This course material is now made available for public usage.
Special acknowledgement to School of Computing, National University of Singapore for allowing Steven to prepare and distribute these teaching materials.

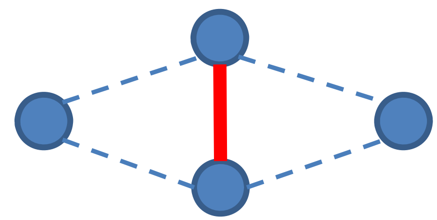
CS3233
Competitive Programming

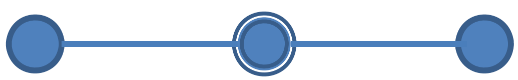
Dr. Steven Halim

Outline

- Mini Contest #6 + Break + Discussion + Admins
- Graph Matching
  - Overview
  - Unweighted MCBM: Max Flow, Augmenting Path, Hopcroft Karp's
    - Relevant Applications: Bipartite Matching (with Capacity), Max Independent Set, Min Vertex Cover, Min Path Cover on DAG
  - Weighted MCBM: Min Cost Max Flow (overview only)
  - Unweighted MCM: Edmonds's Matching
  - Weighted MCM: DP with Bitmask (only for small graph)

Graph Matching

- A matching (marriage) in a graph G (real life) is a subset of edges in G (special relationships) such that no two of which meet at a common vertex (that is, no affair!)
- Thus a.  b.  c. 

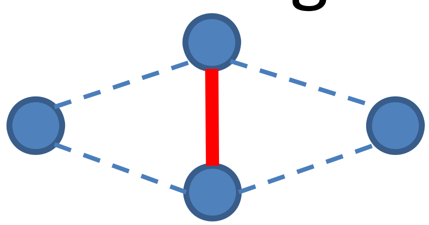
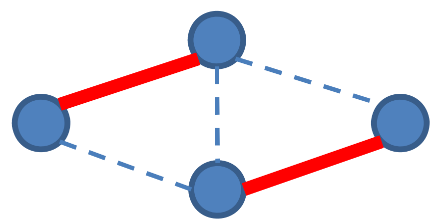
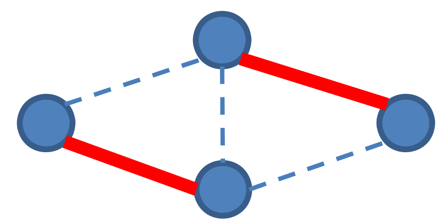
  are matchings (red thick edge),
- But d. 

  is not since there is an overlapping vertex

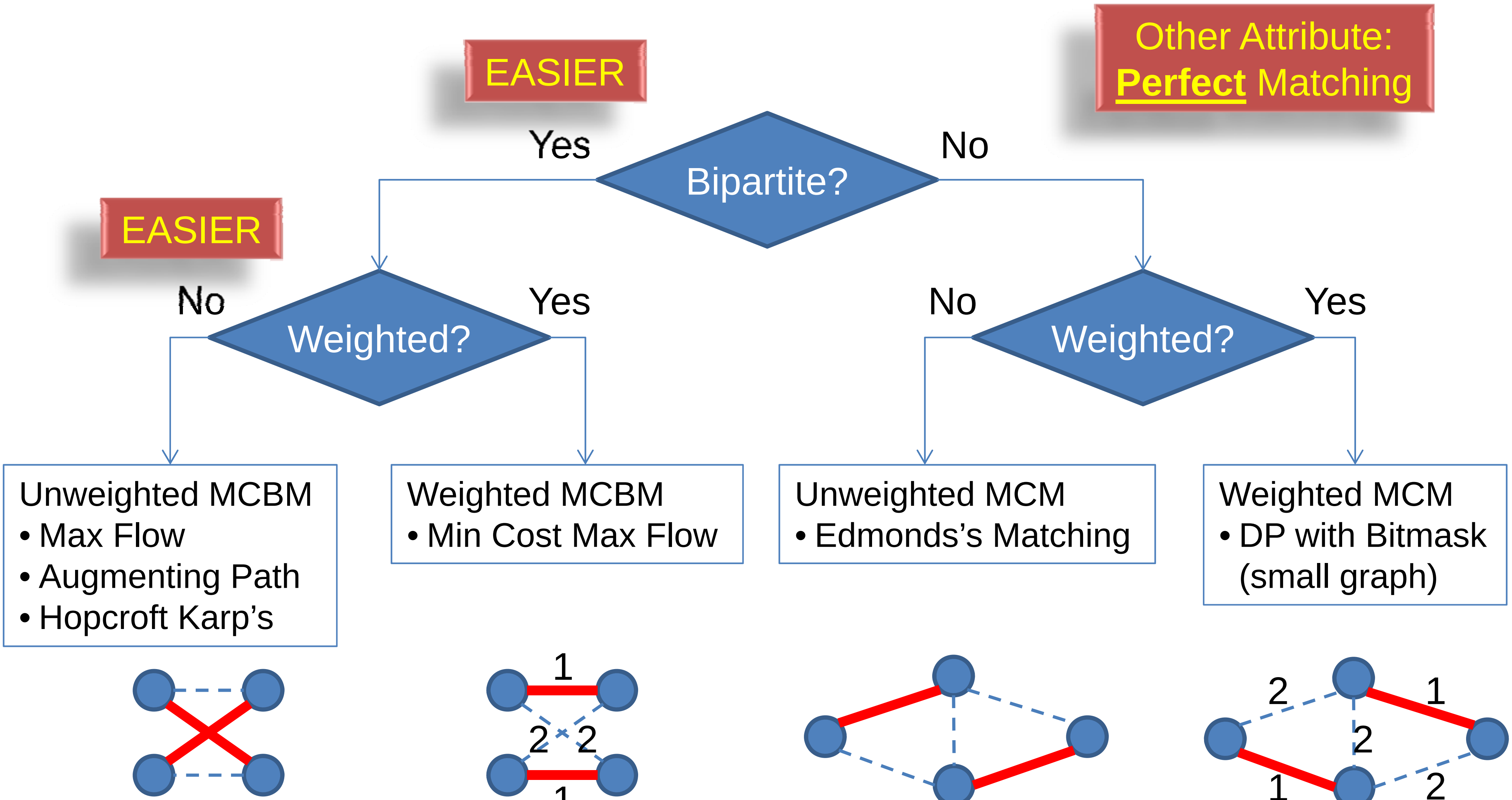
Max Cardinality Matching (MCM)

