]\}$.

Define Ψ as follows. Suppose a sequence α of grammars converges to a grammar p. Then, $\Psi(\alpha)$ converges to $G(\max(W_p))$ (if defined).

It is easy to verify that Θ, Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ *INIT*. ∎

Theorem 7 *INIT $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ iff there exists a recursive function H such that*
(a) $\{W_{H(i)} \mid i \in N\} \subseteq \mathcal{L}$,
(b) $W_{H(i)} \subset W_{H(i+1)}$, and
(c) $\{W_{H(i)} \mid i \in N\}$ is limiting standardizable.

PROOF. For the whole proof, let T_i denote a text, obtained effectively from i, for $\{x \mid x \leq i\}$.

(Only if direction) Suppose $INIT \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ via Θ, Ψ.

Define H and E as follows.

$W_{H(i)} = \text{content}(\Theta(T_i))$.

$E(p)$ is defined as follows. Suppose $\alpha_p = p, p, p, \ldots$. Suppose $\Psi(\alpha_p)$ converges to w. Then $E(p) = \max(W_w)$ (if any).

It is easy to verify that H satisfies requirements (a) and (b) of the theorem, and E witnesses requirement (c) of the theorem.

(If direction) Suppose that H (satisfying requirements (a) and (b) of the theorem), and E (witnessing requirement (c) of the theorem) are given.

Define Θ as follows.

$\Theta(L) = \bigcup_{i \in L} W_{H(i)}$.

Let p_i denote a grammar (obtained effectively from i), for content$(\Theta(T_i))$.

Define Ψ as follows. Suppose a sequence α of grammars converges to grammar q. Then, $\Psi(\alpha)$ converges to a grammar for $\{x \mid x \leq i\}$, such that $E(q) = E(p_i)$ (if there is any such i).

It is easy to verify that Θ and Ψ witness that $INIT \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$. ∎

Theorem 8 $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ *COINIT iff there exist F, a partial recursive down-mapping and G, a partial limit recursive mapping from N to N, such that*

(a) For any $L \in \mathcal{L}$, $F(L)\downarrow$.

(b) For all $L \in \mathcal{L}$, $G(F(L))$ converges to a grammar for L.

PROOF. (Only if direction) Suppose $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ *COINIT* via Θ and Ψ. Define F and G as follows.

$F(S, j) = \min(\bigcup\{\text{content}(\Theta(\tau)) \mid |\tau| \leq j \text{ and content}(\tau) \subseteq S\})$. Clearly, F is a partial recursive down-mapping.

Define $G(w)$ as follows: Let p be a grammar for $\{x \mid x \geq w\}$. Let $G(w) =$ limit (if any) of $\Psi(\alpha_p)$, where $\alpha_p = p, p, p, \ldots$.

It is easy to verify that F, G satisfy the requirements (a) and (b) of the theorem.

(If direction) Suppose F (a partial recursive down-mapping), and G (a partial limit recursive mapping) satisfying requirements (a) and (b) of the theorem are given.

Define Θ as follows.

$\Theta(L) = \{x \mid (\exists \text{ finite } S \subseteq L)(\exists j)[F(S, j)\downarrow \leq x]\}$.

Define Ψ as follows. Suppose a sequence α of grammars converges to a grammar p. Then $\Psi(\alpha)$ converges to $G(\min(W_p))$ (if defined).

It is easy to verify that Θ and Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ *COINIT*. ∎

Theorem 9 *COINIT* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ *iff there exists a recursive function H, such that*

(a) $\{W_{H(i)} \mid i \in N\} \subseteq \mathcal{L}$,

(b) $W_{H(i+1)} \subset W_{H(i)}$, and

(c) $\{W_{H(i)} \mid i \in N\}$ is limit standardizable.

PROOF. Let T_i denote a text, obtained effectively from i, for $\{x \mid x \geq i\}$.

(Only if direction) Suppose *COINIT* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ via Θ and Ψ.

Define H and E as follows.

$W_{H(i)} = \text{content}(\Theta(T_i))$.

$E(p)$ is defined as follows. Suppose $\alpha_p = p, p, p, \ldots$. Suppose $\Psi(\alpha_p)$ converges to w. Then $E(p) = \min(W_w)$, if any.

It is easy to verify that H satisfies requirements (a) and (b) of the theorem, and E witnesses the satisfaction of requirement (c) of the theorem.

(If direction) Now suppose that H, satisfying requirements (a) and (b) of the theorem, and E witnessing the satisfaction of requirement (c) of the theorem are given.

Define Θ as follows.

$\Theta(L) = \bigcup_{i \in L} W_{H(i)}$.

Let p_i denote a grammar (obtained effectively from i) for content($\Theta(T_i)$).

Ψ is defined as follows. Suppose a sequence α of grammars converges to a grammar q. Then, $\Psi(\alpha)$ converges to a grammar for $\{x \mid x \geq i\}$, such that $E(q) = E(p_i)$ (if there is any such i).

It is easy to verify that Θ, Ψ witness that *COINIT* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$. ∎

Definition 18 F, a partial recursive mapping from *FINITE* $\times N$ to $N \times N$ is called *up-to-up-mapping* iff for all finite S, S', and $j, j' \in N$:

If $S \subseteq S'$ and $j \leq j'$, then $F(S,j)\downarrow \Rightarrow [F(S',j')\downarrow$ and $[[\pi_1(F(S,j)) = \pi_1(F(S',j')) < \pi_2(F(S,j)) \leq \pi_2(F(S',j'))]$ or $[\pi_1(F(S,j)) < \pi_2(F(S,j)) \leq \pi_1(F(S',j')) < \pi_2(F(S',j'))]]]$.

For F, an up-to-up-mapping, and $L \subseteq N$, we abuse notation slightly and let $F(L) = i$, if $\lim_{S \to L, j \to \infty} \pi_1(F(S,j)) = i$, and $\lim_{S \to L, j \to \infty} \pi_2(F(S,j)) = \infty$. If no such i exists, then $F(L)$ is undefined.

The above definition reflects the most "natural" way the languages in *COSINGLE* and alike can be learned: one learns such a language in "chunks" $0, 1, 2, \ldots, i-1, i+1, \ldots, k$. While i has not showed up in the input, the lower bound i (the first component a of $(a,b) = F(S,j)$ in our definition) stays the same, while the second-lower bound $k+1$ (the component b) increases. If i shows up in the input text, then the new lower bound i' must be at least $k+1$.

Theorem 10 *$\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ COSINGLE iff there exist F, a partial recursive up-to-up-mapping and G, a partial limit recursive mapping from N to N such that*

(a) For all $L \in \mathcal{L}$, $F(L)\downarrow$.

(b) For all $L \in \mathcal{L}$, $G(F(L))$ is a grammar for L.

Proof. (Only if direction) Suppose $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ *COSINGLE* via Θ and Ψ. Define F and G as follows.

For any set Z, let $\min_1(Z)$ = least element in Z, and $\min_2(Z)$ = second least element in Z.

Define $F(S,j)$ as follows. $F(S,j) = (\min_1(Z), \min_2(Z))$, where Z is complement of $\bigcup\{\text{content}(\Theta(\tau)) \mid \text{content}(\tau) \subseteq S \text{ and } |\tau| \leq j\}$. Clearly, F is a partial recursive up-to-up-mapping.

Define $G(j)$ as follows. Let p be a grammar for $\overline{\{j\}}$. Let $\alpha_p = p, p, p, \ldots$. Then, $G(j)$ converges to the limit of $\Psi(\alpha_p)$, if any.

It is easy to verify that F, G satisfy the requirements (a) and (b) of the theorem.

(If direction) Suppose F (a partial recursive up-to-up mapping) and G (a partial limit recursive mapping) are given satisfying requirements (a) and (b) of the theorem.

Define Θ as follows.

content$(\Theta(\sigma)) = \{x \mid x < i\} \cup \{x \mid i < x < k\}$,

where $i = \max(\{x \mid (\exists y \in N, S \subseteq \text{content}(\sigma), j \leq |\sigma|)[F(S,j) = (x,y)]\})$ and $k = \max(\{y \mid (\exists x \in N, S \subseteq \text{content}(\sigma), j \leq |\sigma|)[F(S,j) = (x,y)]\})$.

Define Ψ as follows. Suppose a sequence α of grammars converges to a grammar p. Then $\Psi(\alpha)$ converges to $G(w)$ (if defined), where w is the minimum element not in W_p (if any).

It is easy to verify that Θ and Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}}$ *COSINGLE*. ∎

Note that if $\mathcal{L}$ contains two languages L_1 and L_2 such that $L_1 \subset L_2$, then $\mathcal{L} \not\leq^{\mathbf{TxtEx}}_{\text{strong}}$ *COSINGLE*. This follows from Proposition 3, since all pairs of languages in *COSINGLE* are incomparable with respect to $\subset$. It follows that *INIT* and *COINIT* are not $\leq^{\mathbf{TxtEx}}_{\text{strong}}$-reducible to *COSINGLE*.

Theorem 11 *COSINGLE* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ *iff there exist recursive functions H and G such that the following properties are satisfied.*

(a) $\{W_{H(i)} \mid i \in N\} \subseteq \mathcal{L}$,

(b) $W_{H(i)} \neq W_{H(j)}$ *for* $i \neq j$.

(c) $W_{H(i)} = \bigcup_{j>i} W_{G(i,j)}$,

(d) $W_{G(i,j)} \subseteq W_{G(k,l)}$, *if* $i < j \leq k < l$, *or* $i = k < j < l$.

(e) $\{W_{H(i)} \mid i \in N\}$ *is limiting standardizable.*

Proof. Let T_i be a text, obtained effectively from i, for $\overline{\{i\}}$.

(Only if direction) Suppose *COSINGLE* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ via Θ, Ψ. For any set Z, let $\min_1(Z)$ denote the least element in Z, and $\min_2(Z)$ denote the second least element in Z.

Define H, G, and E as follows.

$W_{H(i)} = \text{content}(\Theta(T_i))$.

$W_{G(i,j)} = \bigcup\{\text{content}(\Theta(\tau)) \mid \text{content}(\tau) = \{x \mid x < j \text{ and } x \neq i\}\}$.

$E(p)$ is defined as follows. Let $\alpha_p = p, p, p, \ldots$. Suppose $\Psi(\alpha_p)$ converges to w. Then, $E(p) = \min(\overline{W_w})$, if any.

It is easy to verify that G, H satisfy requirements (a) to (d) of the theorem, and E witnesses the satisfaction of requirement (e) of the theorem.

(If direction) Suppose that H, G, E are given such that H and G satisfy requirements (a) to (d) of the theorem and E witnesses satisfaction of requirement (e) of the theorem.

Define Θ as follows.

$\Theta(L) = \bigcup\{W_{G(i,j)} \mid i < j \ \wedge \ \{x < j \mid x \neq i\} \subseteq L\}$.

Let p_i be a grammar (obtained effectively from i) for $\text{content}(\Theta(T_i))$.

Define Ψ as follows. Suppose a sequence α of grammars converges to a grammar q. Then $\Psi(\alpha)$ converges to a grammar for $\overline{\{i\}}$, where $E(q) = E(p_i)$ (if there is any such i).

It is easy to verify that Θ, Ψ witness that *COSINGLE* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$. ∎

Every class we have observed represents certain strategies of learning in the limit. Now let us imagine a "multidimensional" language where every "dimension" is being learned using its specific type of learning strategy, that is *SINGLE*, *COSINGLE*, *INIT*, or *COINIT* like. If this idea can be naturally formalized, the following questions can be asked immediately:

1. Are degrees defined by classes of "multidimensional" languages stronger than the degrees of simple "one-dimensional" classes?

2. Is it possible to characterize these degrees in terms similar to the ones we have used for "one-dimensional" degrees?

Perhaps, the simplest natural way to formalize the above idea is the following

Definition 19 Let $\mathcal{L}_1, \mathcal{L}_2$ be two classes of languages. Then $\mathcal{L}_1 \times \mathcal{L}_2 = \{L_1 \times L_2 \mid L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$.

This definition can be naturally extended to any finite number of dimensions. For example, one can naturally define $\mathit{INIT} \times \mathit{INIT} \times \mathit{COSINGLE}$, etc.

The following propositions trivially follow from the above definition.

Proposition 8 $\mathcal{L}_1 \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}_1 \times \mathcal{L}_2$.

Proposition 9 *Suppose* $\mathcal{L}_1 \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}_2$. *Then* $\mathcal{L}_1 \times \mathcal{L}_3 \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}_2 \times \mathcal{L}_3$.

Proposition 10 $\mathcal{L}_1 \times \mathcal{L}_2 \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}_2 \times \mathcal{L}_1$.

Proposition 11 $(\mathcal{L}_1 \times \mathcal{L}_2) \times \mathcal{L}_3 \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}_1 \times (\mathcal{L}_2 \times \mathcal{L}_3) \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$.

Thus, strong degrees formed using cross product are commutative and associative.

One can easily prove the following facts (we omit most of the proofs) showing that in most cases the strategies for "simple" classes can be applied to more complex classes. To summarize the results below, the number of dimensions can be reduced to 2 in case of different classes in different dimensions, and to 1 in case of the same classes in different dimensions.

Proposition 12 $\mathit{COSINGLE} \times \mathit{COSINGLE} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COSINGLE}$.

Proof. Clearly, $\mathit{COSINGLE} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COSINGLE} \times \mathit{COSINGLE}$. We will define below Θ and Ψ witnessing $\mathit{COSINGLE} \times \mathit{COSINGLE} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COSINGLE}$.

Let $L_i = \{x \mid x \neq i\}$. Let σ be any initial fragment of a text of a language $L \in \mathit{COSINGLE} \times \mathit{COSINGLE}$. Then $\Theta(\sigma) = \{\langle i, j\rangle \mid \langle i, k\rangle \in \text{content}(\sigma)$ for some k or $\langle m, j\rangle \in \text{content}(\sigma)$ for some $m\}$. It is easy to verify that $\Theta(L_i \times L_j) = L_{\langle i,j\rangle}$.

Ψ is defined as follows. Suppose a sequence α of grammars converges to a grammar p. and $\langle i, j\rangle$ is the least element (if any) not in W_p. Then $\Psi(\alpha)$ converges to a grammar for $L_i \times L_j$. It is easy to verify that Θ and Ψ witness that $\mathit{COSINGLE} \times \mathit{COSINGLE} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COSINGLE}$. ∎

The following propositions can be similarly proved.

Proposition 13 $\mathit{INIT} \times \mathit{INIT} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathit{INIT}$.

Proposition 14 $\mathit{COINIT} \times \mathit{COINIT} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COINIT}$

Proposition 15 *For any* $\mathcal{L} \in \mathit{BASIC}$, $\mathit{SINGLE} \times \mathcal{L} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$.