- Usually, the problem asked in graph matching is the size (cardinality) of a maximum matching
- A maximum matching is a matching that contains the largest possible number of edges

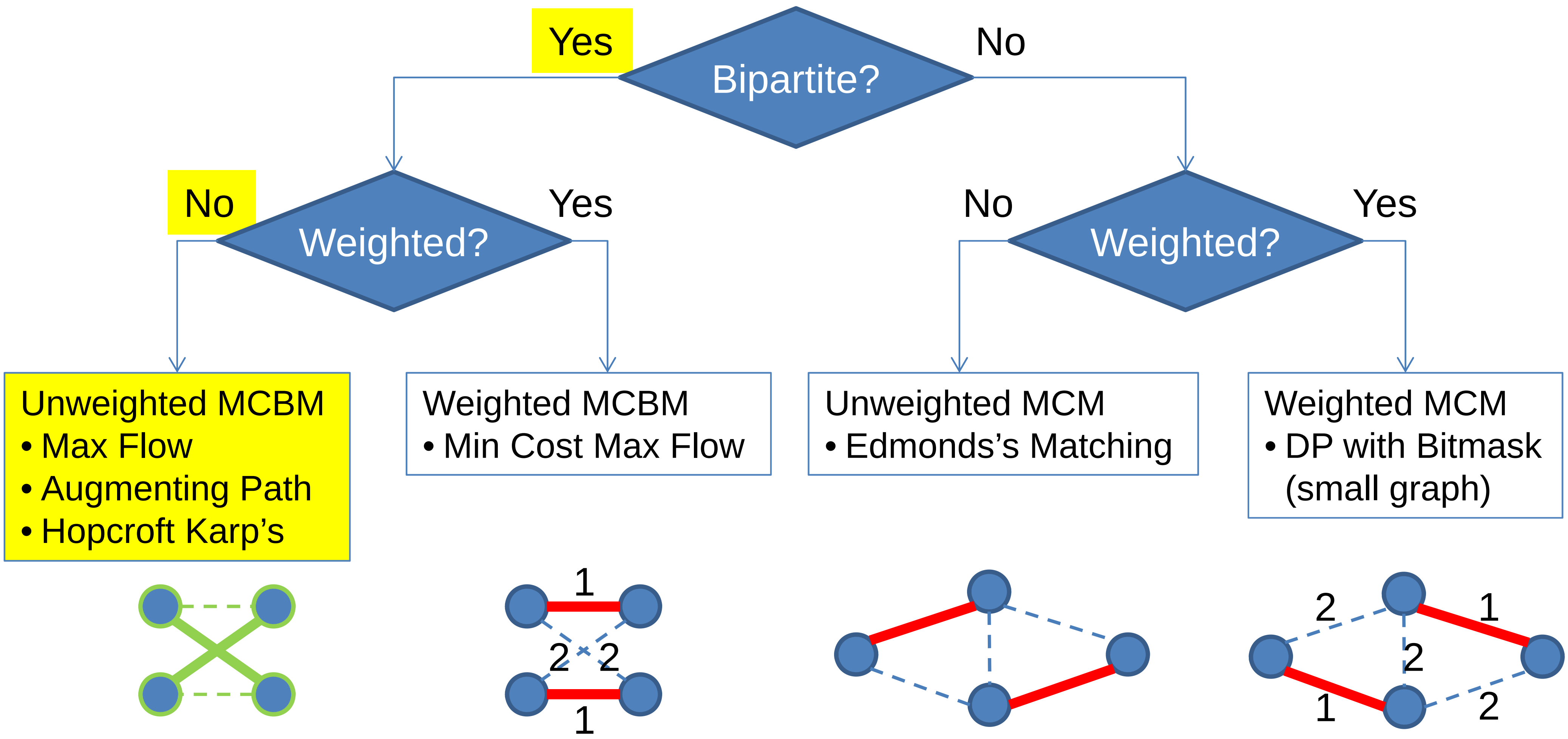
Examples

-  is a maximum matching (0 matching) (no edge to be matched)
-  is also a maximum matching (1 matching) (no other edges to be matched)
- But  is not a maximum matching as we can change it to  or  (2 matchings)

Types of Graph Matching



Types of Graph Matching



Solutions:

Max Flow

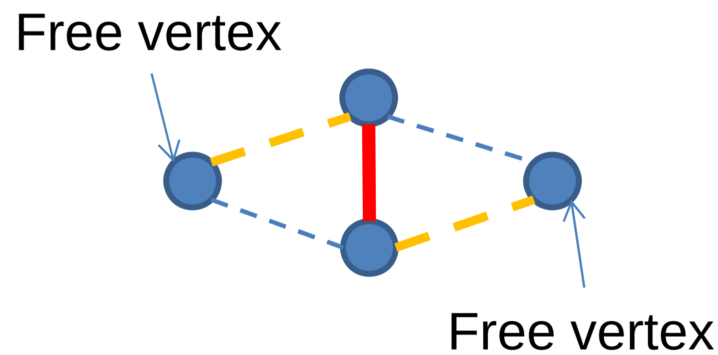
Augmenting Path Algorithm

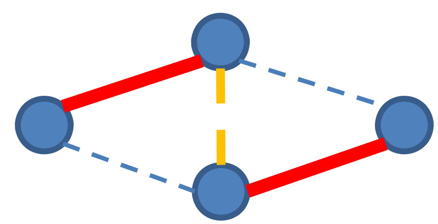
Hopcroft Karp's Algorithm

UNWEIGHTED MCBM

Augmenting Path

- In this graph,
  the path colored
  orange(unmatched)-
  red(matched)-orange
  is an augmenting path
- We can flip the edge
  status to **red-orange-red**
  and the number of
  edges in the matching set
  increases by 1





MC Bipartite Matching (MCBM)