The above results show that *SINGLE* in any combination with other classes can be removed, and any subsequence $\mathcal{L} \times \mathcal{L} \times \ldots \times \mathcal{L}$ (for the same class $\mathcal{L}$) can be reduced to $\mathcal{L}$. Since $\mathit{COSINGLE} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{INIT}$, using Proposition 13, Proposition 8 and Proposition 9 we obtain

Proposition 16 $\mathit{INIT} \times \mathit{COSINGLE} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COSINGLE} \times \mathit{INIT} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathit{INIT}$

Thus, the only classes that remain to be considered are $\mathit{INIT} \times \mathit{COINIT} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COINIT} \times INIT$ and $\mathit{COINIT} \times \mathit{COSINGLE} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COSINGLE} \times \mathit{COINIT}$. We will consider only $\mathit{INIT} \times \mathit{COINIT}$ below. The results for $\mathit{COINIT} \times \mathit{COSINGLE}$ can be formulated and obtained similarly.

Proposition 17 $\mathit{INIT} \times \mathit{COINIT} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{INIT}$ *and* $\mathit{INIT} \times \mathit{COINIT} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{COINIT}$.

PROOF. Proposition easily follows from Proposition 8 and the fact that none of *INIT*, *COINIT* is strong reducible to the other [JS96]. ∎

Now we characterize degrees below and above $INIT \times COINIT$ in terms "combining" those for *INIT* and *COINIT*. Similar results can be obtained for $COINIT \times COSINGLE$.

Definition 20 F, a partial recursive mapping from $FINITE \times N$ to $N \times N$, is called an *up×down-mapping* iff the following conditions hold:

For all finite sets $S, S' \subseteq N$ and $j, j' \in N$,

If $S \subseteq S'$ and $j \leq j'$, Then, $F(S,j)\downarrow \Rightarrow [F(S',j')\downarrow$ and $\pi_1(F(S,j)) \leq \pi_1(F(S',j'))$ and $\pi_2(F(S,j)) \geq \pi_2(F(S',j'))]$.

For F, an up×down-mapping, and $L \subseteq N$, we abuse notation slightly and let $F(L) = (i,j)$, where $i = \lim_{S\to L, j\to\infty} \pi_1(F(S,j))$, and $j = \lim_{S\to L, j\to\infty} \pi_2(F(S,j))$. ($F(L)$ is undefined if no such i and j exist).

Theorem 12 $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} INIT \times COINIT$ *iff there exist F, a partial recursive up×down-mapping, and G, a partial limit recursive mapping from $N \times N$ to N such that*

(a) For all $L \in \mathcal{L}$, $F(L)\downarrow$.

(b) For all $L \in \mathcal{L}$, $G(F(L))$ converges to a grammar for L.

PROOF. (Only if direction) Suppose $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} INIT \times COINIT$, via Θ and Ψ. Define F, G as follows.

$F(S,j) = (i,k)$, where

$i = \max(\{x \mid (\exists\tau \mid \text{content}(\tau) \subseteq S \ \wedge \ |\tau| \leq j)(\exists y)[\langle x,y\rangle \in \text{content}(\Theta(\tau))]\})$, and $k = \min(\{y \mid (\exists\tau \mid \text{content}(\tau) \subseteq S \ \wedge \ |\tau| \leq j)(\exists x)[\langle x,y\rangle \in \text{content}(\Theta(\tau))]\})$. Clearly, F is a partial recursive up×down-mapping.

$G((i,k))$ is defined as follows. Let p be a grammar for $\{x \mid x \leq i\} \times \{y \mid y \geq k\}$. Let $\alpha_p = p, p, p, \ldots$. Then $G((i,k))$ converges to the limit (if any) of $\Psi(\alpha_p)$.

It is easy to verify that F, G satisfy requirements (a) and (b) of the theorem.

(If direction) Suppose F (a partial recursive up×down-mapping) and G (a partial limit recursive mapping) satisfying requirements (a) and (b) of the theorem are given.

Define Θ as follows.

$\Theta(L) = \{\langle x,y\rangle \mid (\exists i,k)(\exists S \subseteq L)(\exists j)[F(L,j) = (i,k) \ \wedge \ x \leq i \ \wedge \ y \geq k]\}$.

Ψ is defined as follows. Suppose a sequence α of grammars converges to a grammar p. Then $\Psi(\alpha)$ converges to $G((i,k))$ (if defined), where $i = \max(\{x \mid (\exists y)[\langle x,y\rangle \in W_p]\})$, and $k = \min(\{y \mid (\exists x)[\langle x,y\rangle \in W_p]\})$.

It is easy to verify that Θ and Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} INIT \times COINIT$. ∎

Theorem 13 $INIT \times COINIT \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ *iff there exists a recursive function H such that*

(a) $\{W_{H(i,j)} \mid i,j \in N\} \subseteq \mathcal{L}$, and $W_{H(i,j)} \neq W_{H(i',j')}$, for $(i,j) \neq (i',j')$,

(b) $W_{H(i,j)} \subset W_{H(i+1,j)}$,

(c) $W_{H(i,j)} \supset W_{H(i,j+1)}$, and

(d) $\{W_{H(i,j)} \mid i,j \in N\}$ is limiting standardizable.

PROOF. Let $T_{i,j}$ be a text, obtained effectively from i and j, for $\{\langle x,y\rangle \mid x \leq i \ \wedge \ y \geq j\}$.

(Only if direction) Suppose $INIT \times COINIT \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ via Θ and Ψ.

Let $W_{H(i,j)} = \text{content}(\Theta(T_{i,j}))$.

$E(p)$ is defined as follows. Suppose $\alpha_p = p, p, p, \ldots$ and $\Psi(\alpha_p)$ converges to q (if it converges). Then, $E(p)$ converges to $\langle i, k\rangle$ (if any) such that

$i = \max(\{x \mid (\exists y)[\langle x, y\rangle \in W_q]\})$, and $k = \min(\{y \mid (\exists x)[\langle x, y\rangle \in W_q]\})$.

It is easy to verify that H satisfies requirements (a) to (c) of the theorem and E witnesses that requirement (d) in the theorem is satisfied.

(If direction) Suppose that H, satisfying requirements (a) to (c) of the theorem, and E witnessing the satisfaction of requirement (d) of the theorem are given.

Define Θ and Ψ as follows:

$\Theta(L) = \bigcup_{\langle i,k\rangle \in L} W_{H(i,k)}$.

Let $p_{i,j}$ be a grammar (obtained effectively from i and j) for content($\Theta(T_{i,j})$). Define Ψ as follows. Suppose a sequence α of grammars converges to a grammar q. Then, let $\Psi(\alpha)$ converge to a grammar for $\{\langle x, y\rangle \mid x \leq i \ \wedge \ y \geq j\}$, where (i, j) (if any) is such that $E(q) = E(p_{i,j})$.

It is easy to verify that Θ and Ψ witness that $\mathit{INIT} \times \mathit{COINIT} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$. ∎

In the above definition of "multidimensional" languages (and respective classes), we assume that every "dimension" is being learned separately. However, as we have shown, there are only two new degrees that can be obtained this way.

Now we consider a more complex way to form "multidimensional" languages. Our approach is based on the following idea: the learner knows in advance to which of the classes from *BASIC* every "dimension" L_k of an "n-dimensional" language L belongs; however, to learn the "dimension" L_{k+1}, one must first learn the codes $i_1, \ldots, i_k$ of the grammars for the languages $L_1, \ldots, L_k$; then L_{k+1} is the $(k+1)$-"projection" $\{x_{k+1} \mid \langle i_1, \ldots, i_k, x_{k+1}, x_{k+2}, \ldots, x_n\rangle \in L\}$ (in case of $L_r \in \mathit{COSINGLE}$, instead of i_r any number $i_r + m, m > 0$ would be used, as this special case calls for; for explanation see discussion of learning strategy for $(\mathit{COSINGLE}, \mathit{INIT})$ below).

For example, suppose it is known that the languages L_k (of the k-th "dimension") are from the class *COINIT*. Then, for any L_k, the number i such that $L_k = \{j \mid j \geq i\}$ can be viewed as a legitimate description of this language. Then this $i = i_k$, together with $i_1, i_2, \ldots, i_{k-1}$ found on the previous phases of the learning process and together with some fixed in advance "pattern" (say, *INIT*) (specifying an appropriate learning strategy) can be used to learn the "dimension" L_{k+1}.

"Patterns" specifying classes of languages in different "dimensions" can be of any nature, as long as they provide sufficient information making the class learnable. In our first formalization of this idea below, we limit "patterns" to come from *BASIC*.

Before we give the general definition for the classes that formalizes the above idea, we demonstrate how to define some classes of "two-dimensional" languages based on the classes from *BASIC*. We hope that these definitions and the following discussion will make the general definition and related results and proofs more transparent.

Definition 21 $(\mathit{COINIT}, \mathit{INIT}) = \{L \mid$ there exist $i, j \in N$ such that $L = \{\langle a, b\rangle \mid a > i$, or $[a = i$ and $b \leq j]\}\}$.

$(\mathit{INIT}, \mathit{COINIT}) = \{L \mid$ there exist $i, j \in N$ such that $L = \{\langle a, b\rangle \mid a < i$, or $[a = i$ and $b \geq j]\}\}$.

$(\mathit{INIT}, \mathit{INIT}) = \{L \mid$ there exist $i, j \in N$ such that $L = \{\langle a, b\rangle \mid a < i$, or $[a = i$ and $b \leq j]\}\}$.

$(\mathit{COINIT}, \mathit{COINIT}) = \{L \mid$ there exist $i, j \in N$ such that $L = \{\langle a, b\rangle \mid a > i$, or $[a = i$ and $b \geq j]\}\}$.

$(\mathit{COSINGLE}, \mathit{INIT}) = \{L \mid$ there exist $i, j \in N$ such that $L = \{\langle a, b\rangle \mid a < i$, or $[a > i$ and $b \leq j]\}\}$.

Figure 1: $L_{i,j}^{COINIT,INIT}$

To justify our definition, we briefly discuss the "natural" strategies that learn the classes defined above.

Consider a language $L \in (COINIT, INIT)$ (see figure 1, where i, j denote the parameters/descriptors of the language L). To learn a language in this class, one first uses a *COINIT*-like strategy, and once the first "descriptor" i of the language has been learned, "changes its mind" to a *INIT*-like strategy to learn the second "descriptor" j. More specifically, imagine the area representing a language in $(COINIT, INIT)$: it consists of the infinite rectangle containing all points $\langle a, b\rangle$ with $a > i$ for some i (apparently, the rectangle is open upword and to the right) and a string of points $\langle i, b\rangle, b \leq j$ just left of the rectangle. The learner first tries to determine the left border i of the rectangle. If some $\langle r, b\rangle$ shows up in the input, $r+1$ can be discarded as a candidate for such i; accordingly, $r+1$ cannot represent the "column" containing the second "dimension" of the language, and, consequently, all pairs $\langle r+1, b\rangle, b \in N$ belong to L, which makes this part of the language easily learnable by *COINIT*-type strategy (only the first "dimension" matters). Once i has been identified (in the limit), the learner, using the "column" $\langle i, \cdot\rangle$, may start to learn the parameter j. Here, if some pair $\langle i, s\rangle$ showed up in the input, $s-1$ can be discarded as a candidate for the parameter j. All discarded pairs $\langle a, b\rangle$ can be viewed as the "terminating" part of the language in question, while $\langle i, j\rangle$ can be viewed as its "propagating" part ("propagating" means "the part of the language representing its description, subject to possible change in the limit").

Similar considerations can be applied to $(INIT, COINIT)$, $(INIT, INIT)$, and $(COINIT, COINIT)$.

To learn a language $L \in (COSINGLE, INIT)$, the learner again first tries to identify the i specifying the first component. At any moment, the learner keeps a "chunk" $\{0, 1, 2, \ldots, i-1, i+1, \ldots, n\}$ (provided by the input text) that suggests the given i as the description for the language L. The numbers $r < i$ are already discarded as possible "candidates" for the description i; accordingly, the language in question contains all pairs $\langle r, b\rangle$, thereby the "rows" $\langle r, \cdot\rangle$ cannot be used to learn the second "dimension", while this part of the language can be easily learned itself by *COSINGLE*-type strategy (only the first "dimension" matters). While $r \in \{i+1, \ldots, n\}$ obviously can also be discarded as possible "candidates" for i, their status is different: they are used to learn the second parameter j, because pairs $\langle i, b\rangle$ are "prohibited" from the language L by the definition of the class *COSINGLE*. Still, if a pair $\langle r, s\rangle$ with $r > i$ shows up in the input, $s - 1$ is discarded as a possible conjecture for j; thus, the "column" $\langle r, j\rangle$ with $r > i$ is considered to be the "propagating" part of the language L representing the current candidate for its description (this approach is formally somewhat different from the one used for classes such as (*COINIT*, *INIT*), where the "rows" i rather than $r > i$ are used to learn the second "dimension", but it is naturally dictated by the specifics of the class *COSINGLE*: the "descriptor" i is not present in the language, so we consider all $s > i$ as "representatives" of the "descriptor" in the language L).

In some sense, any language L in the above classes consists of two parts:

1. *Terminating* part $T(L)$ consisting of the discarded "conjectures".

2. *Propagating* part $P(L)$ consisting of those pairs in L that represent the current hypothesis-"descriptor" of L.

Now we are ready to give the general definition of "multidimensional" classes formalizing the above approach.

Let R be any subset of N. To make our definition as general as possible, we consider the following variants of *SINGLE*, *COSINGLE*, *INIT*, *COINIT* relative to any such R (we give the variant only for *INIT*; similar variant for other classes in *BASIC* can be defined similarly):

$INIT.R = \{L \mid \text{there exists } i \in R \text{ such that } L = \{k \mid k \leq i\}\}$.

For any tuples X and Y, let $X \cdot Y$ stand for the concatenation of X and Y (that is, $X \cdot Y$ is the tuple, where the first tuple is appended by the components of the second tuple).

Recall that $BASIC = \{SINGLE, COSINGLE, INIT, COINIT\}$.

Definition 22 Suppose $k \geq 1$. Let $Q \in BASIC^k$. Let $I \in N^k$. Then inductively on k, we define the languages L_I^Q and $T(L_I^Q)$ and $P(L_I^Q)$ as follows.
If $k = 1$, then
(a) if $Q = (SINGLE)$ and $I = (i)$, then
$T(L_I^Q) = \emptyset$, $P(L_I^Q) = \{\langle i\rangle\}$, and $L_I^Q = T(L_I^Q) \bigcup P(T_I^Q)$.
(b) if $Q = (COSINGLE)$ and $I = (i)$, then
$T(L_I^Q) = \{\langle x\rangle \mid x < i\}$, $P(L_I^Q) = \{\langle x\rangle \mid x > i\}$, and $L_I^Q = T(L_I^Q) \bigcup P(L_I^Q)$.
(c) if $Q = (INIT)$ and $I = (i)$, then
$T(L_I^Q) = \{\langle x\rangle \mid x < i\}$, $P(L_I^Q) = \{\langle i\rangle\}$, and $L_I^Q = T(L_I^Q) \bigcup P(L_I^Q)$.
(d) if $Q = (COINIT)$ and $I = (i)$, then
$T(L_I^Q) = \{\langle x\rangle \mid x > i\}$, $P(L_I^Q) = \{\langle i\rangle\}$, and $L_I^Q = T(L_i^Q) \bigcup P(L_i^Q)$.