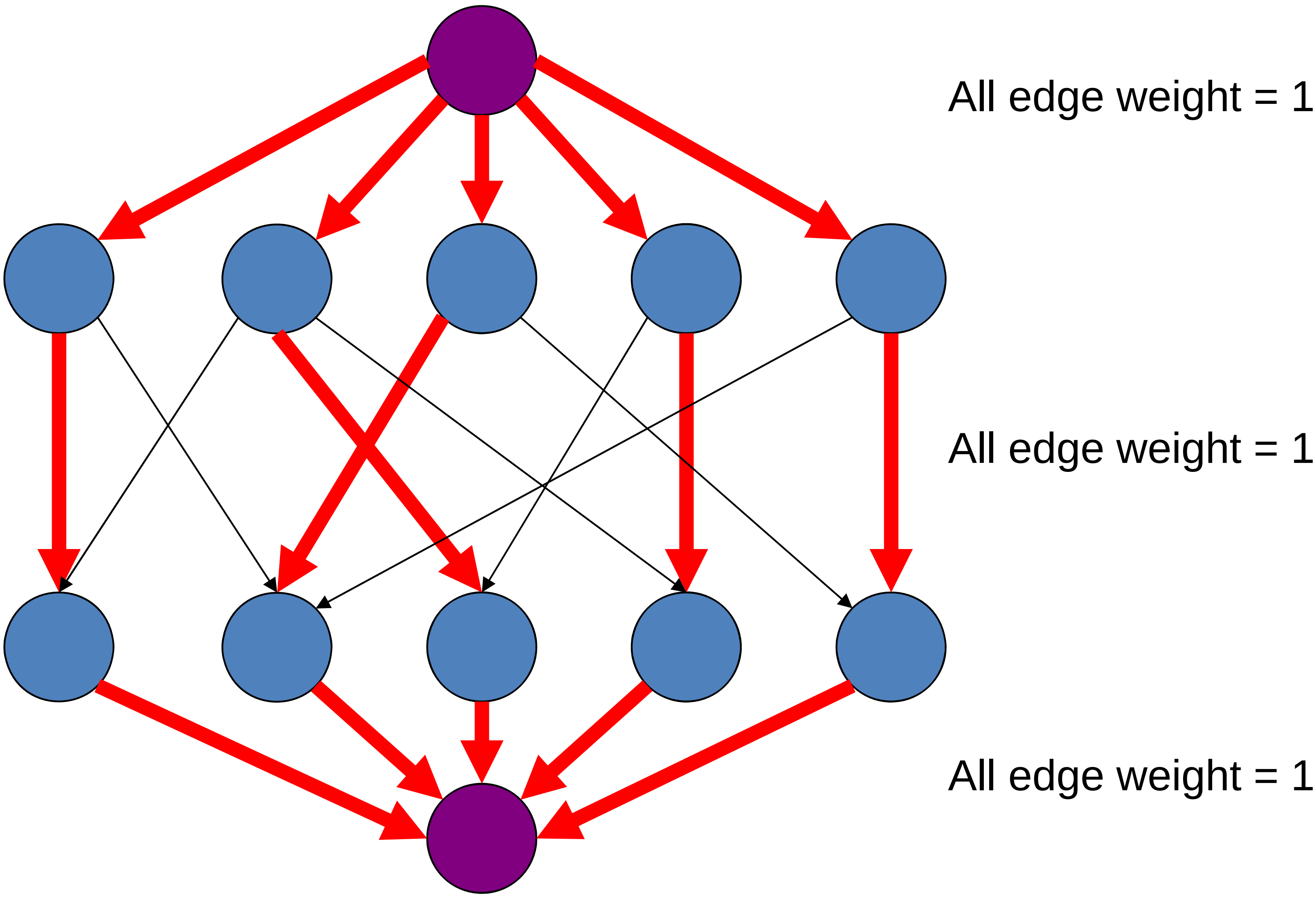
- A Bipartite graph is a graph whose vertices can be divided into two disjoint sets X and Y such that every edge can only connect a vertex in X to one in Y
- Matching in this kind of graph is **a lot easier** than matching in general graph

Finding MCBM by reducing this problem into

MAX FLOW

http://www.comp.nus.edu.sg/~stevenha/visualization/matching.html

Max Flow Solution for MCBM



Time Complexity: Depends on the chosen Max Flow algorithm

Finding MCBM via

AUGMENTING PATH ALGORITHM

Augmenting Path Algorithm

- Lemma (Claude Berge 1957):

> A matching M in G is maximum
> iff there is no more augmenting path in G

- Augmenting Path Algorithm is a simple $O(V*(V+E)) = O(V^2 + VE) \sim= O(VE)$ implementation of that lemma

The Code (1) ☺

```
vi match, vis; // global variables

int Aug(int l) { // return 1 if ∃ an augmenting path
  if (vis[l]) return 0; // return 0 otherwise
  vis[l] = 1;
  for (int j = 0; j < (int)AdjList[l].size(); j++) {
    int r = AdjList[l][j].first;
    if (match[r] == -1 || Aug(match[r])) {
      match[r] = l;
      return 1; // found 1 matching
  } }
  return 0; // no matching
}
```

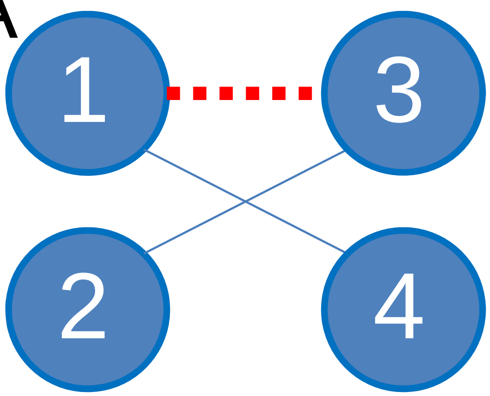
The Code (2) ☺

```
// in int main(), build the bipartite graph
// only directed edge from left set to right set is needed

  int MCBM = 0;
  match.assign(V, -1);

  for (int l = 0; l < Vleft; l++) {
    vis.assign(Vleft, 0);
    MCBM += Aug(l);
  }

  printf("Found %d matchings\n", MCBM);
```

http://www.comp.nus.edu.sg/~stevenha/visualization/matching.html

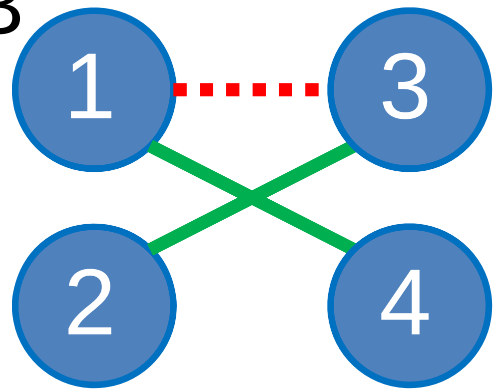
Augmenting Path Algorithm

A



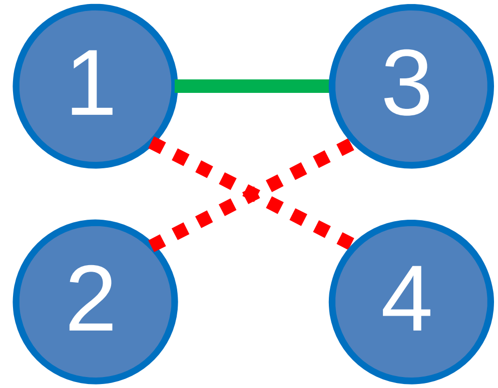
Easy Assignment
1 matching (dotted line)