Now suppose we have already defined L_I^Q for $k \leq n$. We then define L_I^Q for $k = n+1$ as follows. Suppose $Q = (q_1, \ldots, q_{n+1})$ and $I = (i_1, \ldots, i_{n+1})$. Let $Q_1 = (q_1)$ and $Q_2 = (q_2, \ldots, q_{n+1})$. Let $I_1 = (i_1)$ and $I_2 = (i_2, \ldots, i_{n+1})$. Then,
$T(L_I^Q) = \{X \cdot Y \in N^{n+1} \mid X \in T(L_{I_1}^{Q_1})$, or $[X \in P(L_{I_1}^{Q_1})$ and $Y \in T(L_{I_2}^{Q_2})]\}$,
$P(L_I^Q) = \{X \cdot Y \in N^{n+1} \mid X \in P(L_{I_1}^{Q_1})$ and $Y \in P(L_{I_2}^{Q_2})\}$, and
$L_I^Q = T(L_I^Q) \bigcup P(L_I^Q)$.

For ease of notation we often write $L_{(i_1, i_2, \ldots, i_k)}^Q$ as $L_{i_1, i_2, \ldots, i_k}^Q$.

Definition 23 *Let $Q \in BASIC^k$ and $R = R_1 \times R_2 \times \cdots \times R_k \subseteq N^k$, for $k \geq 1$. Then the class $\mathcal{L}^{Q,R}$ is defined as*
$\mathcal{L}^{Q,R} = \{L_I^Q \mid I \in R\}$.

For technical convenience, for $Q = ()$, $I = ()$, $R = \{I\}$, we also define $T(L_I^Q) = \emptyset$, $P(L_Q^I) = \{\langle\rangle\}$, and $L_I^Q = T(L_I^Q) \bigcup P(L_I^Q)$, and $\mathcal{L}^{Q,R} = \{L_I^Q\}$.

Note that we have used a slightly different notation for defining the classes $\mathcal{L}^{Q,R}$ (for example instead of $(INIT, INIT)$, we now use $\mathcal{L}^{(INIT,INIT),N^2}$). This is for clarity of notation.

Also, our main interest is for R_i's being N, though it doesn't matter as long as R_i is an infinite recursive subset of N (or contains an infinite recursive subset) as the following proposition shows. The usage of general R is more for ease of proving our theorems.

Proposition 18 *Suppose $k \geq 1$. Let $Q \in BASIC^k$. Let $R = R_1 \times R_2 \times \cdots \times R_k$, where each R_i is an infinite recursive subset of N. Then, $\mathcal{L}^{Q,R} \equiv^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q,N^k}$.*

For ease of notation, if $R = N^{|Q|}$, we drop R from $\mathcal{L}^{Q,R}$, using just $\mathcal{L}^Q$.

One can easily see that the definitions of the "pair"-type classes comply with the general definition. The immediate question is which of the $Q \in BASIC^*$ represent different strong degrees.

Proposition 19 *Suppose $Q = (COSINGLE, COSINGLE)$, and $Q' = (COSINGLE)$. Then $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.*

PROOF. Define Θ and Ψ as follows:

content$(\Theta(\tau)) = \{\langle i,j\rangle \mid \langle i,j\rangle \leq |\tau| \wedge [(\exists y)[\langle i,y\rangle \in \text{content}(\tau)] \vee [\langle i+1,j\rangle \in \text{content}(\tau)]]\}$. It is easy to verify that $\Theta(L_{i,j}^Q) = L_{\langle i,j\rangle}^{Q'}$.

Now let Ψ be defined as follows. Suppose a sequence α of grammars converges to a grammar p, and $\langle i,j\rangle = \min(\{\langle x,y\rangle \mid \langle x,y\rangle \notin W_p\})$. Then, $\Psi(\alpha)$ converges to a grammar for $L_{i,j}^Q$. It is easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$. ∎

Proposition 20 *Suppose $Q = (COSINGLE, INIT)$, and $Q' = (INIT)$. Then $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.*

PROOF. Define Θ and Ψ as follows.

Let content$(\Theta(\sigma))$ contain $\langle i,j\rangle$ iff for all $\langle k,l\rangle < \langle i,j\rangle$ EITHER

(a) there exists x, such that $\langle k,x\rangle$ in content(σ) OR

(b) there exists a $x > k$, there exists a $y > l$ such that $\langle x,y\rangle \in \text{content}(\sigma)$.

Intuitively above Θ reduces $L_{i,j}^Q$ to $L_{\langle i,j\rangle}^{Q'}$.

Now let Ψ be defined as follows. Suppose a sequence α of grammars converges to a grammar p and $\langle i,j\rangle$ is the maximum element in W_p. Then $\Psi(\alpha)$ converges to a grammar for $L_{i,j}^Q$. It is easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$. ∎

Proposition 21 *Suppose $X \in BASIC$. Suppose $Q = (SINGLE)$, and $Q' = (X)$. Then $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.*

PROOF. Obvious. ∎

Proposition 22 *Suppose $Q = (COSINGLE, SINGLE)$, and $Q' = (COSINGLE)$. Then $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.*

PROOF. Follows using Proposition 19 and 21. ∎

Proposition 23 *Suppose $X \in BASIC$. Suppose $Q = (SINGLE, X)$, and $Q' = (X)$. Then $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.*

PROOF. We only show the case for $X = INIT$. Other cases can be similarly proved. Define Θ and Ψ as follows.

Let content($\Theta(\sigma)$) contain $\langle i,j\rangle$ iff there exists a $\langle k,l\rangle \geq \langle i,j\rangle$ such that $\langle k,l\rangle \in$ content(σ). Intuitively above Θ reduces $L^Q_{i,j}$ to $L^{Q'}_{\langle i,j\rangle}$.

Now let Ψ be defined as follows. Suppose a sequence α of grammars converges to a grammar p, and $\langle i,j\rangle$ is the maximum element in W_p. Then $\Psi(\alpha)$ converges to a grammar for $L^Q_{i,j}$. It follows that $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$. ∎

Proposition 24 *(Based on [JS96]) Suppose $Q = (COSINGLE)$, and $Q' = (INIT)$. Then $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.*

The following proposition can essentially be proved along the lines of above propositions.

Proposition 25 *Suppose $Q_1 = (q_1, \ldots, q_k, q_{k+1}, \ldots, q_l, q_{l+1}, \ldots, q_n)$, and $Q'_1 = (q_1, \ldots, q_k, q', q_{l+1}, \ldots, q_n)$, where each q_i and $q' \in BASIC$. Suppose in one of the Propositions 19 to 24 above, we have shown that, for $Q = (q_{k+1}, \ldots, q_l)$ and $Q' = (q')$, $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.*

Then, $\mathcal{L}^{Q_1} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'_1}$.

Thus for Q's with components from *BASIC*, for the study of $\leq^{\mathbf{TxtEx}}_{\text{strong}}$-reduction one may assume without loss of generality that *COSINGLE* is never followed by *COSINGLE*, *INIT*, or *SINGLE*; and *SINGLE* is not followed by any $X \in BASIC$.

Proposition 26 *(Based on [JS96]) Suppose $Q = (COSINGLE)$, $R = R_1$, $Q' = (COINIT)$, and $R' = R'_1$, where R_1 and R'_1 are infinite subsets of N. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.*

Proposition 27 *(Based on [JS96]) Suppose $Q = (COINIT)$, $R = R_1$, $Q' = (INIT)$ and $R' = R'_1$, where R_1 and R'_1 are infinite subsets of N. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.*

Proposition 28 *(Based on [JS96]) Suppose $Q = (INIT)$, $R = R_1$, $Q' = (COSINGLE)$ and $R' = R'_1$, where R_1 and R'_1 are infinite subsets of N. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.*

Proposition 29 *Suppose $X \in \{INIT, COINIT\}$. $Q = (X, SINGLE)$, $R = R_1 \times R_2$, $Q' = (X)$ and $R' = R'_1$, where R_1, R_2 and R'_1 are infinite subsets of N. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.*

PROOF. We consider the case of $X = INIT$. The proof can be easily modified to work for $X = COINIT$. Suppose by way of contradiction Θ and Ψ witness that $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$. Let i_1 be the minimal element in R_1. Consider $L^Q_{i_1,i_2}$ such that $i_2 \in R_2$. Now $\Theta(L^Q_{i_1,i_2})$ must be different for different i_2. It follows that $\bigcup_{i_2 \in R_2} \Theta(L^Q_{i_1,i_2})$ must be $\{\langle x\rangle \mid x \in N\}$. Note that for $i'_1 \in R_1$, and $i_2, i'_2 \in R_2$, $i'_1 > i_1$, $L^Q_{i'_1,i'_2} \supseteq L^Q_{i_1,i_2}$. Thus, $\Theta(L^Q_{i'_1,i_2}) \notin \mathcal{L}^{Q',R'}$, for any $i'_1 > i_1$, $i'_1 \in R_1$, and $i_2 \in R_2$ (since $\{\langle x\rangle \mid x \in N\} \subseteq \Theta(L^Q_{i'_1,i_2})$). Proposition follows. ∎

Definition 24 We say that a sequence $Q = (q_1, q_2, \ldots, q_k)$ is a *subsequence* of $Q' = (q'_1, q'_2, \ldots, q'_l)$, iff there exist $i_1, i_2, \ldots, i_k$ such that $1 \leq i_1 < i_2 < \ldots < i_k \leq l$, and for $1 \leq j \leq k$, $q_j = q'_{i_j}$.

Definition 25 Suppose $Q, Q' \in \mathit{BASIC}^*$. Q is said to be a *pseudo-subsequence* of Q' iff there exists a Q'', which is subsequence of Q' such that Q'' can be obtained from Q by substituting some *COSINGLE*s in Q with *INIT*, and some *SINGLE*s in Q with *COSINGLE*, *INIT*, or *COINIT*.

Proposition 30 *Suppose $Q, Q' \in \mathit{BASIC}^*$. Suppose Q is a pseudo-subsequence of Q'. Then, $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.*

Proof. Follows from Definition 25 and Proposition 21, Proposition 24 and Proposition 25. ∎

Proposition 31 *Suppose $Q = (q_1, q_2, \ldots, q_k)$ and $Q' = (q'_1, q'_2, \ldots, q'_l)$, where each $q_i, q'_i \in \mathit{BASIC}$. If Q is not a pseudo-subsequence of Q', $\mathcal{L}^{(q_1)} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{(q'_1)}$, and $\mathcal{L}^{(q_2)} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{(q'_1)}$, then QQ obtained from Q by dropping q_1, is not a pseudo-subsequence of Q'.*

Proof. Suppose the hypothesis. Suppose by way of contradiction that QQ is a pseudo-subsequence of Q'. Let $QQ'' = (q''_2, q''_3, \ldots, q''_k)$ be obtained from QQ by replacing some *SINGLE*s by *INIT*, *COINIT* or *COSINGLE*, and replacing some *COSINGLE*s by *INIT*, such that QQ'' is a subsequence of Q'. Clearly, $q''_2 \neq q'_1$ (since otherwise $\mathcal{L}^{(q_2)} \leq \mathcal{L}^{(q'_1)}$). Thus, QQ'' is a subsequence of $(q'_2, q'_3, \ldots, q'_l)$. It follows that $(q'_1, q''_2, q''_3, \ldots, q''_k)$ is a subsequence of Q'. Thus, Q is a pseudo-subsequence of Q' (since one may obtain $(q'_1, q''_2, q''_3, \ldots, q''_k)$ from Q by replacing q_1 by q'_1, in addition to the replacements done in going from QQ to QQ''). This is a contradiction to the hypothesis of the Proposition. ∎

Theorem 14 *Suppose $Q = (q_1, \ldots, q_k) \in \mathit{BASIC}^k$ and $Q' = (q'_1, \ldots, q'_l) \in \mathit{BASIC}^l$, with the property that COSINGLE is never followed by COSINGLE, INIT, or SINGLE; and SINGLE is not followed by any $X \in \mathit{BASIC}$. Let $R = R_1 \times R_2 \times \cdots \times R_k$, $R' = R'_1 \times R'_2 \times \cdots \times R'_l$, where each R_i, R'_i is an infinite subset of N. If Q is not a pseudo-subsequence of Q' then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.*

(Here if $k = 0$, then we take $R = \{()\}$. Similarly, if $l = 0$, then we take $R' = \{()\}$.)

Proof. We prove the theorem by double induction (first on k and then on l). For $k = 0$ or $l = 0$ the theorem clearly holds. Suppose by induction that the theorem holds for $k \leq m$, $l \in N$, and for $k = m + 1$, $l \leq r$. We then show that the theorem holds for $k = m + 1$ and $l = r + 1$. Suppose by way of contradiction that Θ (with Ψ) witnesses that $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.

We consider the following cases:

Case 1: $q_1 = \mathit{SINGLE}$

In this case $k = 1$. Thus, Q' must be $()$. Thus, clearly, $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.

Case 2: $q_1 = \mathit{COSINGLE}$.

Case 2.1: $q'_1 = \mathit{INIT}$ or *COSINGLE*.

In this case $k \geq 2$. Also, q_2 cannot be *SINGLE*, *COSINGLE* or *INIT* by hypothesis of the theorem. Thus, q_2 must be *COINIT*. Thus, by definition of pseudo-subsequence we know that QQ obtained from Q by dropping q_1 from Q is not a pseudo-subsequence of Q'. Thus, we are done by induction hypothesis.

Case 2.2: $q'_1 = \mathit{SINGLE}$.

In this case $Q' = (\mathit{SINGLE})$. Since $\mathit{COSINGLE} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{SINGLE}$, we have $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$.

Case 2.3: $q'_1 = \mathit{COINIT}$.

Consider σ, which minimizes i such that $\langle i, \ldots\rangle \in \text{content}(\Theta(\sigma))$. Let j be the maximum number such that $\langle j, \ldots\rangle \in \text{content}(\sigma)$. It follows that, for any $j' > j$, $j' \in R_1$, $\Theta(L^Q_{j',\ldots})$ (for any value of other parameters) is of the form $L^{Q'}_{i,\ldots}$ (for some value of other parameters). Thus, Θ (with Ψ) essentially witnesses that $\mathcal{L}^{Q,RR} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq j\}$, QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1. Now we are done by induction hypothesis.