B



Augmenting Path
2-3-1-4

D

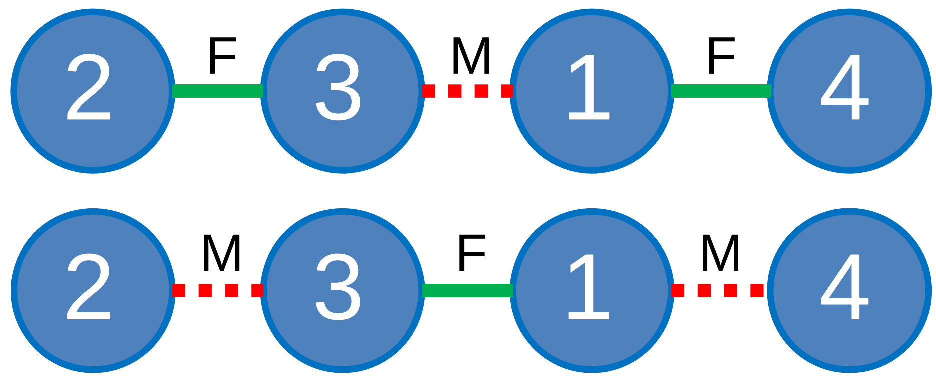


After Flip
2 matchings (dotted lines)

C

An augmenting path
F=Free, M=Matched

Flip to increase matching
from 1 to 2 matchings



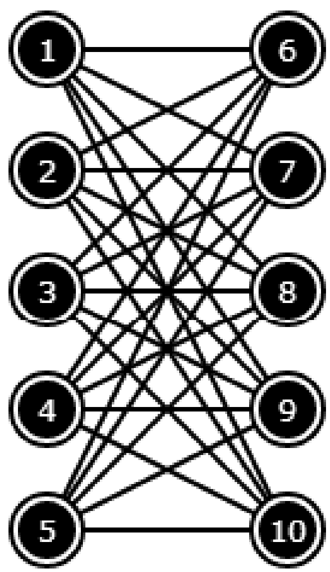
Finding MCBM via

HOPCROFT KARP'S ALGORITHM

http://www.comp.nus.edu.sg/~stevenha/visualization/matching.html

An Extreme Test Case…

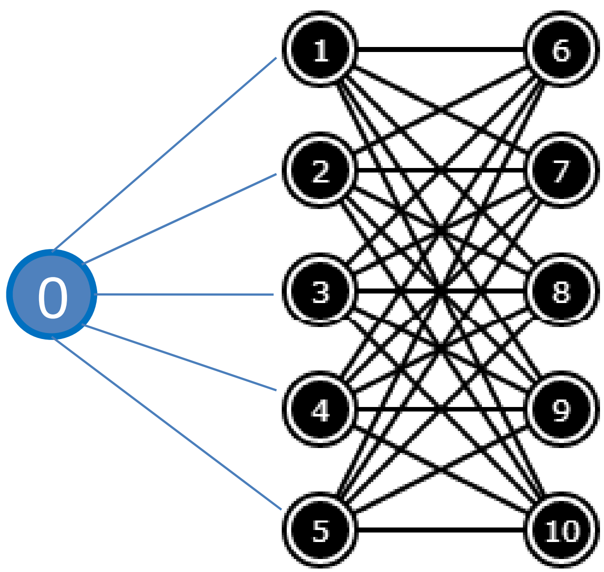
- A Complete Bipartite Graph $K_{n,m}$, **V=n+m** & **E = n*m**
- Augmenting Path algorithm → O((n+m)*n*m)
  - If m = n, we have an $O(n^3)$ solution, OK for n ≤ 200
- Example with n = m = 5



http://www.comp.nus.edu.sg/~stevenha/visualization/matching.html

Hopcroft Karp's Algorithm (1973)

- Key Idea:
  - Find the shortest augmenting paths first from all free vertices (with BFS)
  - Run similar algorithm as the Augmenting Path Algorithm earlier (DFS), but now using this BFS information



http://www.comp.nus.edu.sg/~stevenha/visualization/matching.html

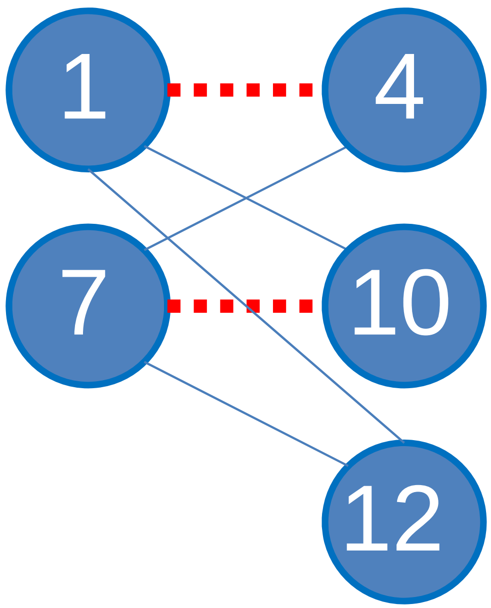
Hopcroft Karp's Algorithm (1973)

- Hopcroft Karp's runs in $O(E\sqrt{V})$, proof omitted
  - For the extreme test case in previous slide, this is O(n*m*sqrt(n+m))
  - With $m = n$, this is about $O(n^{5/2})$, OK for $n \leq 600$
- Question: Is this algorithm **must be learned** in order to do well in programming contest?