Case 3: $q_1 = \mathit{INIT}$.

Case 3.1: $q'_1 = \mathit{COSINGLE}$ or SINGLE.

In case of $q'_1 = \mathit{SINGLE}$, we are done (since $\mathit{INIT} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathit{SINGLE}$). Thus, only the case $q'_1 = \mathit{COSINGLE}$ remains. Fix $i \in R_1$. Suppose $\Theta(L^Q_{(i,0,0,0,\ldots)}) = L^{Q'}_{(j,\ldots)}$ (for some value of other parameters). Note that $L^Q_{i_1,\ldots}$ (for any value of other parameters) is a superset of $L^Q_{i,0,0,\ldots}$, for all $i_1 > i$, $i_1 \in R_1$. Thus, $\Theta(L^Q_{i_1,\ldots})$ (for any value of other parameters) is of form $L^{Q'}_{j,\ldots}$ (for some value of other parameters), for all $i_1 > i$, $i_1 \in R_1$. Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{Q,RR} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq i\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1. Now we are done by induction hypothesis.

Case 3.2: $q'_1 = \mathit{COINIT}$.

Consider σ, which minimizes i such that $\langle i, \ldots\rangle \in \text{content}(\Theta(\sigma))$. Let j be the maximum number such that $\langle j, \ldots\rangle \in \text{content}(\sigma)$. It follows that, for any $j' > j$, $j' \in R_1$, $\Theta(L^Q_{j',\ldots})$ (for any value of other parameters) is of the form $L^{Q'}_{i,\ldots}$ (for some value of other parameters). Thus, Θ (with Ψ) essentially witnesses that $\mathcal{L}^{Q,RR} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq j\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1. Now we are done by induction hypothesis.

Case 3.3: $q'_1 = \mathit{INIT}$.

In this case $k \geq 2$.

Case 3.3.1: $q_2 = \mathit{SINGLE}$. In this case Q must be $(\mathit{INIT}, \mathit{SINGLE})$ and $Q' = (\mathit{INIT})$. Thus, $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.

Case 3.3.2: $q_2 = \mathit{COSINGLE}$ or INIT.

Fix $i_1 \in R_1$, and consider $\bigcup_{i_2 \in R_2,\ldots} \Theta(L^{Q,R}_{i_1,i_2,\ldots})$. If this set contains $\langle i'_1, \ldots\rangle$, for arbitrarily large i'_1, then we are done (since then Θ cannot reduce $L_{ii_1,0,0,\ldots}$ to a language in $\mathcal{L}^{Q',R'}$ for any $ii_1 > i_1$).

So let i'_1 be the maximum value such that some element of form $\langle i'_1, \ldots\rangle$ is in $\bigcup_{i_2 \in R_2,\ldots} \Theta(L^Q_{i_1,i_2,\ldots})$.

Let σ be such that $\text{content}(\sigma) \subseteq \{\langle i_1, x_2, \ldots, x_k\rangle \mid (\forall j : 2 \leq j \leq k)[x_j \in N]\}$, and $\Theta(\sigma)$ contains an element of form $\langle i'_1, \ldots\rangle$. Let i_2 be the maximum value such that some element of form $\langle i_1, i_2, \ldots\rangle$ is in $\text{content}(\sigma)$. It follows that, for all $ii_2 > i_2$, $ii_2 \in R_2$, $\Theta(L^Q_{i_1,ii_2,\ldots})$ (for any value of other parameters) is of form $L^{Q'}_{i'_1,\ldots}$ (for some value of other parameters). Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{QQ,RR} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{QQ',RR'}$, where QQ is obtained from Q by dropping q_1, QQ' is obtained from Q' by dropping q'_1, RR' is obtained from R' by dropping R'_1 and RR is obtained from R by dropping R_1 plus changing R_2 to $R_2 - \{x \mid x \leq i_2\}$. Now we are done by induction hypothesis.

Case 3.3.3: $q_2 = \mathit{COINIT}$.

In this case, QQ obtained from Q by dropping q_1 is not a pseudo-subsequence of Q' (by Proposition 31). Thus we are done by induction hypothesis.

Case 4: $q_1 = \mathit{COINIT}$. This case is very similar to Case 3. We give the analysis for completeness sake.

Case 4.1: $q'_1 = COSINGLE$ or $SINGLE$ or $INIT$.

Let i be minimum value such that $\Theta(L^Q_{\ldots}) = L^{Q'}_{i,\ldots}$, for some values of the parameters. Let $j \in R_1$ be such that $\Theta(L^Q_{j,\ldots}) = L^{Q'}_{i,\ldots}$, for some values of the parameters. It follows that for all $j' > j$, $j' \in R_1$, $\Theta(L^Q_{j',\ldots})$ (for any value of other parameters) is of form $L^{Q'}_{i,\ldots}$ (for some value of other parameters).

Thus, Θ (with Ψ) essentially witnesses that $\mathcal{L}^{Q,RR} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leq j\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1. Now we are done by induction hypothesis.

Case 4.2: $q'_1 = COINIT$.

In this case $k \geq 2$.

Case 4.2.1: $q_2 = SINGLE$. In this case Q must be $(COINIT, SINGLE)$ and $Q' = (COINIT)$. Thus, $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q',R'}$.

Case 4.2.2: $q_2 = COSINGLE$ or $INIT$.

In this case, QQ obtained from Q by dropping q_1 is not a pseudo-subsequence of Q' (by Proposition 31). Thus we are done by induction hypothesis.

Case 4.2.3: $q_2 = COINIT$. Fix $i_1 \in R_1$, and consider $\Theta(L^Q_{i_1,\ldots}) = L^{Q'}_{i'_1,\ldots}$. If i'_1 achieves arbitrary large value (for some values of other parameters) then we are done (since then Θ cannot reduce $L_{ii_1,0,0,\ldots}$ to a language in $\mathcal{L}^{Q',R'}$, for any $ii_1 > i_1$).

So let i'_1 be maximum value such that for some value of other parameters, $\Theta(L^{Q,R}_{i_1,i_2,\ldots}) = L^{Q',R'}_{i'_1,\ldots}$. It follows that, for all $ii_2 > i_2$, $\Theta(L^{Q,R}_{i_1,ii_2,\ldots})$ (for any value of other parameters) is of form $L^{Q',R'}_{i'_1,\ldots}$ (for some value of other parameters). Thus, Θ (along with Ψ) essentially witnesses that $\mathcal{L}^{QQ,RR} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{QQ',RR'}$, where QQ is obtained from Q by dropping q_1, QQ' is obtained from Q' by dropping q'_1, RR' is obtained from R' by dropping R'_1 and RR is obtained from R by dropping R_1 plus changing R_2 to $R_2 - \{x \mid x \leq i_2\}$. Now we are done by induction hypothesis. ∎

Theorem 15 (Q-hierarchy Theorem) *Suppose $Q = (q_1, \ldots, q_k) \in BASIC^k$ and $Q' = (q'_1, \ldots, q'_l) \in BASIC^l$, with the property that COSINGLE is never followed by COSINGLE, INIT, or SINGLE; and SINGLE is not followed by any $X \in BASIC$. Then, $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'}$ iff Q is a pseudo-subsequence of Q'.*

Proof. Follows from Proposition 30 and Theorem 14. ∎

Note that above Theorem thus gives the relationship (with respect to $\leq^{\mathbf{TxtEx}}_{\text{strong}}$) between $\mathcal{L}^Q$ and $\mathcal{L}^{Q'}$, for all $Q, Q' \in BASIC^*$ (since by Proposition 25, one may assume without loss of generality that in Qs, *COSINGLE* is never followed by *COSINGLE*, *INIT*, or *SINGLE*; and *SINGLE* is not followed by any $X \in BASIC$).

Also, Theorem 15 immediately shows that none of $\mathcal{L}^Q$ is $\leq^{\mathbf{TxtEx}}_{\text{strong}}$-complete.

The above Q-hierarchy can be applied to quantify intrinsic complexity of learning other classes from texts. Consider, for example, *open semi-hulls* representing the space consisting of all points (x, y) with integer components x, y in the first quadrant of the plane bounded by the y-axis and the "broken" line passing through some points $(0,0), (a_1, c_1), \ldots, (a_n, c_n)$ with $a_i < a_{i+1}$ (the line is straight between any of the points $(a_i, c_i), (a_{i+1}, c_{i+1})$); further, assume that the slope of the broken line is monotonically non-decreasing (where, for technical convenience, we assume that the first slope is 0: that is $c_1 = 0$). Any such open semi-hull can be easily learned in the limit by the following strategy: given growing finite sets of points in the open semi-hull, learn the first "break" point (a_1, c_1), then the first slope $(c_2 - c_1)/(a_2 - a_1)$, then the second "break" point (a_2, c_2), then the second slope $(c_3 - c_2)/(a_3 - a_2)$, etc. Is this learning strategy optimal? A more general question is: how to measure complexity of learning open semi-hulls? Note that natural complexity measures such as the number of mind changes or memory size would not work, since none of them can be bounded while learning open semi-hulls. One can rather try to determine how many "mind changes" are required in much more general sense: how many times ought a strategy change from *INIT*-like learning to, say, *COINIT*-like learning and back? This is where our hierarchy can be applied. For example, suppose all open semi-hulls with two "angles" are in the class $(\mathit{INIT}, \mathit{COINIT}, \mathit{INIT}, \mathit{COINIT})$. Then there exists a learning strategy that "changes its mind" from *INIT*-like strategy to *COINIT*, then back to *INIT*, and then one more time to *COINIT* (as a matter of fact, such a strategy for learning the above open semi-hulls exists, and it is somewhat "better" than the natural strategy described above). On the other hand, one can show that no $(\mathit{COINIT}, \mathit{INIT}, \mathit{COINIT}, \mathit{INIT})$-type strategy (that is, the one that starts like *COINIT*, "changes its mind" to *INIT*, then back to *COINIT*, and then again to *INIT*) can learn open semi-hulls with two "angles". Upper and lower bound of similar kind are obtained for open semi-hulls and other geometrical concepts in [JK99].

Proposition 32 *Suppose* $Q = (\mathit{INIT}, \mathit{COINIT})$, $Q' = (\mathit{COINIT}, \mathit{INIT})$. *Then*
(a) $\mathit{INIT} \times \mathit{COINIT} \leq^{\mathbf{TxtEx}}_{\mathrm{strong}} \mathcal{L}^Q$.
(b) $\mathit{INIT} \times \mathit{COINIT} \leq^{\mathbf{TxtEx}}_{\mathrm{strong}} \mathcal{L}^{Q'}$.

Proof. We show only part (a). Part (b) can be shown similarly.

Define Θ as follows.

$\Theta(X) = X \cup \{\langle x, y\rangle \mid (\exists i > x)(\exists j)[\langle i, j\rangle \in X]\}$.

It is easy to verify that $\Theta(\{x \mid x \leq i\} \times \{y \mid y \geq j\}) = L^Q_{i,j}$.

Define Ψ as follows. Suppose a sequence α of grammars converges to grammar p. Then, $\Psi(\alpha)$ converges to a grammar for $\{x \mid x \leq i\} \times \{y \mid y \geq j\})$, where $i = \max(\{x \mid (\exists y)[\langle x, y\rangle \in W_p]\})$, and $j = \min(\{y \mid \langle i, y\rangle \in W_p\})$.

It is easy to verify that Θ and Ψ witness that $\mathit{INIT} \times \mathit{COINIT} \leq^{\mathbf{TxtEx}}_{\mathrm{strong}} \mathcal{L}^Q$. ∎

Proposition 33 *Suppose* $Q \in \{(\mathit{INIT}, \mathit{COINIT}), (\mathit{COINIT}, \mathit{INIT}), (\mathit{INIT}, \mathit{COSINGLE}), (\mathit{COINIT}, \mathit{COSINGLE}), (\mathit{COSINGLE}, \mathit{COINIT})\}$. *Then,*
$\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}}_{\mathrm{strong}} \mathit{INIT} \times \mathit{COINIT}$.

Proof. Let $Q' = (\mathit{INIT}, \mathit{COINIT})$ and $Q'' = (\mathit{COINIT}, \mathit{INIT})$. Then, $\mathit{INIT} \times \mathit{COINIT} \leq^{\mathbf{TxtEx}}_{\mathrm{strong}} \mathcal{L}^{Q'}$, and $\mathit{INIT} \times \mathit{COINIT} \leq^{\mathbf{TxtEx}}_{\mathrm{strong}} \mathcal{L}^{Q''}$ (by Proposition 32).

However, either $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}}_{\mathrm{strong}} \mathcal{L}^{Q'}$ or $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}}_{\mathrm{strong}} \mathcal{L}^{Q''}$ (Theorem 15). Thus, $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}}$ $\mathit{INIT} \times \mathit{COINIT}$. ∎

We now give characterizations for degrees below and above an arbitrary degree $\mathcal{L}^Q$.

For the sake of simplicity we consider only Qs with components from $\{\mathit{INIT}, \mathit{COINIT}\}$. (The formulations of characterizations for Qs including *COSINGLE* become technically too complex).

Definition 26 Suppose $Q = (q_1, \ldots, q_k)$, where each $q_i \in \{\mathit{INIT}, \mathit{COINIT}\}$, for $1 \leq i \leq k$. Let $Q' = (q_2, \ldots, q_k)$. We say that $\langle i_1, \ldots, i_k\rangle \leq_Q \langle j_1, \ldots, j_k\rangle$ iff

(a) if $q_1 = \mathit{INIT}$, then $[i_1 < j_1]$ or $[i_1 = j_1$ and $\langle i_2, \ldots, i_k\rangle \leq_{Q'} \langle j_2, \ldots, j_k\rangle]$;

(b) if $q_1 = \mathit{COINIT}$, then $[i_1 > j_1]$ or $[i_1 = j_1$ and $\langle i_2, \ldots, i_k\rangle \leq_{Q'} \langle j_2, \ldots, j_k\rangle]$.

Here, for $Q = ()$, we assume that $\langle\rangle \leq_Q \langle\rangle$.

Note that $\leq_Q$ gives a total order on $N^{|Q|}$. We say that $I_1 <_Q I_2$ iff $I_1 \leq_Q I_2$, but $I_1 \neq I_2$. We say that $I_1 \geq_Q I_2$ iff $I_2 \leq_Q I_1$. Similarly, $I_1 >_Q I_2$ iff $I_2 <_Q I_1$.

Definition 27 Suppose $Q = (q_1, \ldots, q_k)$, where each $q_i \in \{INIT, COINIT\}$, for $1 \leq i \leq k$. We say that $I = \langle i_1, i_2, \ldots, i_k \rangle$ is Q-maximum element of a set S (denoted $\max_Q(S)$) iff $I \in S$, and $(\forall \langle x_1, \ldots, x_k \rangle \in S)[\langle x_1, \ldots, x_k \rangle \leq_Q \langle i_1, \ldots, i_k \rangle]$.

Q-maximum element of $\emptyset$ is undefined.

Note that every non-empty finite set has a Q-maximum element. Also, for any set S, if Q-maximum element exists, then it is unique. (For some infinite sets, Q-maximum element may not exist).

Definition 28 Suppose $Q = (q_1, \ldots, q_k)$, where each $q_i \in \{INIT, COINIT\}$, for $1 \leq i \leq k$. F, a partial recursive mapping from $FINITE \times N$ to N^k, is called an Q- *order-mapping* iff for all finite sets $S, S' \subseteq N$, for all $j, j' \in N$:

$$\text{If } S \subseteq S' \text{ and } j \leq j', \text{ then } F(S,j)\downarrow \Rightarrow [F(S',j')\downarrow \geq_Q F(S,j)].$$

For a Q- order-mapping F and $L \subseteq N$, we abuse notation slightly and let $F(L) = \lim_{S \to L, j \to \infty} F(S,j)$.

Note that $F(L) = \max_Q(\{F(S,j) \mid S \subseteq L, j \in N, F(S,j)\downarrow\})$.

Theorem 16 *Suppose* $Q = (q_1, \ldots, q_k)$, *where each* $q_i \in \{INIT, COINIT\}$, *for* $1 \leq i \leq k$. $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^Q$ *iff there exist* F, G, *where* F *is a partial recursive* Q- *order-mapping and* G *is a partial limit recursive mapping from* N^k *to* N *such that*

(a) For $L \in \mathcal{L}$, $F(L)$ *is defined.*

(b) For $L \in \mathcal{L}$, $G(F(L))$ *converges to a grammar for* L.

Proof. (Only if direction) Suppose Θ and Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^Q$.

Define F, G as follows.

$F(S,j) = \max_Q(\bigcup\{\text{content}(\Theta(\sigma)) \mid \text{content}(\sigma) \subseteq S \ \wedge \ |\sigma| \leq j\})$, where we assume that $\max_Q(\emptyset)$ is undefined.

Clearly, F is a partial recursive Q- order-mapping.

$G(I = \langle i_1, \ldots, i_k \rangle)$ is defined as follows. Let p be a grammar for L^Q_I. Let $\alpha_p = p, p, p, \ldots$. Then, $G(I)$ converges to the limit (if any) of $\Psi(\alpha_p)$.

It is easy to verify that F, G satisfy requirements (a) and (b) of the theorem.

(If direction) Suppose F, a partial recursive Q- order-mapping and G, a partial limit recursive mapping satisfying requirements (a) and (b) of the theorem are given. Then we construct Θ and Ψ witnessing $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^Q$ as follows.

$\Theta(L) = \{\langle x_1, \ldots, x_k \rangle \mid (\exists S \subseteq L)(\exists j)[\langle x_1, \ldots, x_k \rangle \leq_Q F(S,j)\downarrow]\}$.

$\Psi(\alpha)$ is defined as follows. Suppose a sequence α of grammars converges to grammar p, and $\langle i_1, \ldots, i_k \rangle$ is Q-maximum element of W_p. Then $\Psi(\alpha)$ converges to $G(\langle i_1, \ldots, i_k \rangle)$ (if defined). It is easy to verify that Θ and Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^Q$. ∎

Theorem 17 *Suppose* $Q = (q_1, \ldots, q_k)$, *where each* $q_i \in \{INIT, COINIT\}$, *for* $1 \leq i \leq k$. $\mathcal{L}^Q \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ *iff there exists a recursive function* H *such that*

(a) $\{W_{H(i_1,\ldots,i_k)} \mid i_1, \ldots, i_k \in N\} \subseteq \mathcal{L}$, *where* $W_{H(I)} \neq W_{H(J)}$, *if* $I \neq J$, *for any vectors* I *and* J *of length* k.

(b) If $\langle i_1, \ldots, i_k \rangle \leq_Q \langle j_1, \ldots, j_k \rangle$, *then* $W_{H(i_1,\ldots,i_k)} \subseteq W_{H(j_1,\ldots,j_k)}$.

(c) $\{W_{H(i_1,\ldots,i_k)} \mid i_1, \ldots, i_k \in N\}$ *is limiting standardizable.*

PROOF. For $I = (i_1, \ldots, i_k)$, let T_I denote a text, obtained effectively from I, for L_I^Q.

(Only if direction) Suppose $\mathcal{L}^Q \leq_{\text{strong}}^{\mathbf{TxtEx}} \mathcal{L}$ via Θ and Ψ.

Define H and E as follows.

$W_{H(I)} = \text{content}(\Theta(T_I))$.

$E(p)$ is defined as follows. Suppose $\alpha_p = p, p, p, \ldots$, and $\Psi(\alpha_p)$ converges to w. Then, $E(p) = \max_Q(W_w)$ (if any).

It is easy to verify that H satisfies requirements (a) and (b) in the theorem, and E witnesses satisfaction of requirement (c) of the theorem.

(If direction) Suppose that H, E are given such that H satisfies requirements (a) and (b) of the theorem, and E witnesses satisfaction of requirement (c) of the theorem.

Define Θ as follows.

$\Theta(L) = \bigcup_{\langle i_1,\ldots,i_k\rangle \in L} W_{H(i_1,\ldots,i_k)}$.

Let p_I denote a grammar (obtained effectively from I) for content$(\Theta(T_I))$.

Define Ψ as follows. Suppose a sequence α of grammars converges to grammar q. Then, $\Psi(\alpha)$ converges to a grammar for L_I^Q, such that $E(q) = E(p_I)$ (if there is any such I).

It is easy to verify that Θ and Ψ witness that $\mathcal{L}^Q \leq_{\text{strong}}^{\mathbf{TxtEx}} \mathcal{L}$. ∎

One could generalize Definitions 26, 27, and 28 and Theorems 16 and 17 by allowing *COSINGLE* and *SINGLE* in the Qs. Generalization for *SINGLE* is easy. Generalization involving *COSINGLE* is technically messy (for example see the characterization of degrees below and above *COSINGLE* in Theorems 10 and 11), since the definition of $\leq_Q$ and $\max_Q$ become somewhat complicated. We omit this generalization to keep the presentation simple.

In our definition of the classes $\mathcal{L}^Q$ we assumed that the "patterns" for different "dimensions" of a "multidimensional" language come from the set *BASIC*. This gave us opportunity to formalize classes (and degrees) requiring rather complex yet "natural" learning strategies. Now we are going to make another step and define classes of "multidimensional" languages, where such "patterns" come from the whole set of vectors Q. Moreover, the grammar for every "dimension" L_k determines which "pattern" Q must be used to learn L_{k+1}.

Note that there exists a recursive bijective mapping, code_k (obtainable effectively in k) from the set of all possible Q (with components from *BASIC*) onto N^k.

Suppose $Q \in BASIC^k$. Let L_i^Q denote the language $L_{i_1,i_2,\cdots,i_k}^Q$, where $i = \langle i_1, \cdots, i_k\rangle$.

Let code be a mapping from $\bigcup_{k=1}^{\infty} BASIC^k$ to N. Let Q^i denote the Q with code i.

Definition 29 *Suppose* $S_i = \{i\}$.

$\mathcal{Q}^0 = \{S_i \mid i \in N\}$.

Let $L_{i_0,i_1,\cdots,i_m}^{\mathcal{Q}^m} = S_{i_0} \times L_{i_1}^{Q^{i_0}} \times \cdots L_{i_m}^{Q^{i_{m-1}}}$.

$\mathcal{Q}^m = \{L_{i_0,i_1,\cdots,i_m}^{\mathcal{Q}^m} \mid i_0, i_1, \cdots, i_m \in N\}$.

We can thus consider $i_0, i_1, \cdots, i_m$ as a parameter of the languages in $\mathcal{Q}^m$.

For example, any language $L \in \mathcal{Q}^1$ consists of all pairs $\langle i, x\rangle$ such that all components x form a language in $\mathcal{L}^{Q^i}$.

Obviously, every class $\mathcal{L}^Q$ is strongly reducible to $\mathcal{Q}^1$. On the other hand, it easily follows from the hierarchy established in Theorem 14 that the degree $\mathcal{Q}^1$ is above any $\mathcal{L}^Q$. It can be shown that $\mathcal{Q}^2 \not\leq_{\text{strong}}^{\mathbf{TxtEx}} \mathcal{Q}^1$. However, we have not been able to find out if the classes $\mathcal{Q}^m$ with $m > 1$ form an infinite hierarchy.

Let $\mathcal{Q}^* = \bigcup_{m=1}^{\infty} \mathcal{Q}^m$. [1]

It follows from the next theorem that all of $\mathcal{Q}^m$, as well as $\mathcal{Q}^*$, are not $\leq_{\mathrm{strong}}^{\mathbf{TxtEx}^a}$-complete.

Theorem 18 *For any $a \in N \cup \{*\}$, $RINIT_{0,1} \not\leq_{\mathrm{strong}}^{\mathbf{TxtEx}^a} \mathcal{Q}^*$.*

PROOF. Suppose by way of contradiction that Θ (along with Ψ) witnesses that $RINIT_{0,1} \leq_{\mathrm{strong}}^{\mathbf{TxtEx}^a} \mathcal{Q}^*$.

Suppose $\Theta(X_1) = L^{\mathcal{Q}^m}_{p_0,p_1,\ldots,p_m}$.

Note that the above implies that $\Theta(X_c) \in \mathcal{Q}^m$, for all $0 \leq c \leq 1$.

For ease of writing the following proof, we will define two functions $I(c,j)$ and $I'(c,j,k)$, where c is a rational number between 0 and 1, and j,k are some natural numbers (j and k would be bounded as seen in the definition below). Intuitively, $I(c,j)$, $I'(c,j,k)$, would give the parameters of the language $\Theta(X_c)$.

Suppose $\Theta(X_c) = L^{\mathcal{Q}^m}_{i_0,i_1,\ldots,i_m}$.

Then, (i) For $0 \leq j \leq m$, $I(c,j) = i_j$ (we will not be defining or using $I(c,j)$ for $j > m$).

Intuitively, $I(c,j)$ gives the j-th parameter of the language $\Theta(X_c)$.

(ii) Suppose $1 \leq j \leq m$. Suppose $1 \leq k \leq |Q^{i_{j-1}}|$. Suppose $I(c,j) = i_j = \langle x_1, x_2, \ldots, x_{|Q^{i_{j-1}}|}\rangle$. Then, $I'(c,j,k) = x_k$.

Intuitively, $I'(c,j,k)$ gives the k-th component of the j-th parameter of $\Theta(X_c)$.

Now we proceed with the proof. Initially let $s = 0.1$ and $r = 0.9$. The idea is to iteratively cause "stabilization" of each of the parameters by progressively narrowing down the range $[s,r]$. Eventually this would give us that $\Theta(X_s) = \Theta(X_r) = \Theta(X_c)$ for $s \leq c \leq r$, for some $s < r$, causing a contradiction.

The following construction is non-effective (effectiveness is not needed for the argument). The following method of diagonalization cannot be made effective, though we do not know if there are other effective ways of doing the diagonalization.

For $i = 1$ to m do

(* Invariant 1: For all $c, d \in \mathbf{rat}$ such that $s \leq c < d \leq r$, $I(c,t) = I(d,t)$, for $t < i$. *)

(* Note that invariant 1 trivially holds for $i = 1$, since $I(c,0) = I(d,0)$. *)

(* At the end of last (m-th) iteration of the loop the above invariant 1 will hold for $i = m+1$ *)

(* This loop tries to stabilize the i-th parameter of the reduction. *)

Let $Q = Q^{I(s,i-1)}$.

For $w = 1$ to $|Q|$ do

(* Invariant 2: For all $c, d \in \mathbf{rat}$ such that $s \leq c < d \leq r$, $I'(c,i,t) = I'(d,i,t)$, for $1 \leq t < w$. *)

(* Note that invariant 2 trivially holds for $w = 1$, since there is no t with $1 \leq t < w$. *)

(* At the end of last ($|Q|$-th) iteration of the loop the above invariant 2 will hold for $w = |Q| + 1$ *)

(* This inner loop tries to stabilize w-th component of the i-th parameter of the reduction *)

(* If w-th component of Q is *SINGLE* or *COSINGLE*, then the following is not needed, since invariant 2 is already satisfied for the next iteration. However we need this when w-th component of Q is *INIT* or *COINIT*. *)

[1] For the definition of $\mathcal{Q}^*$ we assume that there is some uniform way in which one can determine the size of the tuples, for example by coding any tuple x in N^k, as $\langle k, x\rangle$.

Let $diff = ABS(I'(s,i,w) - I'(r,i,w))$ (where ABS gives the absolute value).
Let $c_1, c_2, \ldots, c_{diff+2}$, be such that
$s < c_1 < c_2 < \cdots < c_{diff+2} < r$.
Now by invariants 1 and 2, and by monotonicity of Θ (with respect to monotonicity of the input language), there must exist l, $1 \leq l \leq diff + 1$, such that
$I'(c_l, i, w) = I'(c_{l+1}, i, w)$.
Let $s = c_l$ and $r = c_{l+1}$.
(* Note that invariant 2 is satisfied, for the next iteration. *)

EndFor

(* Note that invariant 1 is satisfied, for the next iteration. *)

EndFor

End

Now note that each loop is executed only finitely many times. Thus at the end of the algorithm we will have $s < r$, and by invariant 1,

for all $c, d \in \mathbf{rat}$ such that $s \leq c < d \leq r$, $I(c,i) = I(d,i)$, for $i < m+1$.

Thus, $\Theta(X_s) = \Theta(X_r) = \Theta(X_c)$ for $s \leq c \leq r$. A contradiction to Θ (along with Ψ) witnessing $RINIT_{0,1} \leq^{\mathbf{TxtEx}^a}_{\text{strong}} \mathcal{Q}^*$. ∎

6 Weak Degrees and Their Characterizations

In this section we will consider the structure of degrees of weak reducibility (or simply weak degrees) above and below the classes considered in the previous section.

First note that for weak-reductions, $INIT$, $FINITE$, $COSINGLE$ are $\leq^{\mathbf{TxtEx}}_{\text{weak}}$-complete [JS96]. We will be giving a characterization of $\leq^{\mathbf{TxtEx}}_{\text{weak}}$-complete classes in Subsection 6.2 below. Consequently, in Subsection 6.1 we focus on classes involving $COINIT$ and $SINGLE$. We will give a characterization of classes which are weak-reducible to $COINIT$ or $SINGLE$, and the classes to which $COINIT$ or $SINGLE$ are weak-reducible. We will also consider a hierarchy for $\mathcal{L}^{Q,R}$ when each component of Q belongs to $\{COINIT, SINGLE\}$.