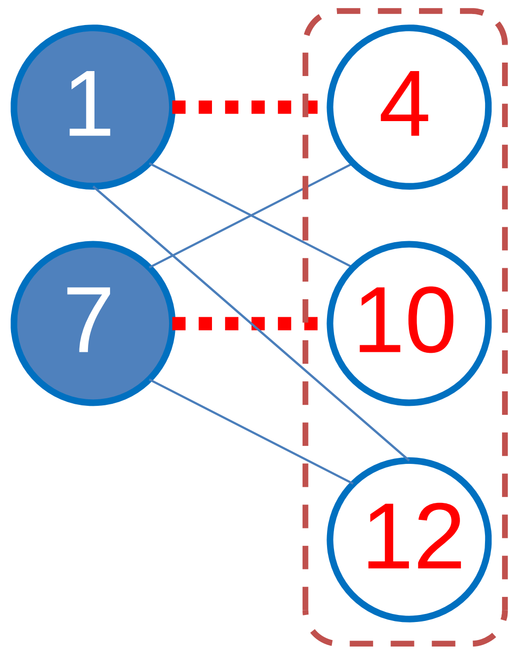
EXAMPLES OF MCBM IN PROGRAMMING CONTESTS

Popular Variants

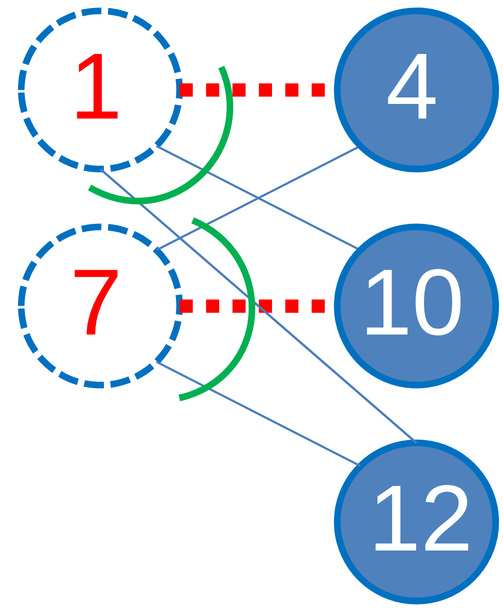
Max Independent Set / Min Vertex Cover



A. MCBM



B. Max Independent Set
MIS: V – MCBM

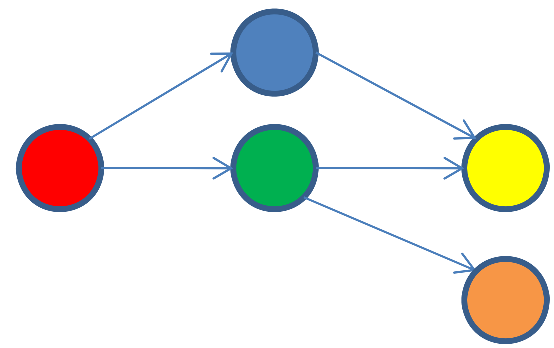


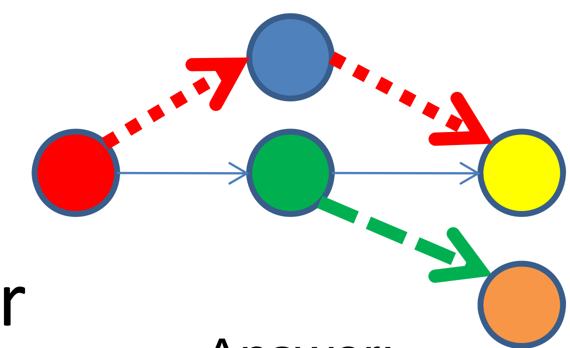
C. Min Vertex Cover
MVC: MCBM

(König's theorem)

Min Path Cover
in DAG

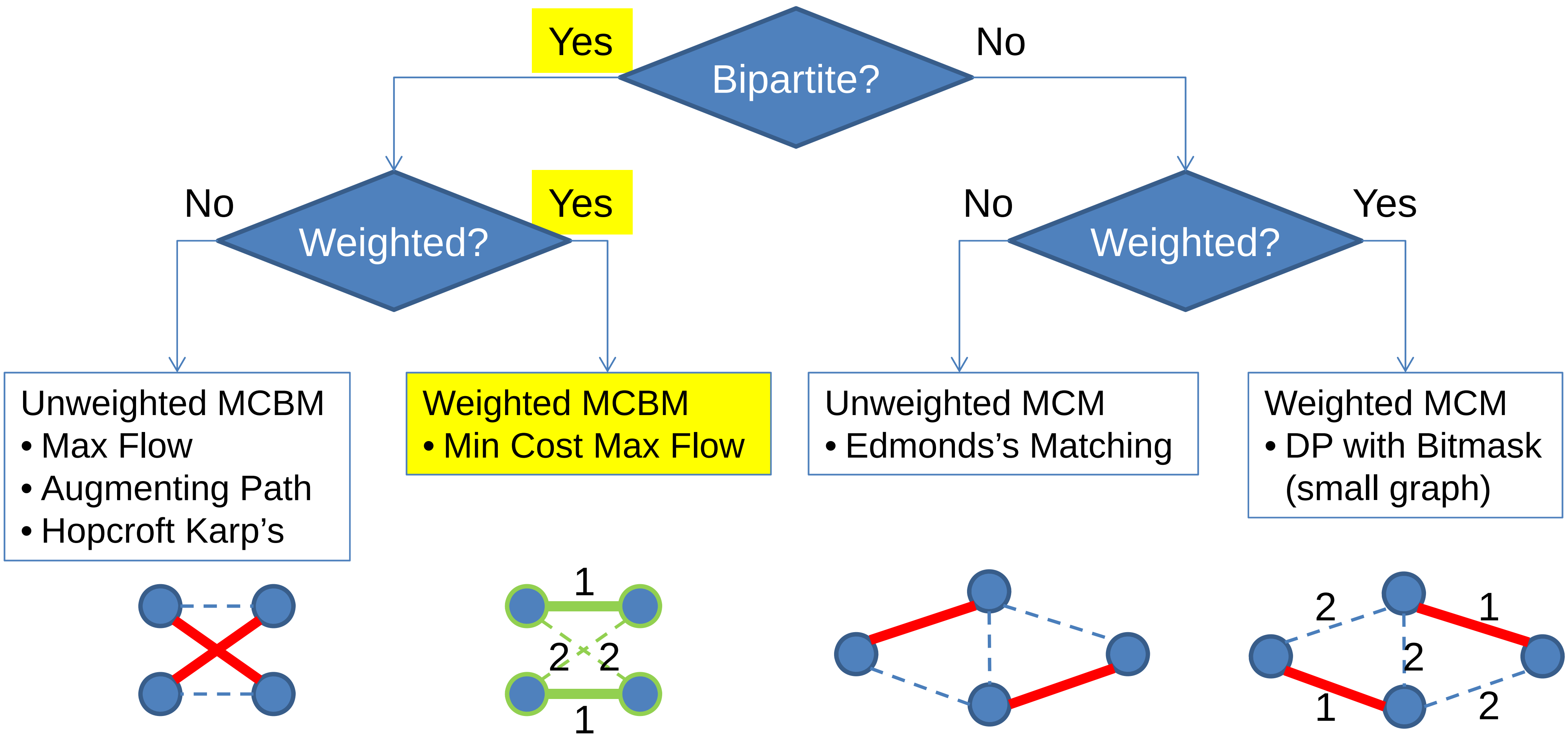
- Illustration:
  – Imagine that vertices are passengers, and draw edge between two vertices if a single taxi can satisfy the demand of both passengers on time…
  – What is the minimum number of taxis that must be deployed to serve all passengers?
- This problem is called: Min Path Cover
  – Set of directed paths s.t. every vertex in the graph belong to at least one path (including path of length 0, i.e. a single vertex)





Answer:
2 Taxis!

Types of Graph Matching



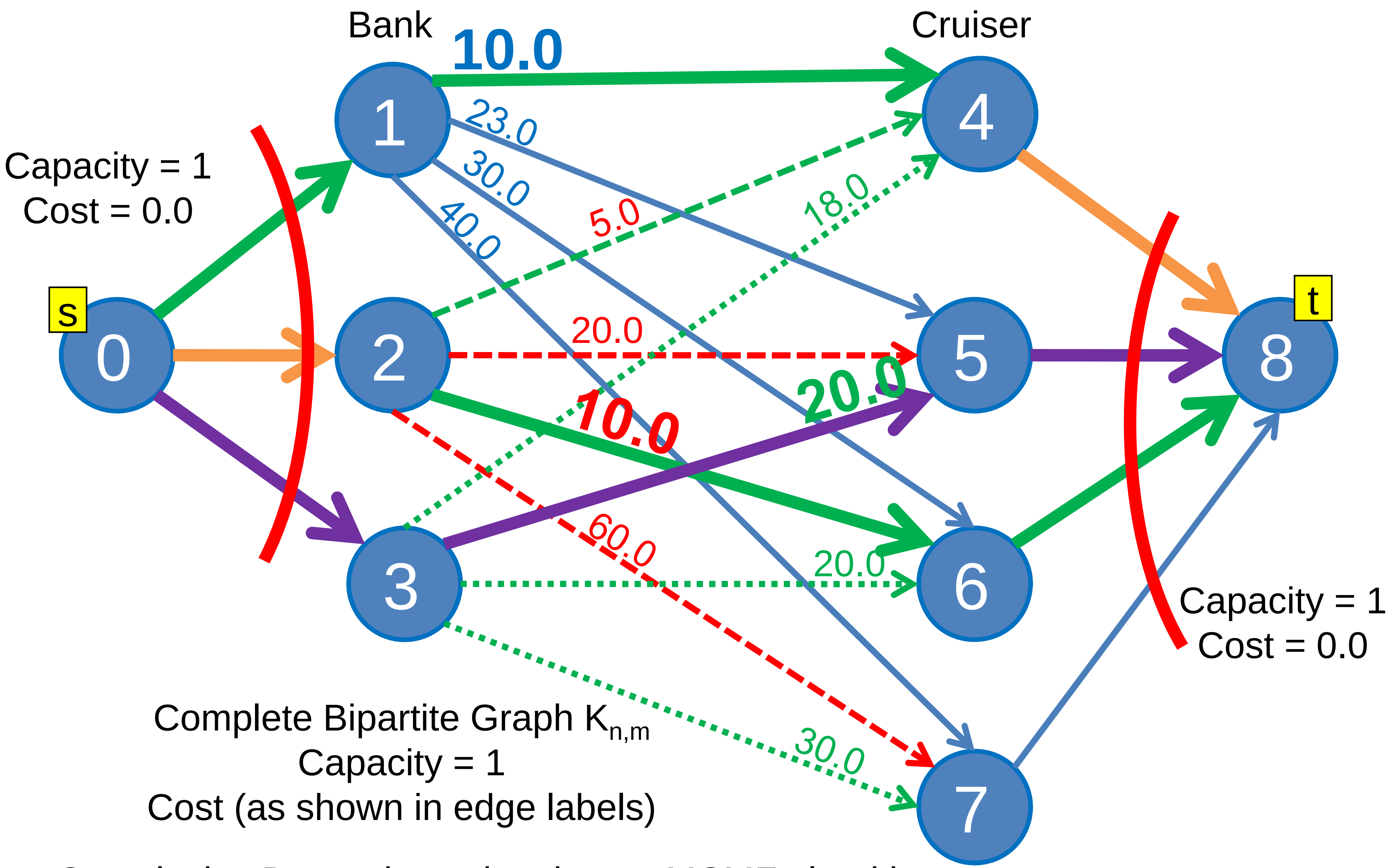
Solution:

Min Cost Max Flow (Overview Only)

WEIGHTED MCBM

UVa 10746 (Solution)