6.1 Incomplete Weak Degrees

The following theorem gives the characterization of classes of languages which are weak-reducible to $SINGLE$.

Theorem 19 $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{weak}}$ *SINGLE iff there exist F, a partial recursive function from SEQ to N, and G, a partial limit recursive mapping from N to N, such that*

(a) For all σ, τ, if $\sigma \subseteq \tau$, then $[\mathbf{F}(\sigma)\downarrow \Rightarrow F(\tau)\downarrow = F(\sigma)]$.

For any text T, let $F(T) = \lim_{n\to\infty} F(T[n])$.

(b) For any text T for $L \in \mathcal{L}$, $F(T)$ is defined.

(c) For any text T for $L \in \mathcal{L}$, $G(F(T))$ converges to a grammar for L.

Proof. (Only if direction) Suppose $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{weak}}$ $SINGLE$ via Θ and Ψ. Without loss of generality assume that $\text{card}(\text{content}(\Theta(\sigma))) \leq 1$, for all σ. Define F and G as follows.

$F(\sigma) = x$, if $\text{content}(\Theta(\sigma)) = \{x\}$. $F(\sigma)$ is undefined if $\text{content}(\Theta(\sigma)) = \emptyset$.

Define $G(w)$ as follows: Let p be a grammar for $\{w\}$. Let $G(w) = $ limit (if any) of $\Psi(\alpha_p)$, where $\alpha_p = p, p, p, \ldots$.

It is easy to verify that F, G satisfy requirements (a) to (c) of the theorem.

(If direction) Suppose F, G satisfying requirements (a) to (c) of the theorem are given. To show that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{weak}}$ *SINGLE*, define Θ and Ψ as follows.

$\Theta(T)$ is defined so that $\text{content}(\Theta(T)) = \{F(T[n]) \mid F(T[n])\downarrow\}$.

Suppose a sequence α of grammars converges to grammar p. Then, $\Psi(\alpha)$ converges to $G(\min(W_p))$ (if defined).

It can be easily verified that the above Θ, Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{weak}}$ *SINGLE*. ∎

The following theorem gives the surprising result that classes of languages to which *SINGLE* is weak-reducible are the same as the classes of languages to which *SINGLE* is strong-reducible. Thus, one can get a characterization of classes to which *SINGLE* is weak-reducible by using the characterization of classes to which *SINGLE* is strong-reducible. Note that in contrast to the following result, the *lower* cones of weak and strong reducibility (with respect to *SINGLE*) *differ*. To see this, consider: $\mathcal{L} = \{L \mid L \neq \emptyset \wedge (\forall x \in L)[W_x = L]\}$. Clearly, $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{weak}}$ *SINGLE*. However, it was shown in [JS97] that $\mathcal{L} \not\leq^{\mathbf{TxtEx}}_{\text{strong}}$ *FINITE*, and thus $\mathcal{L} \not\leq^{\mathbf{TxtEx}}_{\text{strong}}$ *SINGLE*.

Theorem 20 *SINGLE* $\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}$ *iff SINGLE* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$.

Proof. Clearly, *SINGLE* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$ implies *SINGLE* $\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}$. Now suppose *SINGLE* $\leq^{\mathbf{TxtEx}}_{\text{weak}}$ $\mathcal{L}$, as witnessed by Θ and Ψ. Define Θ' as follows.

$\Theta'(\#^k) = \#^k$.

$\Theta'(\#^k i\{\#, i\}^j) = \#^k\Theta(i^{j+1})$.

(Θ' on other inputs doesn't matter).

It is easy to verify that Θ', Ψ witness that *SINGLE* $\leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}$. ∎

We now consider classes which are $\leq^{\mathbf{TxtEx}}_{\text{weak}}$-reducible to *COINIT*. Note that the next theorem is similar in spirit to Theorem 8. However, in the case of weak reductions, texts replace languages.

Theorem 21 $\mathcal{L} \leq^{\mathbf{TxtEx}}_{weak}$ *COINIT, iff there exist two functions, F, a partial recursive mapping from SEQ to N, and partial limiting recursive G mapping N to N such that:*

(i) For all τ extending σ, $F(\sigma)\downarrow \Rightarrow F(\tau)\downarrow \leq F(\sigma)$.

For any text T, let $F(T)$ denote $\lim_{n\to\infty} F(T[n])$.

(ii) For any text T for $L \in \mathcal{L}$, $F(T) < \infty$.

(iii) For any text T for $L \in \mathcal{L}$, $G(F(T))$ is a grammar for L.

Proof. (Only if direction) Suppose $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{weak}}$ *COINIT* via Θ and Ψ. Define F and G as follows:

$F(\sigma) = \min(\text{content}(\Theta(\sigma)))$, where if $\text{content}(\Theta(\sigma))$ is empty, then $\min(\text{content}(\Theta(\sigma)))$ is not defined.

$G(w)$ is defined as follows. Let p be a grammar for $\{x \mid x \geq w\}$. Then, $G(w)$ converges to the limit (if any) of $\Psi(\alpha_p)$, where $\alpha_p = p, p, p, \ldots$.

It is easy to verify that F, G satisfy the requirements (i) to (iii) of the theorem.

(If direction) Suppose F and G satisfying requirement (i) to (iii) of the theorem are given.

Let $\Theta(T)$ be defined so that $\text{content}(\Theta(T)) = \{x \mid (\exists n)[x \geq F(T[n])\downarrow]\}$.

Suppose a sequence α of grammars converges to grammar p. Then $\Psi(\alpha)$ converges to $G(\min(W_p))$ (if defined).

It is easy to verify that above Θ and Ψ witness that $\mathcal{L} \leq^{\mathbf{TxtEx}}_{\text{weak}}$ *COINIT*. ∎

A characterization of classes to which *COINIT* is $\leq^{\mathbf{TxtEx}}_{\text{weak}}$-reducible turns out to be quite complex. The main reason for this is that different texts for the same language may be mapped to quite different languages in $\leq^{\mathbf{TxtEx}}_{\mathrm{weak}}$ reduction. The order of presentation of text thus becomes important in the reduction. Before presenting our characterization for classes to which *COINIT* is weak-reducible we consider some definitions.

Definition 30 Let $\mathit{VALID} = \{(x_1, t_1, \ldots, x_k, t_k) \mid k \in N \ \wedge \ x_i > x_{i+1} \text{ for } 1 \leq i < k\}$.

Definition 31 Suppose $(x_1, t_1, \ldots, x_k, t_k)$ and $(x'_1, t'_1, \ldots, x'_l, t'_l) \in \mathit{VALID}$.

Then, $(x_1, t_1, \ldots, x_k, t_k) \leq_{\mathit{VALID}} (x'_1, t'_1, \ldots, x'_l, t'_l)$ iff $[[k \leq l] \ \wedge \ (\forall i \leq k)[x_i = x'_i] \ \wedge \ (\forall i < k)[t_i = t_{i'}] \ \wedge \ [t_k \leq t'_k]]$.

Definition 32 A text T is said to be *nice* iff

$$(\forall n)[T(n+1) = T(n) = \# \ \vee \ T(n+1) = T(n) + 1 \ \vee \ T(n+1) < \min(\mathrm{content}(T[n+1]))]$$

Let $\mathit{NICETEXTS} = \{T \mid T \text{ is nice }\}$.

One can effectively transform a text T into a nice text T' with the following property:

$$\text{If content}(T) \in \mathit{COINIT}, \text{ then content}(T) = \text{content}(T').$$

This can be done as follows:
$T'(0) = T(0)$.
If $T'(n) = \#$ and $T(n+1) = \#$, then $T'(n+1) = \#$;
ElseIf $T(n+1) < \min(\mathrm{content}(T'[n+1]))$, then let $T'(n+1) = T(n+1)$. (Here $\min(\emptyset) = \infty$).
Else (note that in this case $T(n+1) \geq \min(\mathrm{content}(T'[n+1]))$) let $T'(n+1) = T'(n) + 1$.
It is easy to verify that
(1) For all T: $T' \in \mathit{NICETEXTS}$.
(2) If content$(T) \in \mathit{COINIT}$, then content$(T') =$ content(T).

Theorem 22 *COINIT* $\leq^{\mathbf{TxtEx}}_{\mathrm{weak}} \mathcal{L}$ *iff there exists a recursive function H mapping VALID to N, such that*

(a) Let $V_1 = (x_1, t_1, x_2, t_2, \ldots, x_k, t_k)$ and $V_2 = (x'_1, t'_1, x_2, t_2, \ldots, x'_l, t'_l) \in \mathit{VALID}$.
If $V_1 \leq_{\mathit{VALID}} V_2$, then $W_{H(V_1)} \subseteq W_{H(V_2)}$.
(b) $\{\bigcup_{t'_k \in N} W_{H(x_1, t_1, x_2, t_2, \ldots, x_k, t'_k)} \mid (x_1, t_1, x_2, t_2, \ldots, x_k, 0) \in \mathit{VALID}\} \subseteq \mathcal{L}$.
(c) For $(x_1, t_1, \ldots, x_k, t_k)$ and $(x'_1, t'_1, \ldots, x'_l, t'_l) \in \mathit{VALID}$, if $x_k \neq x'_l$, then
$\bigcup_{t'_k \in N} W_{H(x_1, t_1, x_2, t_2, \ldots, x_k, t'_k)} \neq \bigcup_{t'_l \in N} W_{H(x'_1, t'_1, x'_2, t'_2, \ldots, x'_l, t'_l)}$.
(d) Suppose $\mathcal{C} = \{\bigcup_{t'_k \in N} W_{H(x_1, t_1, x_2, t_2, \ldots, x_k, t'_k)} \mid (x_1, t_1, x_2, t_2, \ldots, x_k, 0) \in \mathit{VALID}\}$. Then there exists a partial limit recursive E such that

(i) For all p, p' such that $W_p = W_{p'} \in \mathcal{C}$, $E(p) = E(p')$.
(ii) For all p, p', if for some $(x_1, t_1, x_2, t_2, \ldots, x_k, 0)$ and $(y_1, s_1, y_2, s_2, \ldots, y_l, 0) \in \mathit{VALID}$, with $x_k \neq y_l$,

$$W_p = \bigcup_{t'_k \in N} W_{H(x_1, t_1, x_2, t_2, \ldots, x_k, t'_k)}, \text{ and } W_{p'} = \bigcup_{s'_l \in N} W_{H(y_1, s_1, y_2, s_2, \ldots, y_l, s'_l)},$$

then $E(p) \neq E(p')$.

Proof. For any $V = (x_1, t_1, \ldots, x_k, t_k) \in \mathit{VALID}$, define $\sigma_V = (x_1 + 0, x_1 + 1 \ldots, x_1 + t_1, x_2 + 0, x_2 + 1, \ldots, x_2 + t_2, \ldots, x_k + 0, x_k + 1, \ldots, x_k + t_k)$.

(Only if direction) Suppose *COINIT* $\leq^{\mathbf{TxtEx}}_{\mathrm{weak}} \mathcal{L}$ via Θ and Ψ.
Define H and E as follows.
$W_{H(V)} = \mathrm{content}(\Theta(\sigma_V))$.

$E(p)$= limiting value (if any) of $\min(W_{\Psi(\alpha_p)})$, where $\alpha_p = p, p, p, \ldots$.

It is easy to verify that H satisfies requirements (a) to (c) of the theorem and E witnesses the satisfaction of requirement (d) in the theorem.

(If direction) Suppose that H (satisfying requirements (a) to (c) of the theorem) and E (witnessing the satisfaction of requirement (d) of the theorem) are given. By discussion just before the theorem, it is enough to weak-reduce the class of texts *NICETEXTS* $- \{\#^\infty\}$ to $\mathcal{L}$. We may further ignore the presence of $\#$ in the σ for the construction of Θ (one may just map $\#^\infty$ to $\#^\infty$ and $\#^i\sigma$ to $\#^i\Theta(\sigma)$, where σ is a non-empty sequence which doesn't contain $\#$).

Define Θ as follows.

(Recall that $W_{i,s}$ is W_i enumerated within s steps)

$\text{content}(\Theta(\sigma_V)) = W_{H(V),|\sigma_V|}$.

For each $V = (x_1, t_1, x_2, t_2, \ldots, x_k, 0) \in \mathit{VALID}$, let p_V be a grammar (obtained effectively from V) for $\text{content}(\Theta(\bigcup_{t_k \in N} \sigma_{x_1,t_1,x_2,t_2,\ldots,x_k,t_k}))$.

Define Ψ as follows. Suppose a sequence α of grammars converges to grammar q. Then, $\Psi(\alpha)$ converges to a grammar for $\{x \mid x \geq x_k\}$, such that $E(q) = E(p_V)$, where $V = (x_1, t_1, x_2, t_2, \ldots, x_k, 0) \in \mathit{VALID}$.

(If several different V's satisfy the above, then it doesn't matter which one is picked).

It is easy to verify that Θ, Ψ witness that *COINIT* $\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}$. ∎

We now turn our attention to the hierarchy formed for $\mathcal{L}^{Q,R}$. Note that if a component of Q is *INIT* or *COSINGLE*, then $\mathcal{L}^{Q,R}$ would be $\leq^{\mathbf{TxtEx}}_{\text{weak}}$-complete. Thus, the only cases of interest are when components of Q are only from *COINIT*, *SINGLE*. Moreover, by using Proposition 23, we may assume that *SINGLE* appears only at the end of Q.

Proposition 34 *Suppose $Q = (\mathit{COINIT}, \mathit{SINGLE})$, $R = R_1 \times R_2$, $Q' = (\mathit{COINIT})$ and $R' = R_1'$, where R_1, R_2 and R_1' are infinite subsets of N. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q',R'}$.*

Proof. Suppose by way of contradiction Θ and Ψ witness that $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q',R'}$. Let i_1 be a non-minimal element in R_1. Consider $L^Q_{i_1,i_2}$ such that $i_2 \in R_2$. Let σ be such that $\text{content}(\sigma) \subseteq L^Q_{i_1,i_2}$, and $\text{content}(\Theta(\sigma)) \neq \emptyset$. Now, there are infinitely many languages in $\mathcal{L}^{Q,R}$ containing $\text{content}(\sigma)$ (for example, all languages $L^Q_{i_1',j}$ such that $i_1' < i_1$, $i_1' \in R_1$ and $j \in R_2$), but only finitely many languages in $\mathcal{L}^{Q',R'}$ containing $\text{content}(\Theta(\sigma))$. It follows that Θ, Ψ cannot witness $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q',R'}$. Proposition follows. ∎

We now consider the hierarchy for $\mathcal{L}^{Q,R}$, for weak-reduction when components of Q are from *COINIT*, *SINGLE*. The hierarchy result is given by Corollary 4 below.

Theorem 23 *Suppose $Q = (q_1, \ldots, q_{k+1})$, and $Q' = (q_1', \ldots, q_{k+1}', q_{k+2}')$, where $q_i = q_i' = \mathit{COINIT}$, for $1 \leq i \leq k+1$, and $q_{k+2}' = \mathit{SINGLE}$. Suppose $R = R_1 \times R_2 \times \cdots \times R_{k+1}$, $R' = R_1' \times R_2' \times \cdots \times R_{k+2}'$, where each R_i, R_i' is an infinite subset of N, except for R_1' which is finite. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q',R'}$.*

Proof. We prove the theorem by induction on k.

We first consider the base case of $k = 0$. Thus, we need to show that $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q',R'}$, where $Q = (\mathit{COINIT})$, $Q' = (\mathit{COINIT}, \mathit{SINGLE})$ and $R = (R_1)$, $R' = (R_1', R_2')$, where R_1' is finite, and R_1, R_2' are infinite. Now if $L^{Q'}_{i,j} \subset L^{Q'}_{i',j'}$, then $i > i'$. Now, suppose f and g are functions such that $\Theta(L^Q_i) = L^{Q'}_{f(i),g(i)}$. Then it follows that $i < i'$ implies $f(i) < f(i')$. But this is impossible, since domain of f is infinite, but range of f is finite.

Now suppose by induction that the theorem holds for $k = n$. We show that the theorem holds for $k = n + 1$.

Suppose by way of contradiction that Θ and Ψ witness $\mathcal{L}^{Q,R} \leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q',R'}$.

Claim 1 *There exists an $i_1 \in R_1$ (which is not the least element of R_1), and a σ such that*

(1) $\text{content}(\sigma) \subseteq L^{Q,R}_{i_1,0,0,\ldots}$.

(2) $\text{content}(\Theta(\sigma))$ *contains $\langle i'_1, \ldots\rangle$, for some $i'_1 \in R'_1$.*

(3) Let ii_1 be the maximum element in R_1 which is less than i_1. Then, for all $i_2 \in R_2, i_3 \in R_3, \ldots$, for any τ extending σ such that $\text{content}(\tau) \subseteq L^{Q,R}_{ii_1,i_2,\ldots}$, $\text{content}(\Theta(\sigma))$ *does not contain any element of the form $\langle x, \ldots\rangle$, with $x < i'_1$. In other words, Θ maps any text extending σ for $L^{Q,R}_{ii_1,i_2,\ldots}$, to a text for a language of form $L^{Q',R'}_{i'_1,\ldots}$.*

Proof. Consider $m_1 \in R_1$ such that there are more elements in R_1 below m_1 than the number of elements in R'_1. Now, if the claim is false, then one could start with a σ_{m_1}, such that $\text{content}(\sigma_{m_1}) \subseteq L^{Q,R}_{m_1,0,0,\ldots}$, and $\text{content}(\Theta(\sigma_{m_1}))$ contains $\langle x_1, \ldots\rangle$, $x_1 \in R_1$. Then, one can inductively define $\sigma_{m_2}, \sigma_{m_3}, \ldots$ (where $m_2, m_3, \ldots$ are elements of R_1 smaller than m_1 in descending order), along with $x_2, x_3, \ldots$, such that $\text{content}(\sigma_{m_w}) \subseteq L^{Q,R}_{m_w,0,0,\ldots}$, and $\text{content}(\Theta(\sigma_{m_w}))$ contains $\langle x_w, \ldots\rangle$, $x_w \in R'_1$, where $x_1 > x_2 > x_3 \ldots$. But this is impossible (since R_1 has more elements below m_1 than the number of elements in R'_1). This proves the claim. □

Now fix i_1, i'_1, σ as in the Claim. Let i'_2 be such that $\text{content}(\Theta(\sigma))$ contains $\langle i'_1, i'_2, \ldots\rangle$. Thus, it immediately follows that Θ (along with Ψ) can be used for a $\leq^{\mathbf{TxtEx}}_{\text{weak}}$-reduction from $\mathcal{L}^{QQ,RR}$ to $\mathcal{L}^{QQ',RR'}$, where QQ, QQ' are obtained from Q, Q' by dropping the first *COINIT*, RR is obtained from R by dropping R_1, and RR' is obtained from R' by dropping R'_1 and changing R'_2 to $R'_2 \cap \{x \mid x \leq i'_2\}$. A contradiction to the induction hypothesis. ∎

Corollary 2 *Suppose $Q = (q_1, q_2, \ldots, q_k)$, and $Q' = (q'_1, q'_2, \ldots, q'_k)$, where for $1 \leq i < k$, $q_i = q'_i = COINIT$, and $q_k = COINIT$, and $q'_k = SINGLE$. Then, $\mathcal{L}^{Q} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q'}$.*

Let $Q = (SINGLE)$, $R = R_1$, $Q' = (COINIT)$, and $R' = R'_1$, where R_1 is infinite and R'_1 is finite. Then it can be easily seen that $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q',R'}$.

Now by using similar induction as in the proof of Theorem 23, one can show

Theorem 24 *Suppose $Q = (q_1, \ldots, q_{k+1})$, and $Q' = (q'_1, \ldots, q'_{k+1})$, where $q_i = q'_i = COINIT$, for $1 \leq i < k+1$, $q_{k+1} = SINGLE$ and $q'_{k+1} = COINIT$. Suppose $R = R_1 \times R_2 \times \cdots \times R_{k+1}$, $R' = R'_1 \times R'_2 \times \cdots \times R'_{k+1}$, where each R_i, R'_i is an infinite subset of N, except for R'_1 which is finite. Then $\mathcal{L}^{Q,R} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q',R'}$.*

Corollary 3 *Suppose $Q = (q_1, q_2, \ldots, q_k, q_{k+1})$, and $Q' = (q'_1, q'_2, \ldots, q'_k)$, where for $1 \leq i \leq k$, $q_i = q'_i = COINIT$, and $q_{k+1} = SINGLE$. Then, $\mathcal{L}^{Q} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q'}$.*

Corollary 4 *Suppose $Q = (q_1, q_2, \ldots, q_k), Q' = (q'_1, q'_2, \ldots, q'_l)$, where each q_i, $1 \leq i < k$, and q'_i, $1 \leq i < l$, is COINIT, and q_k, q'_l are members of $\{COINIT, SINGLE\}$. Then, $\mathcal{L}^{Q} \leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q'}$ iff $k < l$ OR $[k = l$ and $q_k = COINIT \Rightarrow q'_l = COINIT]$.*

Proof. If direction follows from Theorem 15. We prove the only if direction.

We consider the following two cases:

Case 1: $k > l$.

Let $Q'' = (q''_1, q''_2, \ldots, q''_l, q''_{l+1})$ and $Q''' = (q''_1, q''_2, \ldots, q''_l)$, where, for $1 \leq i \leq l$, $q''_i = COINIT$ and $q''_{l+1} = SINGLE$. Then, by Corollary 3 we have that $\mathcal{L}^{Q''} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q'''}$. Since $\mathcal{L}^{Q'} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q'''}$ and $\mathcal{L}^{Q''} \leq^{\mathbf{TxtEx}}_{\text{strong}} \mathcal{L}^{Q}$ (by Theorem 15), we have that $\mathcal{L}^{Q} \not\leq^{\mathbf{TxtEx}}_{\text{weak}} \mathcal{L}^{Q'}$.

Case 2: $k = l$, $q_k = COINIT$, and $q_l = SINGLE$.

Then, from Corollary 2 we have that $\mathcal{L}^Q \not\leq^{\mathbf{TxtEx}}_{\mathrm{weak}} \mathcal{L}^{Q'}$.

Corollary follows from above two cases. ∎

Note that above gives $\leq^{\mathbf{TxtEx}}_{\mathrm{weak}}$-relation among all the classes $\mathcal{L}^Q$, with components of Q coming from $\{COINIT, SINGLE\}$. This is so, since by Proposition 23 and Proposition 25, $SINGLE$ can be assumed to appear in Q at most once and at the end (if it appears).

One can get characterizations of classes below and above $\mathcal{L}^Q$ with respect to weak-reduction in a spirit similar to that of Theorems 21 and 22. However, since they become technically quite complex, we omit them here.

6.2 Complete Weak Degrees

We first give a characterization of $\leq^{\mathbf{TxtEx}^a}_{\mathrm{weak}}$-complete classes, for all $a \in N$.

Definition 33 A non-empty class $\mathcal{L}$ of languages is called *quasi-dense* iff

(a) $\mathcal{L}$ is 1–1 recursively enumerable.

(b) For any $L \in \mathcal{L}$ and any finite $S \subseteq L$, there exists an $L' \in \mathcal{L}$, such that $S \subseteq L'$, but $L \neq L'$.

Note: (b) can be equivalently replaced by

(b') For any finite set S, either there exists no language in $\mathcal{L}$ extending S, or there exist infinitely many languages in $\mathcal{L}$ extending S.

Definition 34 A non-empty r.e. class $\mathcal{T}$ of texts is called *quasi-dense* iff

(a) For distinct $T, T' \in \mathcal{T}$, $\mathrm{content}(T) \neq \mathrm{content}(T')$.

(b) For each σ, either there exists no text in $\mathcal{T}$ extending σ, or there exist at least two distinct texts in $\mathcal{T}$ extending σ.

Note: (b) can be equivalently replaced by

(b') For each σ, either there exists no text in $\mathcal{T}$ extending σ, or there exist infinitely many texts in $\mathcal{T}$ extending σ.

Proposition 35 *Suppose $\mathcal{T}$ is a quasi-dense class of texts. Then there exists a quasi-dense class $\mathcal{L}$ such that* $\{\mathrm{content}(T) \mid T \in \mathcal{T}\} \supseteq \mathcal{L}$.

Proof. Suppose $\mathcal{T} = \{T_i \mid i \in N\}$, where T_i can be obtained effectively from i, and $\mathrm{content}(T_i) \neq \mathrm{content}(T_j)$, if $i \neq j$. Let $L_i = \mathrm{content}(T_i)$. Clearly, $\mathcal{L} = \{L_i \mid i \in N\}$ is a quasi-dense class of languages, and $\{\mathrm{content}(T) \mid T \in \mathcal{T}\} = \mathcal{L}$. ∎

Proposition 36 *Suppose $\mathcal{L}$ is quasi-dense class of languages. Then there exists a quasi-dense class of texts, $\mathcal{T}$, such that* $\{\mathrm{content}(T) \mid T \in \mathcal{T}\} \subseteq \mathcal{L}$.

Proof. Let L_0, L_1, ..., be a 1–1 recursive enumeration of all the languages in $\mathcal{L}$.

Define T_0, T_1, ..., as follows:

Let σ_0, σ_1, ..., be a recursive enumeration of all finite sequences such that every finite sequence appears infinitely often in the enumeration.

Let T_0 be a text for L_0.

Let $Used = \{0\}$ (intuitively, $Used$ denotes the L_j's which we have already used in construction of earlier T_i's). Go to stage 0.

Stage i: Definition of T_{i+1}.

Let $j \geq i$, be least number such that σ_j is a prefix of some T_k, $k \leq i$.

Search for L_r such that, $r \notin \mathit{Used}$, and content$(\sigma_j) \subseteq L_r$.

Define T_{i+1} as a text for L_r which extends σ_j. Let $\mathit{Used} = \mathit{Used} \cup \{r\}$.

Go to stage $i+1$.

Clearly, the above sequence of texts is 1–1 (content-wise). We thus only need to show that for any prefix σ of T_j, there is another T_r, which extends σ. For this let $k > j$, be least number such that $\sigma = \sigma_k$. Now T_k must extend σ. ∎

Proposition 37 *Suppose $a \in N \cup \{*\}$. If $\mathcal{L}$ is a-limiting standardizable, then for all $L, L' \in \mathcal{L}$, either $L = L'$ or $L \neq^{2a} L'$.*

Proof. Suppose by way of contradiction that $L, L' \in \mathcal{L}$ are different but $L =^{2a} L'$. Let i be such that $W_i = L$. Let i' be such that $W_{i'} = L'$. Let i'' be such that $W_{i''} =^a L$ and $W_{i''} =^a L'$.

Note that such i, i', i'' exist. Suppose F witnesses a-limiting standardizability of $\mathcal{L}$. Then, $F(i) = F(i'')$, and $F(i'') = F(i')$, but $F(i) \neq F(i')$. A contradiction. ∎

By a slight modification of the definition of $\leq^{\mathbf{TxtEx}^a}_{\text{weak}}$, we say that $\mathcal{T} \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{L}$, if there exist Θ and Ψ such that, for all $T \in \mathcal{T}$, $\Theta(T)$ is a text for some $L \in \mathcal{L}$, and for any infinite sequence α of grammars being $\mathbf{TxtEx}^a$-admissible for $\Theta(T)$, $\Psi(\alpha)$ converges to a grammar for an a-variant of content(T). One can similarly define $\mathcal{T} \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{T}'$, $\mathcal{L} \leq^{\mathbf{TxtEx}^a}_{\text{weak}} \mathcal{T}$ and $\leq^{\mathbf{TxtEx}^a}_{\text{weak}}$-completeness of $\mathcal{T}$.

Let $\mathbf{RESFIN} = \{T \mid (\forall i \mid i = 0 \ \vee \ T(i) \neq T(i-1))[T(i) \in \{\langle x, i\rangle \mid x \in N\}] \ \wedge$ card(content$(T)) < \infty\}$.

Intuitively, **RESFIN** is a subset of texts for languages in *FINITE*, with some special properties. (**RES** in **RESFIN** above stands for restricted). Texts in **RESFIN** "code" every position, where the next element differs from the previous one. The properties that we need include the facts that **RESFIN** is $\leq^{\mathbf{TxtEx}}_{\text{weak}}$-complete, and all the texts in it are pairwise different. For our purposes of characterization it turns out that **RESFIN** is more suitable to use than the class of all the texts for *FINITE* or *INIT*.

It is easy to verify that **RESFIN** is quasi-dense, r.e. class of texts.

For $T \in \mathbf{RESFIN}$, for $a \in N$, let T^a be defined as follows: For $j < a$, and any $i \in N$, let $T^a(a * i + j) = \langle T(i), j\rangle$.

For $a \in N$, let $\mathbf{RESFIN}^a = \{T^a \mid T \in \mathbf{RESFIN}\}$.

Proposition 38 *Suppose $a \in N$.*

(a) $\mathbf{RESFIN}^a$ is a quasi-dense class of texts.

(b) For all distinct texts T, $T' \in \mathbf{RESFIN}^a$, content$(T) \neq^{a-1}$ content(T').

(c) $\{\text{content}(T) \mid T \in \mathbf{RESFIN}^a\} \in \mathbf{TxtEx}$.

(d) $\{\text{content}(T) \mid T \in \mathbf{RESFIN}^{2a+1}\}$ is a-limiting standardizable.

(e) $\mathbf{RESFIN}^{2a+1}$ is $\leq^{\mathbf{TxtEx}^a}_{\text{weak}}$-complete.