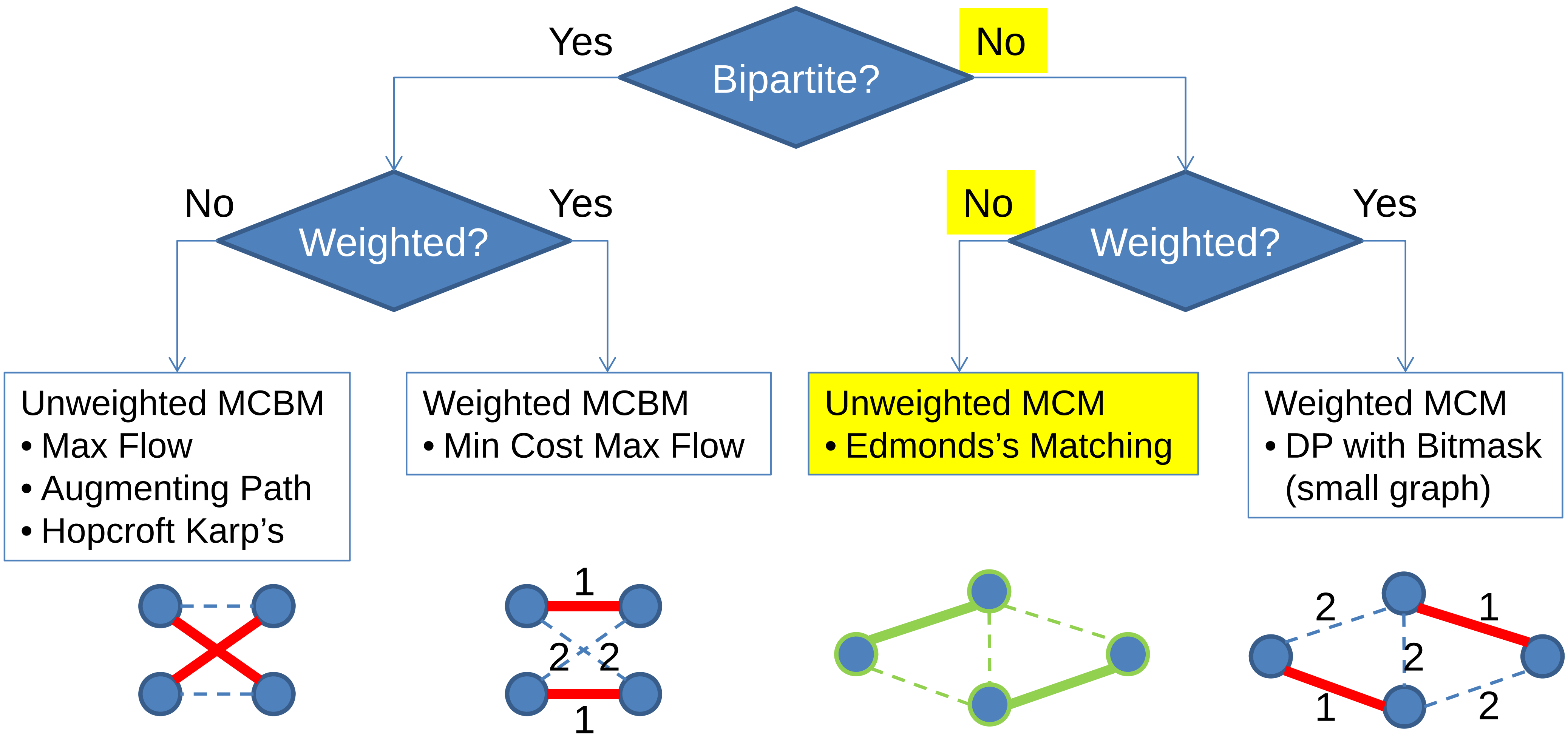
Min Cost so far =
0 + **5.0** + 0 +
0 + 10.0 **- 5.0** + 10.0 + 0 +
0 + 20.0 + 0 = 40.0



Complete Bipartite Graph $K_{n,m}$
Capacity = 1
Cost (as shown in edge labels)

Time Complexity: Depends on the chosen MCMF algorithm

Types of Graph Matching



Solution:

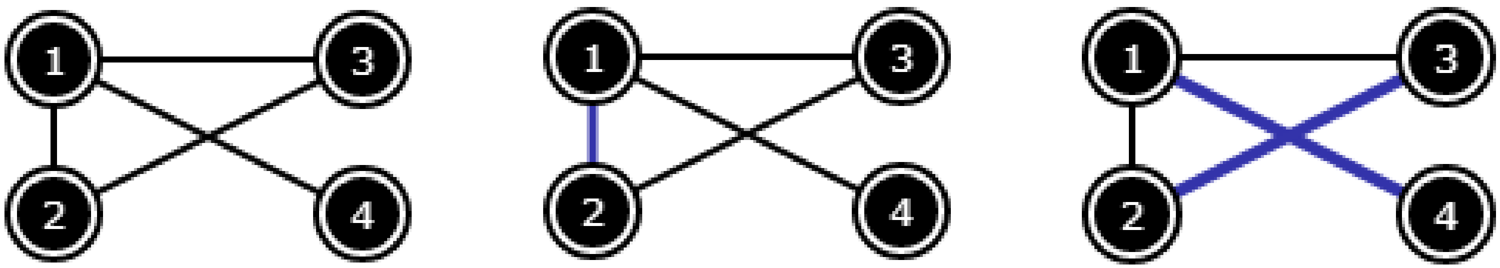
Edmonds's Matching Algorithm

UNWEIGHTED MCM

http://www.comp.nus.edu.sg/~stevenha/visualization/matching.html

Blossom

- A graph is not bipartite if it has at least one odd-length cycle (blossom)
- What is the MCM of this non-bipartite graph?



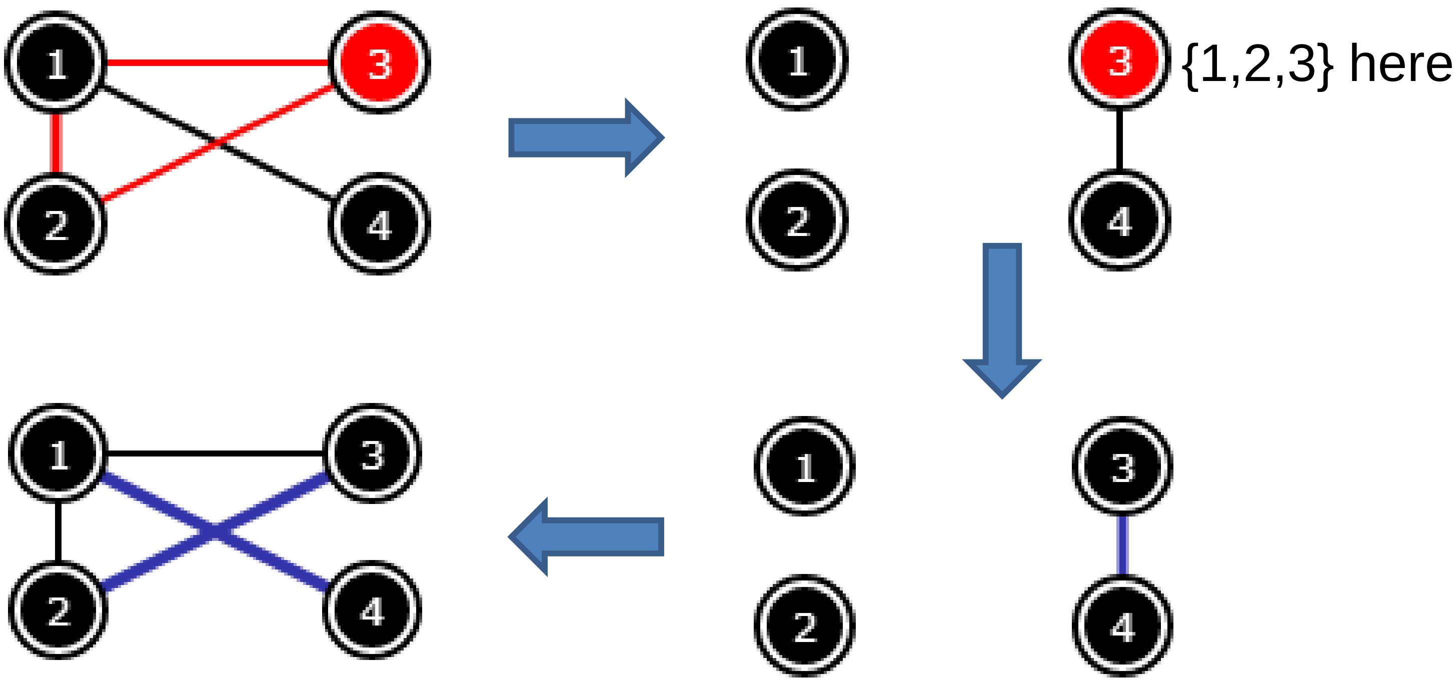
MCM = 2

- Harder to find augmenting path in such graph

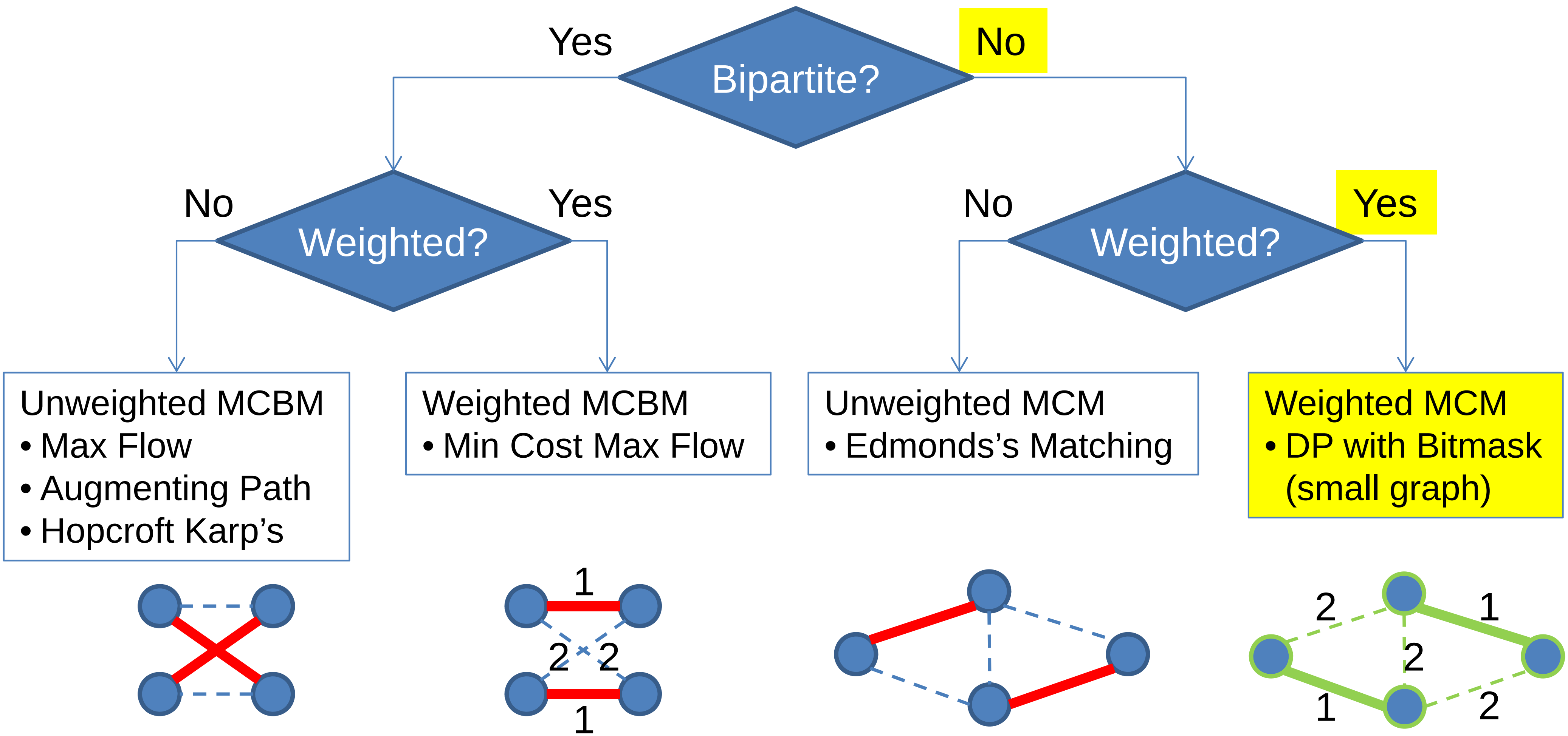
http://www.comp.nus.edu.sg/~stevenha/visualization/matching.html

Blossom Shrinking/Expansion

- Shrinking these blossoms (recursively) will make this problem “easy” again



Types of Graph Matching



Solution:

DP with Bitmask (only for small graph)

WEIGHTED MCM

Graph Matching in ICPC

- Graph matching problem is quite popular in ICPC
  - Sometimes 0 problem but likely 1 problem in the set
  - Perhaps disguised as other problems, e.g. Vertex Cover, Independent Set, Path Cover, etc → reducible to matching
- If such problem appear and your team can solve it, very good ☺
  - Your team will have +1 point advantage over significant # of other teams who are not trained with this topic yet…
- For IOI trainees… all these Graph Matching stuffs…
  - **THEY ARE NOT IN THE SYLLABUS TOO :O:O:O…**

References

- **CP2.9, Section 4.7.4, 9.15 ☺**
- **New write up about Graph Matching**