Proof. Parts (a) to (c) in the above proposition can be easily proved using the definition of $\mathbf{RESFIN}^a$. Part (e) can be shown essentially along the lines of the proof of *FINITE* being $\leq^{\mathbf{TxtEx}}_{\text{weak}}$-complete in [JS96]. We omit the details.

We show part (d). Define F as follows. For any p, let $F(p)$ be the canonical index for $\{x \mid \text{card}(\{y \mid \langle x, y\rangle \in W_p\}) \geq a+1\}$, if W_p is finite. $F(p)$ is undefined if W_p is infinite. It is easy to verify that F is partial limit recursive. Moreover, if $W_p =^a$ content(T^{2a+1}), for some

$T \in \mathbf{RESFIN}$, then $F(p)$ is the canonical index for content(T). It follows that F witnesses a-limiting standardizability of $\{\text{content}(T) \mid T \in \mathbf{RESFIN}^{2a+1}\}$. ▮

Proposition 39 *Suppose $\mathcal{T}$ is quasi-dense class of texts. Suppose Θ is a recursive mapping from $\mathcal{T}$ to $\Theta(\mathcal{T})$, such that for all $T, T' \in \mathcal{T}$,* content$(\Theta(T)) \neq$ content$(\Theta(T'))$*. Then, $\mathcal{T}' = \Theta(\mathcal{T})$ is quasi-dense. Furthermore, if $\mathcal{T}$ is r.e., then so is $\mathcal{T}'$.*

PROOF. Obvious. ▮

Proposition 40 *Suppose $\mathcal{T}$ is an r.e. class of texts, and $\mathcal{T}'$ is an r.e. quasi-dense class of texts. Then, one can define a recursive operator Θ such that (i) $\Theta(\mathcal{T}) \subseteq \mathcal{T}'$, and (ii) for distinct $T, T' \in \mathcal{T}$, $\Theta(T) \neq \Theta(T')$.*

PROOF. Let $T_0, T_1, \ldots,$ be a 1–1 enumeration of $\mathcal{T}$. Let $T'_0, \ldots,$ be a 1–1 enumeration of $\mathcal{T}'$. Let $\sigma_0, \sigma_1, \ldots,$ be a 1–1 enumeration of all the finite sequences.

We will define $\Theta(\cdot)$ in stages below. In odd stage $2s+1$, we would define $\Theta(T_s[m])$, for all $m \in N$. In even stage $2s$ we would define $\Theta(\sigma_s)$ (if not defined already). We will maintain the following invariants:

(i) For all σ, $|\sigma| = |\Theta(\sigma)|$.

(ii) If $\Theta(\sigma)$ has been defined by some stage, then we would have (by that stage), for all $\tau \subseteq \sigma$, $\Theta(\tau) = \Theta(\sigma)[|\tau|]$.

Similarly, for σ replaced by any text $T \in \mathcal{T}$ in previous statement.

(iii) For any σ, there exists a text $T \in \mathcal{T}$, such that $\Theta(\sigma) \subseteq T$.

(iv) Before the start of any stage $2s$ or $2s+1$, Θ would have been defined for $\Theta(T_{s'}[n])$, for $s' < s$, $n \in N$. $\Theta(\sigma)$, would have been defined for only finitely many σ such that $\sigma \not\subseteq T$, for any $T \in \{T_0, T_1, \ldots, T_{s-1}\}$.

We let $\Theta(\Lambda) = \Lambda$.

Stage $2s$:

If $\Theta(\sigma_s)$ has not been defined upto now, then let $\sigma \subseteq \sigma_s$ be the largest prefix of σ_s such that $\Theta(\sigma)$ has been defined upto now. Let $T \in \mathcal{T}$ be such that $\Theta(\sigma) \subseteq T$. Then, for σ' such that $\sigma \subseteq \sigma' \subseteq \sigma'$, let $\Theta(\sigma') = T[|\sigma'|]$.

Stage $2s+1$:

Let $T_s[m]$ be the largest prefix of T_s such that $\Theta(T_s[m])$ has been defined upto now. (Note that there exists such a largest m, since T_s is different from all $T_{s'}$, $s' < s$). Let $T' \in \mathcal{T}'$ be an extension of $\Theta(T_s[m])$ such that T' is different from all of $\Theta(T_j)$, $j < s$. Then, let $\Theta(T_s[m']) = T'[m']$, for $m' > m$. Thus, $\Theta(T_s) = T'$.

It is easy to verify that a Θ such as above can be easily constructed, and Θ satisfies the requirements of the proposition. ▮

Theorem 25 *For any $a \in N$ and $\mathcal{L} \in \mathbf{TxtEx}^a$, $\mathcal{L}$ is $\leq_{\text{weak}}^{\mathbf{TxtEx}^a}$-complete iff there exists an r.e. quasi dense class of texts $\mathcal{T}$ representing a subclass of $\mathcal{L}$ such that* $\{\text{content}(T) \mid T \in \mathcal{T}\}$ *is a-limiting standardizable.*

PROOF. (Only if direction): We need to show that every $\leq_{\text{weak}}^{\mathbf{TxtEx}^a}$-complete class has the properties claimed in the theorem.

Suppose $\mathcal{L}$ is $\leq_{\text{weak}}^{\mathbf{TxtEx}^a}$-complete. Clearly $\mathcal{L} \in \mathbf{TxtEx}^a$. Also, there exist Θ and Ψ witnessing $\{\text{content}(T) \mid T \in \mathbf{RESFIN}^{2a+1}\} \leq_{\text{weak}}^{\mathbf{TxtEx}^a} \mathcal{L}$ (since $\{\text{content}(T) \mid T \in \mathbf{RESFIN}^{2a+1}\} \subseteq$ *FINITE*, and *FINITE* $\in \mathbf{TxtEx}^a$).

We first claim that for any two distinct texts $T, T' \in \mathbf{RESFIN}^{2a+1}$, $\text{content}(\Theta(T)) \neq^{2a} \text{content}(\Theta(T'))$. Suppose by way of contradiction that T, T' are distinct but $\text{content}(\Theta(T)) =^{2a} \text{content}(\Theta(T'))$. Let q be such that $\text{content}(\Theta(T)) =^a W_q =^a \text{content}(\Theta(T'))$. Then, $\Psi(q, q, \ldots)$ must converge to a grammar for an a-variant of both content(T) and content(T'). However, since $\text{content}(T) \neq^{2a} \text{content}(T')$, this is impossible. It follows that any two distinct texts in $\Theta(\mathbf{RESFIN}^{2a+1})$ must have content differing by at least $2a+1$ elements.

Let $\mathcal{T}' = \{\Theta(T) \mid T \in \mathbf{RESFIN}^{2a+1}\}$. It follows using Proposition 39 that $\mathcal{T}'$ is an r.e. quasi-dense class of texts.

Now, suppose F is an a-limiting standardizing function for $\{\text{content}(T) \mid T \in \mathbf{RESFIN}^{2a+1}\}$. Let $F'(q)$ be defined as follows. Suppose $\alpha_q = q, q, q, \ldots$, and $\Psi(\alpha_q)$ converges to w. Then $F'(q) = F(w)$. It is easy to verify that F' is an a-limiting standardizing function for $\{\text{content}(T) \mid T \in \mathcal{T}'\}$.

(If direction): Suppose $\mathcal{T}$ represents a subclass of $\mathcal{L}$ such that $\mathcal{T}$ is an r.e. quasi-dense class of texts and $\{\text{content}(T) \mid T \in \mathcal{T}\}$ is a-limiting standardizable via F.

Recall that $\mathbf{RESFIN}^{2a+1}$ is $\leq_{\text{weak}}^{\mathbf{TxtEx}^a}$-complete by Proposition 38(e). Let Θ be any operator such that (i) $\Theta(\mathbf{RESFIN}^{2a+1}) \subseteq \mathcal{T}$, and (ii) for distinct $T, T' \in \mathbf{RESFIN}^{2a+1}$, $\Theta(T) \neq \Theta(T')$ (by Proposition 40, such a Θ exists).

For $T \in \mathbf{RESFIN}^{2a+1}$, let g_T be a grammar (obtainable effectively from T) for content(T), and h_T be a grammar (obtainable effectively from T) for content$(\Theta(T))$. Note that $F(h_T) \neq F(h_{T'})$, for any two distinct texts $T, T' \in \mathbf{RESFIN}^{2a+1}$ (since $\text{content}(\Theta(T)) \neq \text{content}(\Theta(T'))$).

Let Ψ be defined as follows.

Suppose a sequence α of grammars converges to a grammar q. Then $\Psi(\alpha)$ converges to g_T, such that $F(q) = F(h_T)$ (if there is any such $T \in \mathbf{RESFIN}^{2a+1}$). Note that, for $T \in \mathbf{RESFIN}^{2a+1}$, if q is indeed a grammar for an a-variant of content$(\Theta(T))$, then T is the unique text in $\mathbf{RESFIN}^{2a+1}$, such that $F(q) = F(h_T)$, and this T can be found in the limit.

It is easy to verify that Θ and Ψ witness that $\mathbf{RESFIN}^{2a+1} \leq_{\text{weak}}^{\mathbf{TxtEx}^a} \mathcal{L}$. Thus, $\mathcal{L}$ is $\leq_{\text{weak}}^{\mathbf{TxtEx}^a}$-complete. ∎

The following theorem characterizes complete weak degrees in terms of their algorithmic (standardizability) and topological (quasi-density) potentials.

Theorem 26 *For any $a \in N$ and $\mathcal{L} \in \mathbf{TxtEx}^a$, $\mathcal{L}$ is $\leq_{\text{weak}}^{\mathbf{TxtEx}^a}$-complete iff there exists a quasi-dense subclass of $\mathcal{L}$ which is a-limiting standardizable.*

Proof. The theorem follows from Theorem 25 and Propositions 35 and 36. ∎

We now characterize $\leq_{\text{weak}}^{\mathbf{TxtEx}^*}$-complete classes.

For any $T \in \mathbf{RESFIN}$, define T^* as follows:

$T^*(i) = \langle \text{content}(T[i]), i \rangle$.

(Here and below, for ease of notation we are allowing finite sets as parameters for the pairing function. One could always replace such parameters by the canonical indices for the finite sets).

Let $\mathbf{RESFIN}^* = \{T^* \mid T \in \mathbf{RESFIN}\}$.

Proposition 41 *(a) $\mathbf{RESFIN}^*$ is a quasi-dense class of texts.*

(b) For all distinct texts T, $T' \in \mathbf{RESFIN}^*$, $\text{content}(T) \neq^* \text{content}(T')$.

(c) $\{\text{content}(T) \mid T \in \mathbf{RESFIN}^*\} \in \mathbf{TxtEx}$.

(d) $\{\mathrm{content}(T) \mid T \in \mathbf{RESFIN}^*\}$ *is $*$-limiting standardizable.*
(e) $\mathbf{RESFIN}^*$ *is* $\leq_{\mathrm{weak}}^{\mathbf{TxtEx}^*}$*-complete.*

Proof. Parts (a) to (c) in above proposition can be easily proved using the definition of $\mathbf{RESFIN}^*$. For part (d) define E as follows. $E(p)$ converges to the canonical index for S, if there exists a $j \in N$ such that $\langle S, j\rangle \in W_p$, and for all $S' \in \mathit{FINITE}$, $S' \neq S$, $\langle S', k\rangle \notin W_p$, for $k \geq j$. Now, if W_p is a $*$-variant for content(T^*), for some $T \in \mathbf{RESFIN}$, then, $E(p)$ converges to the canonical index for content(T). Thus, E witnesses $*$-limiting standardizability of $\{\mathrm{content}(T) \mid T \in \mathbf{RESFIN}^*\}$. Part (e) can now be shown essentially along the lines of the proof of *FINITE* being $\leq_{\mathrm{weak}}^{\mathbf{TxtEx}}$-complete in [JS96]. ∎

Theorem 27 *$\mathcal{L}$ is $\leq_{\mathrm{weak}}^{\mathbf{TxtEx}^*}$-complete iff $\mathcal{L} \in \mathbf{TxtEx}^*$ and there exists a r.e. quasi dense class of texts, $\mathcal{T}$, representing a subclass of $\mathcal{L}$ such that $\{\mathrm{content}(T) \mid T \in \mathcal{T}\}$ is $*$-limiting standardizable.*

Proof. The above theorem can be proved similarly to Theorem 25 except that we use $\mathbf{RESFIN}^*$ instead of $\mathbf{RESFIN}^{2a+1}$. We omit the details. ∎

Theorem 28 *$\mathcal{L}$ is $\leq_{\mathrm{weak}}^{\mathbf{TxtEx}^*}$-complete iff $\mathcal{L} \in \mathbf{TxtEx}^*$, and there exists a subclass $\mathcal{L}'$ of $\mathcal{L}$ such that $\mathcal{L}'$ is quasi-dense, and $*$-limiting standardizable.*

Proof. Follows from Theorem 27 and Propositions 35 and 36. ∎

7 Conclusions

The formalisms and results obtained in the paper are of two types:

a) Formalisms, hierarchies, and characterizations for classes of "multidimensional" languages, where information learned from one "dimension" aids to learn another one. The characterizations define set-theoretical and algorithmic properties of such classes.

b) The characterizations of complete degrees. These characterizations specify algorithmic and topological properties of classes in the complete degrees. A new natural powerful class of languages complete for strong reductions has been discovered.

The results for "multidimensional" languages reveal a new variety of learning strategies, which, to learn a "dimension", use previously learned information to find the right "subspace", or a previously learned "pattern" specifying a learning "substrategy" for the next "dimension". As far as the former approach is concerned, the picture of hierarchies based on "core" classes *SINGLE*, *COSINGLE*, *INIT*, *COINIT* (*SINGLE*, *COINIT* for weak reductions) has been completed. The latter approach is implemented in the form of classes $\mathcal{Q}^m$ and $\mathcal{Q}^*$, see Definition 29. There is a number of interesting open problems related to these classes, as well as to the formalism as a whole:

a) Do the classes $\mathcal{Q}^m$ for $m > 1$ form an infinite hierarchy?

b) Is it possible to define a "natural" class of languages based on combinations of classes from *BASIC above* the class $\mathcal{Q}^*$?

c) Is it possible to (naturally) define a type of language classes with a different way of using or learning "patterns"?

The degrees of "core" classes forming *BASIC* are known to contain many of important "practical" learning problems. For example, *COINIT* contains the class of *pattern languages* [JS96]. However, there certainly exist "natural" classes of infinite/finite languages that are probably incomparable, at least in terms of strong reductions, with some/all classes in *BASIC*. One can add these classes to *BASIC* and apply the formalisms developed in the paper. Exploration of, say, Q-classes based on such extensions of *BASIC* can give a deeper understanding of the nature of learning strategies and learning from texts as a whole.

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